also represents the actual business as practiced in the counting-houses of me
om-houses, banks and all kinds of incorporated companies.

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CONTENTS.

PREFACE.

THE cordial reception given to my Normal Mental and Written Arithmetics immediately created a demand for a Higher Arithmetic, written upon the same general plan. This demand has become more and more pressing each year, as the publication of the work, which was known to be in preparation, was delayed. The care of a large institution, in connection with many other professional duties, has occupied so much of my time during the last few years, that it was impossible to have the work ready for the press at an earlier day Indeed, it would have been much longer delayed had it not been for the assistance of Miss Deborah P. Atherton, a former pupil, who has rendered most valuable aid in its preparation.

As now presented, the work is, as its title indicates, a *Higher Arithmetic*. The object has been to give quite a full treatise upon the science of numbers and its most extensive applications. Especial pains have been taken to exhibit the logical relations of the science; to present clear and concise definitions and solutions; to state the rules and principles in a brief, exact, and comprehensive form; and to make an extensive application of its methods to the business practices of the country at the present time. Developed in accordance with this plan, it is thought that the work will commend itself to public favor on account of both its scientific and practical character. Attention is briefly called to a few of its striking peculiarities in both of these respects.

SCIENTIFIC.—The object has been, as stated above, to present a scientific treatise upon the science of numbers. Formerly arithmetic was treated mainly as an art, and the pupil was drilled in mechanical processes, without any conception of the interesting relations of the science and the simplicity of its reasoning processes. A great improvement has been made in this respect within the last quarter of a century. Arithmetic has risen to the dignity of a science, and is beginning to stand in logical completeness beside its sister science, Geometry. I have endeavored to carry out the spirit of modern arithmetic by presenting its principles in logical order, and by mak

ing such contributions as I thought would more fully accomplish this object. With respect to this general feature of the work, attention is invited to the following points:

1. The Logical Outline of Arithmetic, in which it is assumed that all arithmetical processes are embraced under *Synthesis*, *Analysis*, and *Comparison*.

2. The logical presentation of the language of arithmetic, showing the principle of Numeration and its true relation to Notation.

3. The use of the word *term* in the Fundamental Rules, so as to avoid the error of confounding the words *figure* and *number*.

4. The comprehension of the several processes following the Fundamental Rules, under the head of *Secondary Processes* of arithmetic.

5. The treatment of Greatest Common Divisor and Least Common Multiple, and the extension of these processes to Decimals and Denominate Numbers.

6. A New Method of Cube Root, as previously presented in my Elementary Algebra.

7. Important modifications of definitions, as in Multiplication, Division, Fractions, Denominate Numbers, Ratio, Similar Repetends, etc.

8. The logical division of subjects into cases, especially valuable to one preparing to teach arithmetic.

9. Interesting historical notes introduced throughout the entire work.

PRACTICAL.—Though the science of arithmetic is important to the teacher and scholar, the practice of arithmetic, to the man of business, is none the less important. The large majority of those who study arithmetic, need to use it in the transaction of the practical affairs of life; hence a text-book on the subject should be especially practical. It should, so far as is possible, represent the actual business methods of the times. This has been the especial aim in the preparation of this work. While a few of the problems will be found to be mainly intended to illustrate some principle of the science, or to prepare for the more intricate business problems, by far the greater number are inserted for the purpose of showing the application of the science to the actual business transactions of the day.

In this practical character it is believed that the book will be found especially strong and reliable. It has been the aim to represent all the leading business methods and practices of the times. This idea has been carried all through the book, and constitutes one of its most prominent features. In its terms, names of articles, practical examples, methods, forms, etc., it will be found, it is thought, to be

an actual reflection of all the great leading lines of business in this intensely practical and busy age. As examples, we call attention to many of the Practical Problems in the Fundamental Rules, the application of Decimals, the forms of Bills and Accounts, the examples in Denominate Numbers, the varied and extensive applications of Percentage. The articles on Exchange, Custom House Business, Partnership, Insurance, Building Associations, etc., were prepared from material obtained directly from those connected with these various lines of business, and represent the actual business transactions of the present day. The subject of Building Associations is here for the first time presented in a work of this kind.

In this application of the science, it is of course not possible, nor is it desirable, to represent the minor details of every known business interest in the country, since this would require several volumes instead of a single work. The object has been to represent the processes and methods used in all the leading forms of business, so that a pupil trained in these general methods shall be able to apply his knowledge readily to any particular form. In this manner the young man goes out into the business world, not an imitative parrot, capable only of following a particular routine, but an intelligent person, with ability to adopt or originate any process that may be regarded as best for the special case which arises.

These are the principles by which I have been guided in the preparation of this work. Realizing that theory alone renders a man unpractical in life, I have endeavored not to restrict this work to the mere theory of numbers. Realizing also that practice alone gives a person no power to adapt old or originate new processes in particular cases that may arise, I have not confined myself to the presentation of the merely mechanical methods of the counting-house and the market. The object has been to find the golden mean in this respect, and to give that union of theory and practice which shall result in the best mental discipline and the most thorough training. The motto has been, *Theory and Practice, properly combined, give the desirable results of Mental Power and Business Capacity.*

The work does not aim at novelties, but is based upon that system of arithmetic which has grown up in our schools under the wisdom and experience of the best teachers of the last half century. The ambition has been the improvement of the established system, rather than the futile attempt to create a new one. Neither has the work been shaped to meet that spirit of superficiality in arithmetical instruction which is now quite popular among a certain class of educators, but it presents a full and thorough course of instruction in the science. It is designed, not for *superficial*, but for *thorough* teachers

of arithmetic,—for teachers who realize that there is no royal road to mathematics, but that all solid attainments come by hard work, and that one of *the most important elements of an education is the acquisition of the habit of persistent and self-reliant labor.* Above all, the wants of the class-room have been kept constantly in view, and the effort has been continually made to realize and meet the wants of good teachers and earnest students.

With a profound sense of the responsibility of one who attempts to write text-books for the intellectual training of the rising generation, and an earnest desire to measure up to the high demands of this responsibility, this work has been written and is now presented for the consideration of teachers, educators, and directors of public instruction, to whom is intrusted the development of the intellect of the present and future generations.

EDWARD BROOKS.

STATE NORMAL SCHOOL,
Jan. 16, 1876.

THE

NORMAL

HIGHER ARITHMETIC.

INTRODUCTION.

NATURE OF ARITHMETIC.

1. Mathematics is the science of quantity. It treats of the properties and relations of quantity.

2. Quantity is anything that can be measured. It is of two kinds, *Number* and *Extension*.

3. Arithmetic is the science of *Number;* Geometry is the science of *Extension*.

4. Arithmetic **embraces** *ideas* and *truths*. The ideas give rise to *Definitions:* and the truths, to *Principles* and *Problems*.

5. A **Definition** is a concise statement of the distinctive qualities of anything.

6. A **Principle** is a truth of science. *Principles* may be in the form of *Axioms* or *Theorems*.

7. An **Axiom** is a self-evident truth. *Axioms* are the laws which control the reasoning processes.

8. A **Theorem** is a truth which becomes evident by a process of reasoning called a *Demonstration*.

9. A **Demonstration** is a process of reasoning by which the truth of a theorem is proved.

10. A **Problem** is a question requiring some unknown result from that which is known.

11. The **Conditions** of a Problem are the known truths that are given.

12. A **Solution** of a Problem is a process of obtaining the desired result.

13. A **Mental Solution** is one in which the operations are performed without the aid of written characters.

14. A **Written Solution** is one in which the operations are performed by the aid of written characters.

15. A **Rule** is a statement of the method of solving a problem.

16. **Arithmetical Analysis** is the process of solving problems by a comparison of their elements.

17. **Arithmetic** is the science of numbers and the art of computing with them.

18. A **Number** is a unit or a collection of units. Numbers are *Concrete* and *Abstract.*

19. A **Unit** is a single thing or *one*. A single thing is a *concrete unit*; *one* is an *abstract unit.*

20. A **Concrete Number** is one applied to some particular object: as, *two* yards, *five* books, etc.

21. An **Abstract Number** is one not applied to any particular object; as, *two, five,* etc.

22. **Similar Numbers** are those in which the units are alike; as, two *boys* and four *boys*.

23. **Dissimilar Numbers** are those in which the units are unlike; as, two *boys* and four *books*.

24. The **General Classes** of Numbers treated of in Arithmetic are *Integers, Fractions,* and *Denominate Numbers.*

25. An **Integer** is a number of integral units; as, *four, five,* etc.

26. A **Fraction** is a number of the equal divisions of a unit; as, *two-thirds, three-fourths,* etc.

27. A Denominate Number is a number in which the unit is a measure of continuous quantity; as, three *yards*, four *pounds*.

28. These **Three Classes** of numbers admit of the same general processes, and, as subject matter, give rise to a triune division of Arithmetic.

29. The **Processes** of Arithmetic may all be embraced under the three heads, *Synthesis*, *Analysis*, and *Comparison*.

30. The fundamental idea of Arithmetic is the *Unit* or *one*. The synthesis of units gives rise to *Numbers*. Numbers may be subjected to the operations of *synthesis*, *analysis* and *comparison*, and out of these processes arise all the subjects of arithmetic.

31. Fundamental Processes.—A general synthesis is called *Addition*. A special case of Addition, in which the numbers united are all the same, is called *Multiplication*.

A general analysis is called *Subtraction*. A special case of Subtraction, in which the object is to find how many times one number contains another, is called *Division*.

These four processes are called the *Fundamental Operations* of Arithmetic. From these four processes arise others which we may call *Derivative Processes*.

32. Derivative Processes.—A general synthesis of *factors* to form composite numbers may be called *Composition*. A synthesis of *equal* factors is *Involution*. A synthesis of factors to find a number which is one or more times several numbers, is called *Common Multiple*.

An analysis of a number into its *factors* is called *Factoring*. An analysis into *equal* factors is called *Evolution*. The finding of a *common factor* of several numbers is called *Common Divisor*.

These divisions have their origin in synthesis and analysis, and grow out of them. There are several other divisions which have their origin in and grow out of *Comparison*.

33. Comparison.—The comparison of two numbers gives rise to *Ratio*. The comparison of *equal ratios* gives

rise to *Proportion.* The comparison of several numbers differing by a *common ratio* gives rise to *Progression.*

In comparing numbers, we see that we can often change a number from one class of units to another, which gives rise to *Reduction.* In comparing numbers, we may assume some number as a basis of reference, and develop their relation in regard to this basis; when this basis is *a hundred,* we have *Percentage.* Numbers may be compared and their properties investigated, which gives rise to the *Properties of Numbers.*

34. We thus have a complete outline of Arithmetic. It is considered, first, as treating of three classes of numbers, *Integers, Fractions* and *Denominate Numbers.* Its processes are also three-fold, *Synthesis, Analysis* and *Comparison.* The whole science of Arithmetic is an outgrowth of this triune basis.

Logical Outline.
- Synthesis
  - Addition
  - Multiplication
    - Composition
    - Involution
    - Common Multiple
- Analysis
  - Subtraction
  - Division
    - Factoring
    - Evolution
    - Common Divisor
- Comparison
  1. Ratio
  2. Proportion
  3. Progressions
  4. Reduction
  5. Percentage
  6. Properties of Numbers

NOTE.—For a fuller discussion of this subject, see *Brooks's Philosophy of Arithmetic.*

SECTION I.

ARITHMETICAL LANGUAGE.

35. Arithmetical Language is the method of expressing numbers.

36. Arithmetical Language is of two kinds, *Oral* and *Written.* The former is called *Numeration* and the latter *Notation.*

NUMERATION.

37. Numeration is the method of naming numbers, and of reading them when expressed by characters. It is the *oral expression* of numbers.

38. Since it would require too many words to give each number a separate name, numbers are named according to the following simple principle:

Principle.—*We name a few of the first numbers, and then form groups or collections, name these groups, and use the names of the first few numbers to number these groups.*

39. A single thing is named *one;* one and one more are named *two;* two and one more, *three;* three and one more, *four;* and thus we obtain the simple names,

One, two, three, four, five, six, seven, eight, nine, ten.

40. Regarding the collection *ten* as a single thing, we might count *one and ten, two and ten,* etc., as far as *ten and ten,* or *two tens,* which modified by use would give the following numbers:

Eleven, twelve, thirteen, fourteen, fifteen, sixteen, seventeen, eighteen, nineteen, twenty.

41. Proceeding in the same way, we would have *two tens and one, two tens and two, two tens and three,* etc., which modified by use would give the following numbers:

Twenty-one, twenty-two, twenty-three, twenty-four, twenty-five, twenty-six, twenty-seven, twenty-eight, twenty-nine.

42. Continuing in the same manner, we would have *three-*

tens, four-tens, five-tens, etc. By this principle were derived the following ordinary names:

Twenty, thirty, forty, fifty, sixty, seventy, eighty, ninety.

43. A group of *ten tens* is called a *hundred;* a group of *ten hundreds*, a *thousand;* the next group receiving a new name consists of a *thousand thousands*, called a *million;* the next group of a *thousand millions*, called a *billion*, etc.

44. After a *thousand* the two intermediate groups between those receiving a distinct name, are numbered by *tens* and *hundreds*, as *ten thousand* and *hundred thousand.*

NOTES.—1. The above shows the *principle* by which the names of numbers were formed. The names, however, were not derived from the particular expressions given, but originated in the Saxon language.

2. *Eleven* is from the Saxon *endlefen*, or Gothic *ainlif* (*ain, one*, and *lif, ten*); *twelve* is from the Saxon *twelif*, or Gothic *tvalif* (*tva, two*, and *lif, ten*). Some think *eleven* meant *one left after ten*, and *twelve, two left after ten.*

3. *Twenty* is from the Saxon *twentig* (*twegen, two*, and *tig*, a *ten*); *thirty* is from the Saxon *thritig* (*thri, three*, and *tig*, a *ten*), etc.

4. *Hundred* is a primitive word; *thousand* is from the Saxon *thusend*, or Gothic *thusundi* (*thus, ten*, and *hund, hundred*); *million, billion*, etc., are from the Latin.

NOTATION.

45. Notation is the method of writing numbers. Numbers may written in three ways:

1st. By *words*, or common language.

2d. By *figures*, called the *Arabic Method.*

3d. By *letters*, called the *Roman Method.*

ARABIC NOTATION.

46. The **Arabic System** of Notation is the method of expressing numbers by characters called *figures.*

47. In this system numbers are expressed according to the following principle:

Principle.—*We represent the first nine numbers by characters, and then use these characters to number the groups, indicating the group numbered by the position of the character.*

48. Figures. Figures are characters used in expressing numbers. There are ten figures used, as follows:

FIGURES.	1,	2,	3,	4,	5,	6,	7,	8,	9,	0,
NAMES AND VALUES.	one,	two,	three,	four,	five,	six,	seven.	eight,	nine.	naught, cipher or zero.

49. By the combination of these figures all numbers may be expressed; hence they are appropriately called the *alphabet* of *arithmetic*.

50. Combination. These figures are combined according to the following principles:

1. *A figure standing alone, or in the first place at the right of other figures, expresses* UNITS *or* ONES.

2. *A figure standing in the second place, counting from the right, expresses* TENS; *in the third place,* HUNDREDS; *in the fourth place,* THOUSANDS, etc.; thus:

10 is 1 ten, or ten.	100 is 1 hundred.
20 " 2 tens, or twenty.	200 " 2 hundred.
30 " 3 tens, or thirty.	520 " 5 hundred and twenty.
40 " 4 tens, or forty.	456 " 4 hundred and fifty-six.
56 " 5 tens and six units.	1000 " 1 thousand.
68 " 6 tens and eight units.	2000 " 2 thousand.

51. Periods. For convenience in writing and reading numbers, the figures are arranged in *periods* of three places each. The first three places constitute the *first* or *units period;* the second three places, the *second* or *thousands period,* etc.

52. The terms of each period are considered respectively as the *units, tens* and *hundreds* of that period.

53. The name of each of the first eight periods is represented by the following

NUMERATION TABLE.

	Hundred-sextillions.	Ten-sextillions.	Sextillions.	Hundred-quintillions.	Ten-quintillions.	Quintillions.	Hundred-quadrillions.	Ten-quadrillions.	Quadrillions.	Hundred-trillions.	Ten-trillions.	Trillions.	Hundred-billions.	Ten-billions.	Billions.	Hundred-millions.	Ten-millions.	Millions.	Hundred-thousands.	Ten-thousands.	Thousands.	Hundreds.	Tens.	Units.
	8	8	8,	8	8	8,	8	8	8,	8	8	8,	8	8	8,	8	8	8,	8	8	8,	8	8	8.
PLACES.	24th,	23d,	22d,	21st,	20th,	19th,	18th,	17th,	16th,	15th,	14th,	13th,	12th,	11th,	10th,	9th,	8th,	7th,	6th,	5th,	4th,	3d,	2d,	1st.
PERIODS.	8th.			7th.			6th.			5th.			4th.			3d.			2d.			1st.		

54. This table enables us to read a numerical expression of twenty-four figures. The succeeding periods are *Septillions, Octillions, Nonillions, Decillions, Undecillions, Duodecillions, Tertio-decillions, Quarto-decillions, Quinto-decillions, Sexto-decillions, Septo-decillions, Octo-decillions, Nono-decillions, Vigillions,* etc.

55. The combination of figures to express a number forms a *numerical expression.*

56. The different figures of a numerical expression are called *terms. Terms* are also used to indicate the numbers represented by the figures.

NOTE.—The use of the word *term* to indicate both the figures and numbers represented by them enables us to avoid the error of using the word *figure* for the word *number.* Thus, instead of saying, "add the figures," which is an absurdity, we can say, *add the terms,* meaning the numbers denoted by the figures.

EXERCISES IN NUMERATION.

57. The pupils are now prepared to learn to *read* numbers when expressed by *figures.* From the preceding explanations, we have the following *rule for numeration:*

Rule.—I. *Begin at the right hand and separate the numerical expression into periods of three figures each.*

II. *Then begin at the left hand and read each period in succession, giving the name of each period except the last.*

NOTE.—The name of the last period is usually omitted, it being understood.

1. What number is expressed by 5468217?

OPERATION.

5,468,217

SOLUTION.—Separating the numerical expression into periods of three figures each, beginning at the right hand, we have 5,468,217. The third period is 5 *millions,* the second period is 468 *thousands,* and the first 217 *units;* hence the number is 5 million, 468 thousand, 217.

Read the following numerical expressions:

2.	2356741	**6.**	71390268156
3.	4009637	**7.**	10203004000
4.	41327984	**8.**	40002005071829
5.	502800004	**9.**	304050607090105020804 7

10. Required the names of the following places:

Sixth; twelfth; eighth; tenth; fourteenth; nineteenth; thirteenth; seventeenth; twenty-first; eighteenth; twenty-fourth; twenty-second; twenty-ninth; thirty-fourth.

11. Required the names of the following periods:

First; third; fifth; second; seventh; fourth; ninth; sixth; tenth; eleventh; fourteenth; sixteenth.

12. Required the places of the following periods:

Thousands; millions; ten-thousands; hundred-thousands; ten-millions; billions; hundred-trillions; quintillions; octillions; hundred-sextillions; ten-quintillions; septillions; hundred-quadrillions.

13. Required the period and place of the following:

Billions; hundred-billions; ten-billions; quintillions; ten-trillions; ten-quadrillions; hundred-quintillions; septillions; hundred-sextillions; ten-nonillions; sextillions; ten-octillions; hundred-quadrillions.

NOTE.—After pupils are familiar with reading by dividing into periods, the division may be omitted or performed mentally.

EXERCISES IN NOTATION.

58. Having learned to read numerical expressions, we are now prepared to *write* them. From the principles which have been explained, we have the following rule:

Rule.—I. *Begin at the left and write the hundreds, tens, and units, of each period in their proper order.*

II. *When there are vacant places, fill them with ciphers.*

1. Express in figures the number *three thousand eight hundred* and *six*.

SOLUTION.—We write the 3 thousands in the 4th place, 8 hundreds in the 3d place, and there being no tens, we write a cipher in the 2d place, and 6 units in the 1st place, and we have 3806.

Express the following numbers in figures:

1. Forty-six million and forty-seven thousand.

2. Two hundred and two million and twenty-two.

3. Six hundred and sixty million five hundred and thirty-seven thousand and three.

4. One billion four million and eighty.

5. Two hundred and nine billion sixty-five million four thousand and seven.

6. Seventy trillion eight billion one million and six hundred.

7. One hundred and two quadrillion three billion four hundred thousand and fifty.

8. Thirty-five octillion seven hundred and ten trillion thirty million six hundred and seventeen

9. Twenty undecillion six hundred nonillion ninety-four septillion three hundred and one billion fifty-eight thousand three hundred and four.

10. Seventy duodecillion, nine octillion five hundred and thirty-two sextillion four hundred trillion eight million one hundred and ten.

59. Orders.—Since we may have 2 *tens*, 3 *hundreds*, etc., the same as 2 *apples*, 3 *books*, etc., these different groups may be regarded as *units* of different *orders*; thus,

UNITS	are called	Units of the 1*st order*
TENS	" "	Units of the 2*d order.*
HUNDREDS	" "	Units of the 3*d order.*
THOUSANDS	" "	Units of the 4*th order.*
TEN-THOUSANDS	" "	Units of the 5*th order.*

60. From this it is seen that *ten* units of a lower order make one unit of the next higher order; the system of notation is therefore called the *Decimal System*, from the Latin *decem*, ten.

NOTE.—The pupil should notice carefully the distinction between *periods* and *orders* of *units*. The *first period*, called *units* period, consists of units of the 1st, 2d and 3d order; the *second period*, called *thousands* period, consists of units of the 4th, 5th and 6th orders, etc. Periods increase by *thousands*; orders by *tens*.

EXAMPLES FOR PRACTICE.

Write and read the following:

1. Two units of the 2d order, and four of the 1st.

2. Nine units of the 4th order, and three of the 1st.

3. Five units of the 7th order, four of the 4th, and eight of the 2d.

4. Three units of the 9th order, five of the 5th, two of the 2d, and four of the 1st.

5. Eight units of the 8th order, six of the 6th, three of the 3d, and one of the 1st.

6. One unit of the 11th order, four of the 10th, nine of the 7th, two of the 6th, and seven of the 3d.

7. Five units of the 10th order, two of the 6th, and three of the 1st.

8. Six units of the 13th order, and four of the 5th.

THE DECIMAL SCALE.

61. In Numeration and Notation we have *two classes of units, Simple* and *Collective.*

62. A **Simple Unit** is a *single thing*, or *one; a collective unit* denotes a *group* or *collection*, regarded as a whole

63. The **Orders of Units** are the units represented by the figures in the different places of a numerical expression.

64. **Simple Units** are called units of the *first order*, *tens* are called units of the *second order; hundreds*, units of the *third order*, etc.

65. The **Scale** of a system of notation is the law of relation between its successive orders of units.

66. The **Radix** of the scale is the number which expresses the relation of the successive orders.

67. The **Decimal Scale** of notation is that in which the radix is *ten*. The system of numeration and notation explained is therefore called the *decimal system*.

68. Since figures in different parts of the scale express different units, figures may be regarded as having two values, a *Simple* and *Local* value.

69. The **Simple Value** of a figure is the number of units it expresses when it stands alone, or in units place.

70. The **Local Value** of a figure is the number it expresses when in any other than units place.

71. The **Decimal System** of numeration had its origin in the practice, common to all nations, of counting by groups of tens.

72. The **Arabic System** of notation is based on the simple but ingenious *device of place*. The system would be the same in principle, whatever the radix of the scale.

73. If we fix the place of units by a point (.), we may extend the scale to the right of units place, and have the scale descending as well as ascending.

74. The first place on the right of the point will be one-tenth of units or *tenths*, the second place one-tenth of tenths, or *hundredths*, the third place, *thousandths*, etc.

75. Such terms are called *decimals*, and the point is called the *decimal point*. Thus, 48.375 is read 48 and 3 tenths 7 hundredths and 5 thousandths, or 48 and 375 thousandths.

76. The **Currency** of the United States is expressed by the decimal system in integers and decimals. The *dollar* is the unit and is indicated by the symbol **$**. The first place at the right of the decimal point is called *dimes;* the second place, *cents;* and the third place, *mills.*

77. Dimes and **Cents,** in practice, are read as a *number of cents.* Thus, $4.65 is read 4 dollars and 65 cents; and $72.485 is read 72 dollars 48 cents and 5 mills. Mills are often expressed as a fractional part of cents; thus $8.465 is written $8.46½.

NOTES.—1. Pupils will notice the difference between the Arabic system of notation and the decimal system of numeration. The Roman method of notation bears the same relation to the decimal system as the Arabic.

2. Any number could have been taken as the basis of the scale; hence the *Decimal System* is not *essential*, but merely *accidental* or *conventional.*

3. The decimal scale originated from the custom, among primitive races, of reckoning by counting the fingers, the number on both hands, including the thumbs, being *ten.*

4. The Arabic notation is named from the Arabs, who introduced it into Europe by their conquest of Spain during the 11th century. The Arabs obtained it from the Hindoos, by whom it was probably invented more than 2000 years ago.

5. The first nine of the Arabic characters are called *significant figures*, because they always denote a definite number of units. They are also called *digits*, from the Latin *digitus*, a finger, because they were employed as a substitute for the fingers, with which the ancients used to reckon.

6. The character 0 is called *naught*, because it indicates no value. It is also called *zero*, which is an Italian word, signifying *nothing.* It is also called *cipher*, which is derived from the Arabic *sifr* or *sifrum*, meaning *empty*, *vacant.* The term was subsequently applied to all the Arabic characters, and the use of them was called *ciphering.*

7. There are three theories for the origin of the Arabic characters: 1st, that they are modifications of characters formed by the combination of *straight lines;* 2d, that they are modifications of characters formed by the combination of *angles;* and 3d, that they are derived from the initial *letters* of the Hindoo words for numbers. The last theory is given by Prinseps and indorsed by Max Müller, and is probably the true one. (See *Brooks's Philosophy of Arithmetic.*)

ENGLISH METHOD OF NUMERATION.

78. The method of numeration by dividing numbers into periods of three figures each, is called the *French Method.* There is also another method called the *English Method.*

79. The **English Method** uses periods of six figures each, calling the first period *units*, the second *millions*, the third *billions*, the fourth *trillions*, etc.

30. The places in each period are *units, tens, hundreds, thousands, tens of thousands, hundreds of thousands.* The method is represented in the following table:

Hund. of Thou. of Trillions.	Tens of Thou. of Trillions.	Thousands of Trillions.	Hundreds of Trillions.	Tens of Trillions.	Trillions.	Hund. of Thou. of Billions.	Tens of Thou. of Billions.	Thousands of Billions.	Hundreds of Billions.	Tens of Billions.	Billions.	Hund. of Thou. of Millions.	Tens of Thou. of Millions.	Thousands of Millions.	Hundreds of Millions.	Tens of Millions.	Millions.	Hundreds of Thousands.	Tens of Thousands.	Thousands.	Hundreds.	Tens.	Units.
6	6	6	6	6	6,	6	6	6	6	6	6,	6	6	6	6	6	6,	6	6	6	6	6	6,
4th or Trillions Period.						3d or Billions Period.						2d or Millions Period.						1st or Units Period.					

The remaining periods have the same names as in the French method.

EXAMPLES FOR PRACTICE.

1. Write the following numbers by both the French and English methods and show their difference:

1. One million.
2. One billion.
3. One trillion.
4. One quadrillion.
5. One quintillion.
6. One sextillion.

2. Read the following by both the French and English methods:

1. 468756054.
2. 8630685025.
3. 70685973284.
4. 5637240250167.
5. 76557004032854.
6. 3205056702436057.

3. Write the following by either method, and read the results by the other method:

1. Five million six thousand and one.
2. Six billion five thousand and three million seven hundred and nine.
3. Nine thousand trillion five hundred thousand billion seventeen thousand and three.
4. How many times one trillion French is one trillion English?
5. How many times one decillion French is one decillion English?
6. How many times one quintillion French is one quintillion English?

ROMAN NOTATION.

81. The **Roman Method** of Notation employs seven letters of the Roman alphabet. Thus, I represents *one;* V, *five;* X, *ten;* L, *fifty;* C, *one hundred;* D, *five hundred;* M, *one thousand.*

82. To express other numbers these characters are combined according to the following principles:

1. *Every time a letter is repeated its value is repeated.*

2. *When a letter is placed before one of a greater value, the* DIFFERENCE *of their values is the number represented.*

3. *When a letter is placed after one of a greater value, the* SUM *of their values is the number represented.*

4. *A dash placed over an expression increases its value a thousand fold.* Thus $\overline{\text{VII}}$ denotes seven thousand.

NOTE.—In applying these principles write the different orders of units in succession, beginning with the highest.

83. These principles are exhibited in the following table:

ROMAN TABLE.

I	One.	XI	Eleven.	XC	Ninety.
II	Two.	XIV	Fourteen.	C	One hundred.
III	Three.	XV	Fifteen.	CC	Two hundred.
IV	Four.	XIX	Nineteen.	D	Five hundred.
V	Five.	XX	Twenty.	DC	Six hundred.
VI	Six.	XXX	Thirty.	DCCCC	Nine hundred.
VII	Seven.	XL	Forty.	M	One thousand.
VIII	Eight.	L	Fifty.	MM	Two thousand.
IX	Nine.	LX	Sixty.	MCLX	One thousand one hundred and sixty.
X	Ten.	LXX	Seventy.	MDCCCLIX	1859.

NOTE.—The Roman method is named from the Romans, who invented and used it. It is now employed only to denote the chapters and sections of books, pages of preface and introduction, and in other places for prominence and distinction.

EXAMPLES FOR PRACTICE.

1. Write the following numbers by the Roman Method:

1. Nine hundred and thirty-six. 2. One thousand five hundred and sixteen. 3. Four thousand two hundred and four. 4. Seven thousand and sixty-eight. 5. Thirty thousand and thirteen.

2. Read the following numbers:

LXXXVIII; $\overline{\text{V}}$DLIX; MDCCCLXXV; MMDXC; $\overline{\text{L}}$CCCXXX; XDCLVI; $\overline{\text{LIX}}$CCCCXLIV; $\overline{\text{M}}$MMMXC; cliv; xcvii; clxix.

SECTION II.

FUNDAMENTAL OPERATIONS.

ADDITION.

84. Addition is the process of finding the *sum* of two or more numbers.

85. The **Sum** of several numbers is a number which contains as many units as the numbers added.

86. The **Sign of Addition** is +, and is read *plus*. It denotes that the numbers between which it is placed are to be added.

87. The **Sign of Equality** is =, and is read *equals*. It denotes that the numbers between which it is placed are equal.

NOTES.—1. The *Sign of Addition* consists of two short lines bisecting each other, the one in, and the other perpendicular to, the line of writing.

2. The symbol + was introduced by *Stifelius*, a German mathematician, in a work published in 1544.

3. The symbol = was introduced by Robert Recorde, an English mathematician, in his "*Whetstone of Wit*," a work on algebra published in 1557.

PRINCIPLES.

1. *The numbers added must be similar.*
2. *Units of the same order only can be added directly.*
2. *The sum is a number similar to the numbers added.*
4. *The sum is the same in whatever order the numbers are added.*

PROBLEM.

88. ***To find the sum of two or more numbers.***

1. What is the sum of 571, 395, and 683?

OPERATION.

$$\begin{array}{r} 571 \\ 395 \\ 683 \\ \hline 1649 \end{array} \textit{ Ans.}$$

SOLUTION.—We write the numbers so that terms of the same order stand in the same column, and begin at the right to add. 3 and 5 are 8 and 1 are 9, units; we write the 9 units under the column of units: 8 and 9 are 17 and 7 are 24, tens, or 2 hundreds and 4 tens; we write the 4 tens under the column of tens, and add the 2 hundreds to the column of hundreds: 2 and 6 are 8 and 3 are 11 and 5 are 16, hundreds, or 1 thousand and 6 hundreds; we write the 6 hundreds under the column of hundreds, and place the 1 at the left in the place of thousands. Hence the sum of 571, 395, and 683 is 1649.

Rule.—I. *Write the numbers to be added so that terms of the same order stand in the same column, and draw a line beneath.*

II. *Begin at the units, add the terms of each column separately and write the sum underneath, if less than ten.*

III. *When the sum of any column is ten or more than ten, write the units figure only, and add the tens to the next column.*

IV. *Write the entire sum of the last column.*

Proof.—Begin at the top and add the columns downward, and if the work is correct the two sums will be equal.

SECOND METHOD. Separate the number into two or more parts, add these parts, and then add the sums of these parts; if the work is correct the two results will be equal.

NOTES.—1. We write the figures of the same order in the same column for *convenience* of adding, since only units of the same order can be directly added.

2. We begin at the *right* to add for *convenience*, so that when the sum of any column exceeds 9, we may add the left-hand term of such sum to the next column.

3. Beginning at the bottom of a column to add is mostly a matter of *custom;* cases may arise, however, in which it would be more convenient.

4. The proof by excess of 9's will be given hereafter.

2. What is the sum of 35246+234+7891+50673+75214 +82349? *Ans.* 251607.

3. What is the sum of 5462+37185+989+64732+34785 +6495? *Ans.* 149648.

4. What is the sum of 247923+568172+4136+21975+ 729186+235697? *Ans.* 1807089.

5. What is the sum of 3724679+42531+297346+4965287 +914535+6724913? *Ans.* 16669291.

6. What is the sum of 4273561+391845+72233+99+ 25673981+7253648? *Ans.* 37665367.

Find the sum of the numbers,

7. From 223 to 232 inclusive. *Ans.* 2275.

8. From 3459 to 3475 inclusive. *Ans.* 58939.

9. From 4375 to 4400 inclusive. *Ans.* 114075.

10. From 78437 to 78450 inclusive. *Ans.* 1098209.

11. From 87692 to 87700 inclusive. *Ans.* 789264.

PRACTICAL PROBLEMS.

1. A publisher, issuing a new work, paid $650 for the plates, $250 for the paper, $95.75 for the press-work, and $275.50 for binding; what did the first edition cost?

OPERATION.
$650.00
250.00
95.75
275.50
$1271.25

SOLUTION.—If for the plates he paid $650, for the paper $250, for the press-work $95.75, and for the binding, $275.50, for all he paid the sum of $650, $250, $95.75, and $275.50, which, by addition, we find to be $1271.25; hence the first edition cost $1271.25.

2. A merchant laying in his spring goods, expended for calico, $765.87½; for percales, $1075.37½; for French chintz, $564.75; for Victoria lawn, $375.16; and for book muslin, $256.56; what was his bill? *Ans.* $3037.72.

3. My grocer sends me the following bill: 25 lb. of sugar, $3.50; 7 lb. of tea, $8.75; 5 barrels of apples, $18.75; 12 lb of coffee, $3.50; 2 barrels of flour, $19.50; 100 lb. of buckwheat flour, $3.25, and 25 lb. of corn meal, $0.75; what was the amount? *Ans.* $58.

4. A bought a house and lot for $3000, paid $565 for building a barn, $156.35 for putting the grounds in order, $105.67 for introducing gas and water, and $75 for a new range; for what must he sell this property to make $450.75 on his investment? *Ans.* $4352.77.

5. A bill of goods contains the following items: 1 piece Irish linen, $9.92; 3 pieces Russia crash, $4.81; 3 pieces plaid jaconets, $22.50; 1 piece linen-drill, $13.68; 2 pieces Marseilles vesting, $36.30; 2 pieces linen duck, $124 37; 2 pieces brown corduroy, $46.41; required the amount of the bill. *Ans.* $257.99.

6. Find the sum of 8 trillion 3 billion 1 million 495 thousand, and 6 quadrillion 74 trillion 15 million 4 hundred written by the French method, and also by the English method. *Ans.* 6000082006085003032990800.

7. Find the sum of 950053, 420000, five hundred and one thousand one hundred, MDCCCLXXVI, MDCXCVIII, DCCCCXLIX, and DCCCLI, and express it in the Roman method. *Ans.* $\overline{\text{MDCCCLXXVI}}$DXXVII.

8. Messrs. Watson & Co., rendered the following bill: $\frac{1}{2}$ dozen Jones's L. H. Shovel, \$4.75; $\frac{1}{3}$ dozen auger bits, \$6.30; $\frac{1}{6}$ dozen Hdld socket chisels, \$2.75; $3\frac{1}{2}$ lbs. axe stone, \$0.42; $\frac{1}{2}$ rm. sand paper, ass'd, \$2; 2 quires emery paper, \$3.12; $\frac{1}{3}$ gross table spoons, \$3.75; 1 keg nails, \$15.75; 1 keg horseshoes, \$6.75; 3 pairs brass candlesticks, \$1.35; required the amount of the bill. *Ans.* \$46.94.

9. A dry goods merchant bought silk for \$240, linen for \$375, and woolen goods for \$450; the silk was sold at a profit of \$85, the cloth at a profit of \$75, and the woolen goods at a profit of \$150; what was the whole amount received for the goods? *Ans.* \$1375.

10. A gentleman leaves to each of his three sons \$3500, and to his two daughters \$800 apiece more than to a son, and to his wife \$400 more than to a son and daughter, and the remainder of his estate, which was \$500 more than he bequeathed to his family, he left to an orphan asylum; what was the amount of his estate? *Ans.* \$55,100.

11. Add the following ledger columns:

DR. **CASH.** **CR.**

1874					1874				
July	3	To Sundries,	725	00	July	5	By Merchandise,	625	75
"	9	" Merchandise,	875	75	"	11	" Bills Payable,	433	00
"	10	" Bills Receivable,	625	00	"	"	" Interest,	25	98
"	"	" Interest,	37	50	"	13	" Louis Walton,	250	75
"	12	" James Nelson,	410	25	"	20	" Merchandise,	1015	45
"	13	" Merchandise,	325	65	"	22	" Lehigh Valley		
"	14	" Reading R. R.					R. R. Stock,	2806	00
		Stock,	2806	25	"	"	" Commission,	7	02
"	22	" Rogers & Co.,	1815	85	"	29	" Adams & Co.,	315	72
"	30	" Merchandise,	875	40	"				
Aug.	1	" Am't car'd for'd,					" Am't car'd for'd,		

CONTRACTIONS IN ADDITION.

89. Contractions in Addition are abbreviated methods of adding.

CASE I.

90. ***To add by omitting the names of the numbers added, merely naming results.***

1. Find the sum of 367, 589, 635, and 768.

OPERATION.

367
589
635
768
2359

SOLUTION.—We write the numbers so that terms of the same order stand in the same column, and begin at the right to add: 8, 13, 22, 29; we write the 9 and add the 2 to the next column: 2, 8, 11, 19, 25; we write the 5, and add the 2, etc.

Find the sum of the numbers

2. From 782 to 800 inclusive. *Ans.* 15029.
3. From 649 to 664 inclusive. *Ans.* 10504.
4. From 8234 to 8250 inclusive. *Ans.* 140114.
5. From 9247 to 9263 inclusive. *Ans.* 157335.
6. From 9897 to 9910 inclusive. *Ans.* 138649.

CASE II.

91. ***To add two or more columns at the same time.***

1. Find the sum of 3486, 5267, 6845 and 7654.

OPERATION.

3486
5267
6845
7654
23252

SOLUTION.—54 and 40 are 94 and 5 are 99 and 60 are 159 and 7 are 166 and 80 are 246 and 6 are 252; we write the 52 and add the 2 to the next column; 76 and 2 are 78 and 60 are 138 and 8 are 146 and 50 are 196 and 2 are 198 and 30 are 228 and 4 are 232, which we write.

NOTE.—In practice name only the results, omitting the naming of the numbers added.

Find the sum of the numbers

2. From 496 to 512 inclusive. *Ans.* 8568.
3. From 832 to 848 inclusive. *Ans.* 14280.
4. From 5626 to 5640 inclusive. *Ans.* 84495.
5. From 6987 to 7000 inclusive. *Ans.* 97909.

NOTES.—1. When two or more terms of a column can be easily grouped together, use their sum instead of adding each separately; combining with especial reference to *tens*.

2. When a term is repeated several times in a column, multiply it by the number of times it is repeated, and use the result.

SUBTRACTION.

92. Subtraction is the process of finding the *difference* between two numbers.

93. The **Difference** between two numbers is a number which, added to the less, equals the greater.

94. The **Subtrahend** is the number to be subtracted.

95. The **Minuend** is the number from which we subtract.

96. The **Sign of Subtraction** is —, and is read *minus*. It denotes that the number immediately following it is to be subtracted from the number preceding it.

NOTES.—1. The *Sign of Subtraction* is a short line in the line of writing.
2. The symbol — was introduced by *Stifelius*, a German mathematician, in a work published in 1544.

PRINCIPLES.

1. *Similar numbers only can be subtracted.*
2. *Units of the same order only can be directly subtracted.*
3. *The difference is a number similar to the minuend and subtrahend.*
4. *If the minuend and subtrahend be equally increased or diminished, the difference will remain the same.*

PROBLEM.

97. *To find the difference between two numbers.*

98. There are **Two Methods** of explaining subtraction, called the *Method by Borrowing*, and the *Method by Adding Ten.*

NOTE.—The taking one from the term of the minuend is called *borrowing*, and the adding one to the next term of the subtrahend is called *carrying*.

1. Subtract 365 from 647.

OPERATION.

$$\begin{array}{r} 647 \\ 365 \\ \hline 282 \end{array}$$

SOLUTION BY BORROWING.—We write the subtrahend under the minuend and begin at the right to subtract. 5 units from 7 units leave 2 units, which we write under the units; we cannot subtract 6 tens from 4 tens, we will therefore take 1 hundred from the 6 hundreds and add it to the 4 tens; 1 hundred equals 10 tens, which added to 4 tens, equal 14 tens; 6 tens from 14 tens leave 8 tens, which we write in tens place: 3 hundreds from 5 hundreds (the number of hundreds remaining after taking away 1 hundred) leave 2 hundreds, which we write in the hundreds place.

Solution by Adding Ten.—5 units from 7 units leave 2 units; we cannot take 6 tens from 4 tens, we will therefore add 10 tens to the 4 tens, making 14 tens; 6 tens from 14 tens, leave 8 tens: now since we have added 10 tens, or 1 hundred, to the minuend, our remainder will be 1 hundred too large, hence we must add 1 hundred to the subtrahend; 1 hundred and 3 hundreds are 4 hundreds; 4 hundreds from 6 hundreds leave 2 hundreds.

Rule.—I. *Write the subtrahend under the minuend, placing terms of the same order in the same column, and draw a line beneath.*

II. *Begin at units, and subtract each term of the subtrahend from the corresponding term of the minuend, writing the remainder beneath.*

III. *If any term of the subtrahend is greater than the corresponding term of the minuend, add* 10 *to the latter and then subtract.*

IV. *Add* 1 *to the next term of the subtrahend (or subtract* 1 *from the next term of the minuend), and proceed as before.*

Proof.—Add the difference to the subtrahend, and if the work is correct the sum will equal the minuend.

Second Method.—Subtract the difference from the minuend, and, if the work is correct, the result will equal the subtrahend.

Note.—The method of proof by excess of 9's will be explained hereafter.

2. From 987435 take 369476. *Ans.* 617959.
3. From 70432065 take 64545406. *Ans.* 5886659.
4. From 80035007 take 7640094. *Ans.* 72394913
5. From 95930—53428 take 5036. *Ans.* 37466.
6. From 55159 take 75138—61859. *Ans.* 41880.
7. From 604035—470647 take 90009—78087. *Ans.* 121466.
8. From 888888—99999 take 810000—188888. *Ans.* 167777.

In the four following problems subtract the second number from the first, then from the remainder, etc., till the last remainder is less than the second number.

9. 610000 and 155555. *Ans.* 143335.
10. 1000008 and 245679. *Ans.* 17292

11. 98765432 and 23456789. *Ans.* 4938276.

12. 9834064 and 1277045. *Ans.* 894749.

PRACTICAL PROBLEMS.

1. If a farm was bought for $6770, and sold for $8025, what was the gain?

SOLUTION.—If a farm was bought for $6770, and sold for $8025, the gain was the difference between $6770 and $8025, which is $1255.

OPERATION.

$8025
6770
$1255 *Ans.*

2. Having $7570 in bank, I added enough to make my deposit $12000; how much did I add? *Ans.* $4430.

3. A speculator lost $5675, then gained $4325, and then had $5000; how much had he at first? *Ans.* $6350.

4. I received an invoice of goods amounting to $1876.25, and immediately forwarded a draft for $987.75 on account; what balance do I owe? *Ans.* $888.50.

5. I invested $7680 in California wheat, giving my note for $2875, and paying the balance in cash; how much cash did I pay? *Ans.* $4805.

6. A merchant owed a jobber $4653, and in payment gave him two checks for $1250 each; what balance was due the jobber? *Ans.* $2153.

7. A dealer in stocks gained $25,500 one year, and the following year lost $19,750, and had $50,000 remaining; what was the original capital? *Ans.* $44,250.

8. Plato was born 430 B. C., and Socrates 470 B. C.; the former died 347 B. C., the latter 400 B. C.; what was the difference in their ages? *Ans.* 13 years.

9. What is the difference, in Roman characters, between $\overline{\text{M}}$ and $\overline{\text{LXXV}}\text{DCCIII}$? *Ans.* $\overline{\text{DCCCCXXIV}}\text{CCXCVII}$.

10. What is the difference between 1 duodecillion and 1 quadrillion, and how many figures will express it?

Ans. 39 figures.

11. What is the difference between 28 decillion 143 quadrillion 705 billion 96 million 6 thousand and 5, expressed in the English and French methods?

Ans. A number of 62 figures.

12. If a coal merchant's gains increase $5426 annually for six successive years, and the last year his gains amount to $39,765, what were the first year's gains? *Ans.* $7209.

13. An assignee found the assets of an estate supposed to be worth $20,000, $3575 more than the liabilities, which were $5860 less than the supposed value of the estate; what were the assets? *Ans.* $17,715.

14. What is the difference between 5 septillion 10 sextillion 3 quintillion, expressed in the English and French methods, the answer to be written in the English method?

Ans. 5 septillion 10 sextillion 2 quintillion 999994 quadrillion 989997 trillion.

15. A gentleman having an estate of $253,760, bequeathed $20,200 to each of his two sons, $24,000 to his daughter, to his widow as much as to all the children, and the remainder of his estate to a college; what did the widow and college receive? *Ans.* Widow, $64,400; College, $124,960.

16. A man bought a horse for $200, but seeing another he liked better traded him off, giving $25 to boot; receiving a good offer for this one, he traded again, receiving $75; he traded again, paying $60, and finally sold the last one for $175; what did he lose by his bargains? *Ans.* $35.

17. Find the balance of the following ledger account:

Dr.			Jones & Marston.				Cr.
1875.				1875.			
Dec.	1	To Merchandise,	3475 00	Dec.	11	By Cash,	1200 00
"	8	" Sundries,	750 00	"	"	" Bills Payable,	1500 00
"	12	" Merchandise,	865 50	"	18	" Sundries,	325 00
"	17	" "	476 75	"	25	" Cash,	641 75
"	20	" Interest,	56 35	"	31	" Bal. to new %	... ...
"	"	" Sundries,	274 40				
"	29	" Merchandise,	642 50				
1876.							
Jan.	1	To Balance,					

CONTRACTIONS IN SUBTRACTION.

99. Contractions in Subtraction are abbreviated methods in subtracting.

CASE I.

100. ***To subtract a number by subtracting from 10 and then adding.***

1. Subtract 25368 from 63524.

SOLUTION.—8 from 10 leaves 2, and 4 are 6; 6 and 1 are 7, 7 from 10 leaves 3, and 2 are 5; 3 and 1 are 4, 4 from 5 leaves 1; 5 from 10 leaves 5, and 3 are 8; 2 and 1 are 3, 3 from 6 leaves 3.

OPERATION.

$$\begin{array}{r} 63524 \\ 25368 \\ \hline 38156 \end{array}$$

Rule.—*When a term of the subtrahend is greater than the corresponding term of the minuend, subtract the former from* 10, *and add the remainder to the latter; when not greater, subtract by the ordinary rule.*

NOTES.—1. We may apply this rule to all the terms by omitting to carry when the sum is greater than 9.

2. The reason for this method will be readily seen. The pupil may be required to explain it.

2. Subtract 54062 from 82547; 68975 from 97460; 79240 from 107725; 34651 from 63136. *Ans.* 28485.

3. Subtract 346978 from 543217; 556324 from 752563; 789313 from 985552; 153454 from 349693. *Ans.* 196239.

4. Subtract 54321 from 93213; 679543 from 718435; 439765 from 478657; 974512 from 1013404. *Ans.* 38892.

CASE II.

101. ***To subtract two or more numbers at a single operation.***

1. Subtract 546 and 365 from 1364.

SOLUTION.—5 and 6 are 11, and since 3 more will make 14, we write 3 for the first term of the remainder; 6 and 1 are 7 and 4 are 11, and since 5 more makes 16, we write 5 in the remainder; 3 and 1 are 4 and 5 are 9, and since 4 more makes 13, we write 4 in the remainder.

OPERATION.

$$\begin{array}{r} 1364 \\ \hline 546 \\ 365 \\ \hline 453 \end{array}$$

Rule.—*Write the several subtrahends under the minuend, add each column of the subtrahends, and write for the re-*

mainder a term which, added to this sum, will give a number having for its unit term the corresponding term of the minuend.

Note.—Observe the same rule in carrying as in addition.

(2)	(3)	(4)	(5)
From 7845	From 12978	From 856346	From 1000000
Take 1472	Take 7934	Take 432157	Take 543655
3456	2561	154337	314133
864	1278	123786	142212
		79584	

6. A speculator bought a lot of land for $413, a second lot for $420, a third for $519, and a fourth for $607; he sold the whole for $2200; what was his gain? *Ans.* $241.

7. A man having an estate of $60,523, left $19,500 to his son, and $19,500 to his daughter; several small legacies and the expenses amounted to $1519; and what remained was to be the share of the widow; how much did the widow receive? *Ans.* $20,004.

8. The area of the land surface of the globe is said to be 48,998,388 square miles; the area of Asia is 15,086,000 square miles, of Africa 10,936,000, of North America 8,160,000, of South America 6,552,000, of Europe 3,764,388; what is the area of Oceanica? *Ans.* 4,500,000.

PRACTICAL PROBLEMS.

1. I received a bill of goods amounting to $1575.75, and remitted two drafts of $578.25 and $692.85 respectively; what is the balance? *Ans.* $304.65.

2. A man had in bank $15,000, deposited $3875, drew out $8725, and then put in enough to make his deposits $20,000; what was his last deposit? *Ans.* $9850.

3. A speculator lost $18,000 in oil, then gained $15,750 on grain, and then lost $19,250 in "Erie," which exhausted his capital; what was his capital? *Ans.* $21,500.

4. A speculator gained $1050 one year, the next year he lost as much again as he had gained, and the next year gained as much again as he had lost, and then had $50,000; with what capital did he begin? *Ans.* $46,850

5. A gentleman by his will divided an estate of $40,750 as follows: his son to have $10,000; each of his three daughters $5,475, and the widow the remainder; what did the widow receive? *Ans.* $14,325.

6. Two brothers, A and B, had each $25,000; A loaned B $7,500 and then borrowed of him $15,450, and lost so much of it in speculation that B had $7050 more than A; how much did A lose? *Ans.* $22,950.

7. Dr. Willard receives the following bill from George Jobson: 1 barrel of kerosene, $18.75; 2 barrels of flour, $19.75; 25 lb. of sugar, $3.62½; he presents at the same time his bill as follows: for medical services, $11.50; medicines, $3.45; what is the balance, and in whose favor?

Ans. $27.17½ in Jobson's favor.

8. Hood & Martin bought of Chambers & Rogers the following goods: 1 piece blue anchor cloth, $215.16; 2 pieces black cloth, $91.80; 1 piece doeskin cassimere, $45; 2 pieces black ribbed cloth, $102.69. Chambers & Rogers bought of Hood & Martin as follows: 1 piece red twilled flannel, $24.30; 2 pieces blue twilled flannel, $48.95; 1 piece white domestic flannel, $10.08; what was the balance due Chambers & Rogers? *Ans.* $371.32.

9. Required the balance of the following account in my bank book:

Dr.									Cr.
1875.					1875.				
Jan.	22	To balance,	697	07	Feb.	1	By Chk. fa'r Roth,	150	00
"	26	" Read'g divi'd,	62	50	"	6	" " " Zaun,	48	50
Feb.	4	" Certif. deposit,	1200	00	"	12	" " " Baker,	1200	00
Mar.	31	" 4 checks,	704	00	"	20	" " " Passmore	27	19
Apr.	2	" Cash,	1200	00	Mar	6	" " " Behmer,	242	18
"	12	" Pa. R. R. scrip,	505	62	Apr.	24	" " " Hiestand,	2000	00
"	20	" Cert. and int.,	1013	50	May	12	" " " Hager &		
June	4	" Un'nPac'c div.	69	30			Bro.,	37	90
"	6	" Pa. R. R. div.,	284	00	"	18	" " " Mason &		
"	10	" Check,	1347	74			Hamlin,	117	58
"	20	" Coupons,	189	30	"	20	" " " Insurance		
"	24	" Checks,	2600	00			Co.,	1306	90
					June	15	" Certif. deposit,	1200	00
					"	25	" Railroad stock,	[illegible]	50
					June	30	" Balance,		

MULTIPLICATION.

102. Multiplication is the process of finding the *product* of two numbers.

103. The **Product** of two numbers is the result obtained by taking one number as many times as there are units in another.

104. The **Multiplicand** is the number to be multiplied.

105. The **Multiplier** is the number by which we multiply.

106. The **Sign of Multiplication** is ×, and is read *multiplied by*, *times*, or *into*. When placed between two numbers it denotes that one is to be multiplied by the other.

NOTES.—1. The *Sign of Multiplication* consists of two short lines of equal length bisecting each other at an angle of 45 degrees with the line of writing.
2. The symbol × was introduced by *Wm. Oughtred*, an English mathematician, born in 1574.

PRINCIPLES.

1. *The multiplier is always an abstract number.*

For, the multiplier shows the *number of times* that the multiplicand is taken; hence it cannot be *yards* or *bushels*, or any other concrete number.

2. *The product is always similar to the multiplicand.*

Thus, 4 times 5 *dollars* are 20 *dollars*, and not 20 *yards* or 20 *days*, or anything else besides *dollars*.

3. *The product of two numbers is the same whichever is made the multiplier.*

Thus, the 12 stars in the diagram may be regarded as 3 *fours*, or as 4 *threes;* hence 3 times 4 equals 4 times 3, and the same may be shown of any other number.

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* * * *
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4. *If the multiplicand be multiplied by all the parts of the multiplier, the sum of all the partial products will be the true product.*

This is evident from the axiom that the whole is equal to the sum of all the parts.

NOTE.—From Prin. 1 we see that such problems as "multiply 25 cents by 25 cents," or "2 shillings and 6 pence by itself," are impossible and absurd.

PROBLEM.

107. ***To multiply one number by another.***

1. Multiply 685 by 345.

SOLUTION.—We write the multiplier under the multiplicand, and draw a line beneath. 5 times 5 units are 25 units, or 2 tens and 5 units; we write the 5 units and carry the 2 tens; 5 times 8 tens are 40 tens, plus 2 tens, are 42 tens, or 4 hundreds and 2 tens; we write the 2 tens and carry the 4 hundreds; 5 times 6 hundreds are 30 hundreds, plus 4 hundreds, are 34 hundreds, which we write: in the same way we find 4 tens times 685 are 2740 *tens*, and 3 hundred times 685 are 2055 *hundreds;* adding these partial products we have 5 units+4 tens+3 hundreds times 685, or 345 times 685, which is 236325.

OPERATION.

$$\begin{array}{r} 685 \\ 345 \\ \hline 3425 \\ 2740 \\ 2055 \\ \hline 236325 \end{array}$$

Rule.—I. *Write the multiplier under the multiplicand, placing terms of the same order in the same column, and draw a line beneath.*

II. *Begin at the right, and multiply the multiplicand by each term of the multiplier, writing the first term of each product under the term of the multiplier which produces it.*

III. *Add the partial products, and their sum will be the entire product.*

Proof.—Multiply the multiplier by the multiplicand, and if the work is correct this product will equal the first product.

NOTES.—1. When there are ciphers between the significant terms of the multiplier, pass over them and multiply by the significant terms alone.

2. We begin at the right to multiply, so that when any product exceeds nine, we may add the number expressed by the left hand figure to the next product.

Multiply

2. 4276 by 394. *Ans.* 1684744.
3. 5364 by 207. *Ans.* 1110348.
4. 8075 by 3165. *Ans.* 25557375.
5. 2856 by 6124. *Ans.* 17490144.
6. 31876 by 3256. *Ans.* 103788256.
7. 48306 by 2078. *Ans.* 100379868.
8. 61357 by 3851. *Ans.* 236285807.
9. 86195 by 4287. *Ans.* 369517965.
10. 38056 by 3075. *Ans.* 117022200.
11. 91654 by 41652. *Ans.* 3817572408.
12. 43729 by 50706. *Ans* 2217322674.

13. 384675 by 65078. *Ans.* 25033879650.
14. 8765932 by 850704. *Ans.* 7457213416128.
15. 1234567 by 4600407. *Ans.* 5679510668769.

PRACTICAL PROBLEMS.

1. At 37 dollars a ton, what will be the cost of 796 tons of pig iron?

	OPERATION.
SOLUTION.—If 1 ton cost \$37, 796 tons will cost 796 times \$37, which we find, multiplying 796 by 37 for convenience, to be \$29452.	796 37 \$29452

2. What will be the cost of 691000 Philadelphia pressed brick at \$33 a thousand? *Ans.* \$22,803.

3. A grocer bought 46 barrels of superfine flour at \$12.50 per barrel, and 75 barrels of corn meal at \$4.50 per barrel; what was the whole cost? *Ans.* \$912.50.

4. If a man spends 25 cents a day for cigars, how much will he spend in 40 years, allowing for ten leap years, there being 52 weeks and one day in a year, and 7 days in a week? *Ans.* \$3652.50.

5. In a freight car there are 9 boxes of books, each weighing 465 pounds; 12 barrels of pork, each weighing 200 pounds; and 45 boxes of shoes, each weighing 156 pounds; what was the weight of all? *Ans.* 13605 pounds.

6. A wholesale druggist sold 5 kegs of baking soda containing 112 lbs. each, at 6 cents a lb.; 7 casks of washing soda, containing 300 lbs. each, at 3 cents a lb., 15 boxes of French Castile soap, containing 30 lbs. each, at 16 cents a lb.; and 50 dozen cakes of perfumed toilet soap at 75 cents a dozen; what did the whole amount to? *Ans.* \$206.10.

7. A lady preparing to go to housekeeping purchased at Van Harlingen and Arrison's 1 piece of linen sheeting, containing 38 yards, at 45 cents a yard; 3 pieces of muslin sheeting, containing 43 yards each, at 28 cents a yard; 10 yards of damask table linen at \$1.25 a yard; 15 yards of better quality of table linen at \$2 a yard; 1 dozen napkins at \$2.50 a dozen, and 1 dozen doilies, at \$2 a dozen; what was her bill? *Ans.* \$100.22.

CONTRACTIONS IN MULTIPLICATION.

108. Contractions in Multiplication are abbreviated methods of multiplying.

109. A Composite Number is the product of two or more numbers, each greater than a unit, called *factors.* Thus, 24 is a composite number, whose factors are 4 and 6, or 3 and 8, or 2, 3, and 4.

110. Continued Multiplication is the process of finding the product of three or more numbers by multiplying the first and second, this result by the third, etc. The result is the *Continued Product.*

PRINCIPLES.

1. *The product of two numbers is equal to the continued product of one number by the factors of the other.*

Thus, 27 multiplied by 24 is equal to 27 multiplied by 6 and that product by 4.

2. *The continued product of several factors will be the same in whatever order the factors are taken.*

Thus, $2\times3\times4$ is equal to $2\times4\times3$, or $3\times4\times2$, or to the product in whatever other order the numbers may be taken.

CASE I.

111. ***When the multiplier is a composite number.***

1. Multiply 256 by 24.

SOLUTION.—24 equals 4 times 6, hence 24 times 256 equals 4 times 6 times 256; 6 times 256 equals 1536, and 4 times 1536 equals 6144; therefore 256 multiplied by 24 equals 6144. Hence the

OPERATION.

$$\begin{array}{r} 256 \\ \underline{6} \\ 1536 \\ \underline{4} \\ 6144 \end{array}$$

Rule.—*Multiply the multiplicand by one factor, this product by another factor, and thus continue until all the factors have been used; the last product will be the result required.*

Multiply

2. 367 by 28.	*Ans.* 10276.	**6.** 2057 by 63.	*Ans.* 129591.
3. 764 by 35.	*Ans.* 26740.	**7.** 3895 by 72.	*Ans.* 280440.
4. 596 by 42.	*Ans.* 25032.	**8.** 4168 by 108.	*Ans.* 450144.
5. 4783 by 56.	*Ans.* 267848.	**9.** 4796 by 144.	*Ans.* 690624

10. What cost 42 horses at 175 dollars each?

OPERATION.

SOLUTION.—42 equals 6 times 7: if one horse costs 175 dollars, 7 horses will cost 7 times 175 dollars, which are 1225 dollars; and 6 times 7 horses, or 42 horses, will cost 6 times 1225 dollars, which are 7350 dollars. Therefore, etc.

$$\begin{array}{r} 175 \\ 7 \\ \hline 1225 \\ 6 \\ \hline 7350 \end{array}$$

11. What will 72 yards of cloth cost at the rate of $6.35 a yard? *Ans.* $457.20.

12. What will 84 yoke of oxen cost at the rate of $135 a yoke? *Ans.* $11,340.

13. How much must I pay for grading 132 rods of railroad at $345 a rod? *Ans.* $45,540.

14. A farm containing 144 acres of land was sold at the rate of $296 an acre; for what did it sell? *Ans.* $42,624.

CASE II.

112. ***When ciphers are at the right of one or both factors.***

1. Multiply 2700 by 120.

OPERATION.

SOLUTION.—27 multiplied by 12 equals 324, hence 2700 multiplied by 12 equals 100 times 324, or 32400; and multiplied by 120 will be 10 times as much, or 324000. (Prin. 1. Art. 110.)

$$\begin{array}{r} 2700 \\ 120 \\ \hline 54 \\ 27 \\ \hline 324000 \end{array}$$

Rule.—*Take the product of the numbers denoted by the significant figures, and annex as many ciphers to the result as are found at the right of both factors.*

What is the value

2. Of 3560×360? *Ans.* 1281600.

3. Of 48700×450? *Ans.* 21915000.

4. Of 30900×670? *Ans.* 20703000.

5. Of 28500×8500? *Ans.* 242250000.

6. Of 67400×9600? *Ans.* 647040000.

7. Of 865000×7800? *Ans.* 6747000000.

8. Of 723000×9700? *Ans.* 7013100000.

9. Of 3876000×35100? *Ans.* 136047600000.

10. Of 4264000×20400? *Ans.* 86985600000.

CASE III.

113. ***When one part of the multiplier is a factor of another part.***

1. Multiply 576 by 246.

SOLUTION.—In this example 6, one part of the multiplier, is a factor of 24, the other part; hence we may proceed thus: 6 times 576 equals 3456; and 24 times 576 equals 4 times 3456, or 13824, which we write as tens: taking the sum of the partial products, we have 141696.

OPERATION.

576	
246	
3456	Prod. by 6 units.
13824	Prod. by 24 tens.
141696	*Ans.*

2. Multiply 43526 by 24832.

SOLUTION.—We first multiply by the 8 hundreds, writing the first figure in hundreds place; we then multiply this product by 4, writing the first figure in units place, which gives 32 times the number; we then multiply the first product by 3 and write the first figure in thousands place, which gives 24 thousand times the number: taking the sum of these partial products, we have the entire product.

OPERATION.

```
    43526
    24832
 --------
  348208
   1392832
 1044624
 ---------
1080837632
```

Rule.—I. *Multiply the multiplicand by some term of the multiplier which is a factor of one or more parts of the multiplier.*

II. *Multiply this product by a factor which, taken with the terms used, will produce other parts of the multiplier, and place the right hand term of the product under the right hand term of the part of the multiplier thus used.*

III. *Continue thus until the entire multiplier is used; the sum of all the products will be the entire product.*

What is the value

3. Of 4675×355? *Ans.* 1659625.

4. Of 7608×369? *Ans.* 2807352.

5. Of 13524×428? *Ans.* 5788272.

6. Of 37643×2807? *Ans.* 105663901.

7. Of 57316×35728? *Ans.* 2047786048.

8. Of 618504×24642? *Ans.* 15241175568.

9. Of 730592×408848? *Ans.* 298701078016.

10. Of 395076×576426? *Ans.* 227732078376.

CASE IV.

114. ***When the multiplier differs but little from 100, 1000, 10000, etc.***

1. Multiply 5607 by 996.

SOLUTION.—Since 996 equals 1000 minus 4, 996 times 5607 is the same as 1000 times the number minus 4 times the number; 1000 times 5607 is 5607000, and 4 times 5607 is 22428, and the difference is 5584572. Hence the

OPERATION.

$$\begin{array}{r} 5607 \\ 996 \\ \hline 5607000 \\ 22428 \\ \hline 5584572 \end{array}$$

Rule.—*Annex to the multiplicand as many ciphers as there are terms in the multiplier; multiply the multiplicand by the difference between the multiplier and* 100, 1000, *etc., and add or subtract the two results as the multiplier is greater or less than* 100, 1000, *etc.*

NOTE.—This rule is of especial value when the multiplier is a little less than 100, 1000, etc.

What is the value

2. Of 76573 × 93? *Ans.* 7121289.
3. Of 53781 × 998? *Ans.* 53673438.
4. Of 64336 × 105? *Ans.* 6755280.
5. Of 397842 × 9994? *Ans.* 3976032948.
6. Of 587543 × 9989? *Ans.* 5868967027.
7. Of 473721 × 9970? *Ans.* 4722998370.
8. Of 5654321 × 99980? *Ans.* 565319013580.
9. Of 7733447 × 998800? *Ans.* 7724166863600.

PRACTICAL PROBLEMS.

1. A clerk's salary is $25 a week; he pays $7.75 for his board, and $5.25 for other expenses; how much will he save in a year? *Ans.* $624.

2. A man is north of Cincinnati 50 miles; if he should travel south 15 days at the rate of 35 miles a day, how far would he be from Cincinnati? *Ans.* 475 miles.

3. The library of an academy consists of 4 cases, each containing 16 shelves, and each shelf averaging 77 books; how many books are in the library? *Ans.* 4928 books.

4. Stewart & Co. bought 18 cases French chintzes, each case containing 45 pieces, and each piece 34 yards, at 17 cents a yard; what was their bill? *Ans.* $4681.80.

5. A shipping firm received the following freight: 25 hogsheads tobacco at $10.50 a hhd.; 3000 quarters of wheat

at $2 a quarter, and 4000 barrels of petroleum at $1.10 a barrel; what was the amount received? *Ans.* $10,662.50.

6. G. Pidcock & Co. bought Ohio sheep as follows: 209, averaging 79 lbs. each, at 6 cents per lb.; 108, averaging 100 lbs., at 7 cents per lb.; 68, averaging 56 lbs., at 5 cents per lb.; and 90 Canadian lambs, averaging 77 lbs., at 8 cents per lb.; they sold the whole at an average price per head of $6; what was their profit, deducting $67.20 for expenses? *Ans.* $291.34.

7. A bankrupt failed for $100,000; his assets were as follows: a farm of 450 acres, worth $97 an acre; a house worth $25,000; 55 shares New York Central Railroad, at $101 a share; 99 shares Pacific Mail at $39 a share; and 155 shares Western Union Telegraph at $82; what remains unpaid? *Ans.* $9224.

8. A commission merchant in Philadelphia sold the following consignment from Cincinnati: 200 barrels prime mess pork at $19 per barrel; 1990 lbs. pickled hams at 11 cents per lb.; 996 lbs. smoked hams at 13 cents a lb., and 1080 lbs. of lard at 14 cents per lb.; in return he forwarded 150 barrels crushed sugar, 200 lbs. each, at 11 cents per lb., and the remainder he paid by draft; what was the amount of the draft? *Ans.* $999.58.

9. A wool dealer in New York, on making up his accounts for the week ending December 18, 1875, found his purchase to be as follows: Smyrna unwashed fleeces, 450 lbs., at 20 cents; Syrian washed, 230 lbs., at 33 cents; Donskoi washed, 140 lbs., 31 cents; Cape of Good Hope, 75 lbs., 36 cents. During the same time his sales were as follows: Texas fine, Eastern, 250 lbs., 30 cents; medium, 190 lbs., 29 cents; Texas, Western, 81 lbs., 20 cents; California Spring Clip, superior, unwashed, 150 lbs., 33 cents; coarse, 120 lbs., 22 cents; burry, 75 lbs., 15 cents; what are the amounts of the purchases and sales for the week, and what is the balance of the account? *Ans.* $2.85.

DIVISION.

115. Division is the process of finding the *quotient* of two numbers.

116. The **Quotient** of two numbers is a number which expresses how often one number is contained in another.

117. The **Dividend** is the number to be divided.

118. The **Divisor** is the number by which we divide.

119. The **Remainder** is the number which is sometimes left after dividing.

120. The **Terms** in Division are the *Dividend*, the *Divisor*, and the *Quotient.*

121. The **Sign of Division** is $\div$, and is read *divided by.* It denotes that the number preceding it is to be divided by the number following it.

Division is also indicated by writing the divisor beneath the dividend with a line between them; or by writing the divisor at the left of the dividend with a curved line between them; thus, $\frac{24}{8}$, also 8)24.

NOTES.—The *Sign of Division* is a short line, in the line of writing, with dots above and below the middle of it.

2. The symbol $\div$ was introduced by Dr. John Pell, an English mathematician, born in 1610.

PRINCIPLES.

1. *The divisor and dividend are always similar numbers.*

For, it is evident that any number can be contained only in a similar number; and also that no number of times one concrete number can equal a concrete number of another kind.

2. *The quotient is always an abstract number.*

For, since the quotient shows how many times the divisor is contained in the dividend, it cannot be *apples times* or *peaches times*, but simply an abstract number of times.

3. *The remainder is a number similar to the dividend.*

For, since it is an undivided part of the dividend, it must be of the same unit as the dividend.

4. *If all the parts of the dividend be divided by the divisor, the whole dividend will be divided by it.*

For, all the partial quotients taken together will evidently be equal to the entire quotient.

NOTE.—These principles are theoretically true, though in practice we do sometimes divide by an abstract number and obtain a concrete quotient.

PROBLEM.

122. ***To divide one number by another.***

1. Divide 7872 by 32.

OPERATION.

```
32)7872(246
   64
   ---
   147
   128
   ---
    192
    192
    ---
```

SOLUTION.—32 is not contained in 7 thousands any thousands times, hence there are no thousands in the quotient: 7 thousands and 8 hundreds are 78 hundreds; 32 is contained in 78 hundreds 2 hundreds times; 2 hundreds times 32 are 64 hundreds, which subtracted from 78 hundreds leave 14 hundreds: 14 hundreds and 7 tens are 147 tens; 32 is contained in 147 tens 4 tens times; 4 tens times 32 are 128 tens, which subtracted from 147 tens, leave 19 tens: 19 tens with 2 units are 192 units; 32 is contained in 192 units 6 units times; 6 units times 32 are 192 units. Hence the quotient is 246.

Rule.—I. *Draw curved lines at both sides of the dividend, and place the divisor at the left.*

II. *Divide the number expressed by the fewest terms at the left that will contain the divisor, and place the quotient at the right.*

III. *Multiply the divisor by this quotient, write the product under the partial dividend, subtract, and to the remainder annex the next term of the dividend.*

IV. *Divide as before, and thus continue until all the terms of the dividend have been used.*

V. *If any partial dividend will not contain the divisor, place a cipher in the quotient, annex the next term of the dividend, and proceed as before.*

VI. *When there is a final remainder, annex it, with the divisor written beneath, to the integral part of the quotient.*

NOTE.—When the divisor does not exceed 12, we usually draw a line under the dividend, and write the quotients beneath, doing the rest of the work mentally. This is called *Short Division;* the other method is called *Long Division.*

Proof.—Multiply the integral part of the quotient by the divisor, and add the remainder, if any, to the product; if the work is correct the result will be equal to the dividend.

NOTES.—I. The pupils will notice that there are *five* operations: 1st, *Write the numbers;* 2d. *Divide;* 3d. *Multiply;* 4th. *Subtract;* 5th. *Bring down.*

II. Pupils often have difficulty in finding the correct quotient figure; this difficulty can be greatly diminished by attention to the following suggestions:

1st. Notice how often the left hand term of the divisor is contained in the term or terms of the partial dividend, as far from the right hand term as the left hand term in the divisor is from the right hand term.

2d. If, when we multiply, the product is greater than the partial dividend, the quotient term must be diminished.

3d. If, when we subtract, the remainder is greater than the divisor, the quotient term must be increased.

III. We commence at the left to divide, so that the remainder can be united to the number of units of the next lower order, giving a new partial dividend. The sign + is used to denote a remainder.

Divide

2. 20149917 by 846. *Ans.* 23817+

3. 12840243 by 2135. *Ans.* 6014+.

4. 13824979 by 6734. *Ans.* 2053+.

5. 8074984 by 6328. *Ans.* 1276+.

6. 237925094 by 2222. *Ans.* 107077.

7. 21621825225 by 9009. *Ans.* 2400025.

8. 426510892284 by 60404. *Ans.* 7060971.

9. 213255462816 by 60408. *Ans.* 3530252.

10. 137081638980 by 49335. *Ans.* 2778588.

11. 1378921500 by 55714. *Ans.* 24750.

12. 468761197905 by 555555. *Ans.* 843771.

13. 48924056844 by 69543. *Ans.* 703508.

14. 27844312576 by 73256. *Ans.* 380096.

15. 48384751874346 by 590778. *Ans.* 81900057.

16. 5299770856733656 by 7904207. *Ans.* 670500008.

17. 2016722783975663729 by 41927081. *Ans.* 48100720009.

PRACTICAL PROBLEMS.

123. In **Division** there are two classes of practical problems:

1st. To find the number of equal parts of a number.

2d. To divide a number into equal parts.

CASE I.

124. ***To find the number of equal parts of a number.***

1. At 95 dollars each, how many oxen may be bought for 3040 dollars?

Solution.—If 95 dollars will buy one ox, 3040 dollars will buy as many oxen as 95 dollars are contained times in 3040 dollars, which are 32. Therefore, etc.

2. At 54 dollars a share, how many shares of bank stock can be purchased for 333234 dollars? *Ans.* 6171 shares.

3. If a student, on a pedestrian tour, walks 134 miles a week, how many weeks would it take him to walk 238788 miles? *Ans.* 1782 weeks.

4. If the construction of a railroad cost $116,188,800, how long was the road, provided it was built at the rate of $470,400 a mile? *Ans.* 247 miles.

5. If a banker has a net gain of $7,420 annually, how long will it take him to pay for a farm of 175 acres at $212 per acre? *Ans.* 5 years.

6. A man traded 7425 acres of woodland, worth 48 dollars an acre, for farm land, worth 144 dollars an acre; how many acres did he receive? *Ans.* 2475 acres.

7. Suppose that A and B are 2376 miles apart and approach each other, A traveling 15 miles an hour and B 18 miles an hour; in how many hours will they meet? *Ans* 72 hours.

8. A man has $39,180 which he wishes to invest in land. He buys 246 acres at $145 an acre; how many acres can he buy with the balance of the money, at $130 an acre? *Ans.* 27 acres.

9. How many two-horse phaetons worth $245 apiece could be bought for the value of 348 horses worth $234 each, and $643 in money? *Ans.* 335 phaetons.

10. A man bought 140 acres of land for $10,500, and sold 95 acres at $125 an acre; at what price per acre must he sell the remainder to gain $5,425? *Ans.* $90.

11. A speculator wishes to trade land worth $95 an acre for 73 acres at $125 an acre, and gain $290 on this estimate of values by the exchange; how many acres will he exchange? *Ans.* 93 acres.

CASE II.

125. ***To divide a number into equal parts.***

1. Divide 456 into 6 equal parts.

Solution.—If we divide 456 into 6 equal parts, each part is $\frac{1}{6}$ of 456: $\frac{1}{6}$ of 45 tens is 7 tens and 3 tens remaining; 3 tens and 6 units equal 36; $\frac{1}{6}$ of 36 is 6; hence $\frac{1}{6}$ of 456 is 76, or 76 is one of the 6 equal parts of 456.	OPERATION. 6)456 76

2. Divide 16512 into 12 equal parts. *Ans.* 1376.

3. Divide 42228 into 18 equal parts. *Ans.* 2346.

4. Divide 73260 into 37 equal parts. *Ans.* 1980.

5. Divide 52224 into 64 equal parts. *Ans.* 816.

6. Messrs. Wilson and Co. purchased 24 shawls for $840; what did they cost apiece?

Solution.—If 24 shawls cost $840, one shawl will cost one twenty-fourth of $840, which, by division, we find is $35. Therefore, etc.	OPERATION. 24)840(35 72 120 120

7. Messrs. Taylor & Brother bought 37 sets of furs for $6068; what did they pay per set? *Ans.* $164.

8. A young man shared a legacy of $24,780 with each of his 5 brothers, and another legacy of $14,300 with each of his 4 sisters; what sum did he receive? *Ans.* $6990.

9. I bought a farm of 136 acres for $8568, and sold 93 acres of it at $75 an acre, and the remainder for what it cost; how much did I gain by the bargain? *Ans.* $1116.

10. Miss Atherton bought 235 shares of Northern Central Railroad stock for $10,575, and sold a part off for $7448 at $56 a share; how many shares remained and what was the gain on those sold? *Ans.* Rem., 102 shares; gain, $1463.

11. Bought a farm for $35,380, and having made improvements valued at $3420, I sold one-half of it for $21,750 at $75 an acre; how many acres did I purchase, and at what price per acre? *Ans.* 580 acres; $61 an acre.

12. An army contractor bought some horses for $18,750, sold part of them for $4725 at $135 apiece, and lost $15 on each horse sold; and subsequently sold the remainder so as to gain $375 on the whole; at what rate were the remainder sold? *Ans.* $160 apiece.

CONTRACTIONS IN DIVISION.

126. Contractions in Division are abbreviated forms of dividing.

127. Successive Division is the process of dividing one number by another, the quotient by a second divisor, etc. *Successive division* is the reverse of *continued multiplication.*

PRINCIPLES.

1. *The quotient of two numbers is equal to the quotient derived by the successive division of one of the numbers by the factors of the other.*

2. *The quotient derived by successive division is the same, in whatever order the divisors are taken.*

CASE I.

128. ***When the divisor is a composite number.***

1. Divide 7875 by 35, using the factors 5 and 7.

SOLUTION.—Since 35 times a number equals 7 times 5 times the number, $\frac{1}{35}$ of a number equals $\frac{1}{7}$ of $\frac{1}{5}$ of the number; $\frac{1}{5}$ of 7875 is 1575, $\frac{1}{7}$ of 1575 is 225.

OPERATION.

$$\begin{array}{r} 5)\underline{7875} \\ 7)\underline{1575} \\ 225 \end{array}$$

Rule.—*Divide the dividend by one factor of the divisor, the quotient by another factor, and thus continue for all the factors used; the last quotient will be the quotient required.*

Divide

2. 11112 by 24.	**6.** 974496 by 96.
3. 15624 by 42.	**7.** 450144 by 108.
4. 267848 by 56.	**8.** 966064 by 121.
5. 280440 by 72.	**9.** 690624 by 144.

129. The **True Remainder** in successive division being neither the last remainder nor the sum of all the remainders, it is necessary to explain the method of finding it.

1. Divide 2243 by 84, using the factors 3, 4, and 7.

SOLUTION.—Dividing by 3 we find that 2243 equals 747 *threes*, and 2 remaining; dividing by 4 we find 747 *threes* equals 186 *twelves* and 3 *threes*, or 9 remaining; dividing by 7 we find that 186 *twelves* equals 26 *eighty-fours* and 4 *twelves*, or 48 remaining. Hence the true remainder is $2+9+48=59$.

OPERATION.

$$\begin{array}{lr} 3)\underline{2243} & \\ 4)\underline{747}-2 = & 2 \\ 7)\underline{186}-3 \textit{ threes} = & 9 \\ \quad 26-4 \textit{ twelves} = & \underline{48} \\ & 59 \end{array}$$

Rule.—*Multiply each remainder by all the divisors preceding the one which obtained it, and take the sum of the products and the remainder arising from the first division.*

Divide and find the true remainder.

2. 13225 by 105 (3, 5, 7). *Ans.* 125. *Rem.* 100.
3. 43125 by 126 (2, 7, 9). *Ans.* 342. *Rem.* 33.
4. 141190 by 180 (4, 5, 9). *Ans.* 784. *Rem.* 70.
5. 16199 by 216 (2, 3, 4, 9). *Ans.* 74. *Rem.* 215.
6. 113546 by 378 (2, 3, 7, 9). *Ans.* 300. *Rem.* 146.
7. 363887 by 1512 (2, 3, 4, 7, 9). *Ans.* 240. *Rem.* 1007.

CASE II.

130. ***When there are ciphers at the right of the divisor.***

1. Divide 9856 by 800.

SOLUTION.—8 hundreds are contained in 98 hundreds 12 times with a remainder of 200; 800 is not contained in 56, hence the entire remainder is 200+56, or 256.

OPERATION.

$$\begin{array}{r} 8|00)98|56 \\ \hline 12\text{—}256 \end{array}$$

Rule.—I. *Cut off the ciphers at the right of the divisor, and as many terms at the right of the dividend.*

II. *Divide the remaining part of the dividend by the remaining part of the divisor.*

III. *Prefix the remainder to the part of the dividend cut off, and the result will be the true remainder.*

NOTE.—When the divisor is a unit of any order with ciphers, the remainder will be the figures cut off at the right, and the quotient the figures at the left.

What is the value	QUO.	REM.
2. Of 50483÷700?	*Ans.* 72.	83.
3. Of 43802÷1500?	*Ans.* 29.	302.
4. Of 723456÷2800?	*Ans.* 258.	1056.
5. Of 5793020÷13300?	*Ans.* 435.	7520.
6. Of 87644300÷4570000?	*Ans.* 19.	814300.
7. Of 937659000÷39800000?	*Ans.* 23.	22259000.

CASE III.

131. ***When the remainders are obtained without writing the products and subtracting.***

1. Divide 86795 by 37.

OPERATION.

```
37)86795(2345
    127
     169
      215
       30 Rem.
```

SOLUTION.—We divide 86 by 37 and find a quotient of 2; we then multiply 37 by 2, but instead of writing the product and subtracting it from the partial dividend, we observe what numbers must be added to the product to give the terms of the partial dividend, and write them for the remainder, thus: 37 is contained in 86, 2 times; 2 times 7 are 14 and 2 are 16; we write the 2 under the 6; 2 times 3 are 6 and 1 to carry are 7; 7 and 1 are 8; we write the one under the 8, and bringing down 7, the next figure of the dividend, we have 127 for the next dividend; 37 is contained in 127, 3 times; 3 times 7 are 21, and 6 are 27; hence we write the 6 under the 7; 3 times 3 are 9, and 2 to carry are 11, which increased by 1 make 12; we write the 1 under the 2, and bringing down, we have 169 for the next dividend, etc.

Rule.—I. *Obtain the quotient figures in the usual manner.*

II. *Obtain the remainders by observing what number must be added to each partial product to obtain the terms of the partial dividend.*

III. *Bring down the terms of the dividend in the usual manner, and thus proceed until the division is complete.*

2. Divide 811332 by 372. *Ans.* 2181.
3. Divide 1957413 by 453. *Ans.* 4321.
4. Divide 6419945 by 3007. *Ans.* 2135.
5. Divide 8074528 by 6328. *Ans.* 1276.
6. Divide 97547337 by 3891. *Rem.* 3858.
7. Divide 4223745376 by 180071. *Ans.* 23456.
8. Divide 170627676887 by 413071. *Rem.* 25846.

CASE IV.

132. ***When the divisor is a little less than* 100, 1000, *etc.***

1. Divide 7639521 by 96.

OPERATION.

76395	21
3056	01
122	25
5	13
	33
79578	$\frac{33}{96}$

SOLUTION.—Dividing 7639521 by 100 (or 96 + 4) by cutting off two figures at the right of the dividend, we obtain for the first partial quotient 76395, and a remainder 21. Since the divisor used is 4 more than the real divisor, the remainder is too small by 4 times 76395. Adding 4 times 76395, or 305580, to the remainder, we find it to be $305580 + 21 = 305601$, which contains the divisor. Dividing again, we have a quotient 3056 and a remainder 1. Adding to this remainder 4 times 3056, or 12224, we have a remainder 12225, which still contains the divisor. Dividing again, we have a quotient 122 and remainder 25, to which remainder adding 4 times 122 or 488, we obtain a

fourth remainder 513, which being again divided and increased by 4 times 5, gives the true remainder 33. Adding the several partial quotients, and annexing the remainder, we have $79578\frac{33}{95}$, the quotient required. Hence the following

Rule.—I. *Cut off from the right of the dividend by a vertical line as many terms as there are in the divisor, multiply the part on the left of the line by the difference between the divisor and* 100, 1000, *etc., and add the product to the number on the right for a true remainder, of which we make a new dividend.*

II. *Divide as before, multiply the new quotient by the difference between the divisor and* 100, 1000, *etc., add the product to the remainder for a true remainder, and thus proceed until the remainder is less than the given divisor; the sum of the several quotients with the last remainder, if any, will be the quotient required.*

Divide the following:

2. $65343214 \div 999$. *Ans.* $65408\frac{622}{999}$.
3. $797876541 \div 9994$. *Ans.* $79835\frac{5551}{9994}$.
4. $457637892 \div 9998$. *Ans.* $45772\frac{9436}{9998}$.
5. $6747890343 \div 99930$. *Ans.* $67526\frac{17163}{99930}$.
6. $5434479222 \div 99800$. *Ans.* $54453\frac{69822}{99800}$.

THE PARENTHESIS AND VINCULUM.

133. The **Parenthesis,** ( ), denotes that the quantities included are subjected to the same operation. Thus, $18-(9+5)$ means 18 minus the sum of 9 and 5.

134. The **Vinculum,** or bar, ——, is used for the same purpose as the parenthesis, the numbers under it being considered as one quantity. Thus $12-\overline{9-3}$ means that the difference of 9 and 3 is to be subtracted from 12.

1. What is the value of $(572-14)-\overline{376-35}$?

SOLUTION.—572 — 14 equals 558; 376 — 35 equals 341, and 558 — 341 equals 217. Therefore, etc.

2. Of $(84793-45832)-(76345-46247)$? *Ans.* 8863.
3. Of $(534-46)-\overline{7640-6989}+472-12$? *Ans.* 297.
4. Of $(7000700-2999299)-\overline{40040-37737}+572$? *Ans.* 3999670.

5. Of $\overline{8796-2437+210}\times(8761-5672+6912)$?

Ans. 65696569.

6. Of $(656+397)\div(247-166)+25\times 670$?

Ans. 16763.

7. Of $\overline{7945-5340}\times(549+751)\div(5789-5529)$?

Ans. 13025.

8. Of $\{(9324+2461-7275)\div\overline{3471-2432+1216}\}\times(6789-2507+3364)$?

PRACTICAL EXAMPLES.

1. The product of two numbers is 415638, and one of them is 7697; what is the other? *Ans.* 54.

2. The product of three numbers is 2237984, and two of them are 103 and 97; what is the third? *Ans.* 224.

3. The dividend is 274500, the quotient 983, and the remainder 243; what is the divisor? *Ans.* 279.

4. What is the nearest number to 25000 that can be divided by 575 without a remainder? *Ans.* 24725.

5. What is the nearest number to 37401 that can be divided by 784 without a remainder? *Ans.* 37632.

6. Find the value of $29+348\div 6+217\times 25+438\div 73$ added to $192\div 24+(225-102)\times 26$. *Ans.* 8724.

7. A man paid a debt of $105.45 with an equal number of dollars, dimes and cents; how many were there of each kind? *Ans.* 95.

8. Find the value of $\{(9097+6956-2364)-(8765-2721+2917)\}\div\{\overline{6432+5832\times 99-(3278-118503\div 297-2790)}\div(12965-5273+7391-\overline{8771+3349})\}$.

9. The product of three numbers is 196790480, the smallest is 365, and the product of this and the largest is 396755; required the other two factors. *Ans.* 496; 1087.

10. A New Jersey farmer, wishing to go West, sold his farm of 150 acres at $84 an acre, and bought prairie land in Illinois for $45 an acre; how many acres did his new farm contain? *Ans.* 280 acres.

11. A farmer sold an equal number of ducks and turkeys; for the ducks he received $2 each, and for the turkeys $3.50

each; and the whole amount received was $44; how many of each did he sell? *Ans.* 8.

12. The first edition, 2800 copies, of a book of 480 pages cost me $1543; what did I pay a page for stereotyping, if the press work cost me $125, the paper cost 12½ cents a copy, and the binding 15 cents a copy? *Ans.* $1.35.

13. A cistern containing 13500 gal. is filled by two pipes, one discharging 250 gal. an hour and the other 300 gal., but, by a leak in one of the pipes, 100 gal. are lost in an hour; how long will it take to fill the cistern? *Ans.* 30 h.

14. Prove and illustrate that the sum or difference of two numbers, divided by any number, will equal the sum or difference of the quotients found by dividing those two numbers by the same number.

15. The day before Christmas, a butcher sold an equal number of ducks and turkeys, and three times as many chickens; he received for the chickens $1.75, for the ducks $2.25, and for the turkeys $4 each, and the whole amount was $92; what was the number of each? *Ans.* 24; 8; 8.

16. The keeper of a restaurant, counting the currency received from one day's sales, found it to amount to $31.50, one-ninth being in fifty-cent notes, and the rest made up of an equal number of twenty-five-cent and ten-cent notes; how many were there of each? *Ans.* 7; 80; 80.

17. A farmer's wife took to the store 6 lb. of butter at 45 cents a pound, 4 doz. eggs at 25 cents a dozen, and 2 pair of spring chickens at $1.25 a pair; she received in exchange groceries amounting to $1.75, a pair of scissors at 50 cents, needles and thread at 20 cents, and delaine at 25 cents a yard; how many yards of delaine did she receive? *Ans.* 15.

18. James Green purchased Norristown Railroad stock to the amount of $5814, and sold part of it for $2756 at $53 a share, losing $4 on a share; but some years after, the road being leased by the Reading Railroad, he sold out at a gain on the whole transaction of $1492; for what did he sell a share? *Ans.* $91.

GENERAL PRINCIPLES OF FUNDAMENTAL OPERATIONS.

135. The **Fundamental Operations** of Arithmetic are Addition, Subtraction, Multiplication, and Division. They are fundamental because others depend upon them.

136. The **Principles** of the fundamental operations are the general truths which relate to them.

137. The **Problems** of the fundamental operations are the different classes of questions which arise under them.

NOTE.—The pupils will illustrate the following principles and problems.

PRINCIPLES OF ADDITION.

1. *The sum of all the parts equals the whole.*
2. *The whole, diminished by one or more parts, equals the sum of the other parts.*

PROBLEMS.

1. Given, the parts to find the whole.
2. Given, the whole and all the parts but one, to find that part.
3. When several numbers are given, how do you find their sum?
4. When the sum of several numbers and all of them but one are given, how is that one found?

PRINCIPLES OF SUBTRACTION.

1. *The Remainder equals the Minuend minus the Subtrahend.*
2. *The Minuend equals the Subtrahend plus the Remainder.*
3. *The Subtrahend equals the Minuend minus the Remainder.*

PROBLEMS.

1. Given, the minuend and subtrahend, to find the remainder.
2. Given, the minuend and remainder, to find the subtrahend.
3. Given, the subtrahend and remainder, to find the minuend.
4. When two numbers are given, how is their difference found?
5. When the greater of two numbers and the difference between them are given, how is the less found?
6. When the less of two numbers and the difference between them are given, how is the greater found?

PRINCIPLES OF MULTIPLICATION.

1. *The Product equals the Multiplicand into the Multiplier.*
2. *The Multiplicand equals the Product divided by the Multiplier.*
3. *The Multiplier equals the Product divided by the Multiplicand.*

PROBLEMS.

1. Given, the multiplicand and multiplier, to find the product.
2. Given, the product and multiplier, to find the multiplicand.
3. Given, the product and multiplicand, to find the multiplier.
4. When two or more numbers are given, how is their product found?
5. When the product and one of two factors are given, how is the other found?
6. When the several factors of a number are given, how is the number found?
7. When the continued product of several factors and all the factors but one are given, how is that one found?

PRINCIPLES OF DIVISION.

1. *The Quotient equals the Dividend divided by the Divisor.*
2. *The Dividend equals the Divisor multiplied by the Quotient.*
3. *The Divisor equals the Dividend divided by the Quotient.*
4. *The Dividend equals the Divisor multiplied by the Quotient, plus the Remainder.*
5. *The Divisor equals the Dividend minus the Remainder, divided by the Quotient.*

PROBLEMS.

1. Given, the divisor and dividend, to find the quotient.
2. Given, the divisor and quotient, to find the dividend.
3. Given, the dividend and quotient, to find the divisor.
4. Given, the divisor, quotient, and remainder, to find the dividend.
5. Given, the dividend, quotient, and remainder, to find the divisor.
6. Given, the final quotient of a continued division and the several divisors, to find the dividend.
7. Given, the final quotient of a successive division, the first dividend, and all the divisors but one, to find that divisor.
8. Given, the dividend and several divisors of a successive division, to find the quotient.

PRINCIPLES OF CHANGES OF TERMS.

OF ADDITION.

1. Increasing or diminishing any term by any number, increases or diminishes the sum by that number.

OF SUBTRACTION.

1. Increasing or diminishing the minuend and subtrahend by the same number does not change the remainder.
2. Increasing or diminishing the minuend by any number, increases or diminishes the remainder by that number.
3. Increasing or diminishing the subtrahend by any number, diminishes or increases the remainder by that number.

OF MULTIPLICATION.

1. Multiplying either the multiplicand or multiplier by any number, multiplies the product by that number.

2. Dividing either the multiplicand or multiplier by any number, divides the product by that number.

3. Multiplying both multiplicand and multiplier by a number, multiplies the product by both numbers.

4. Dividing both multiplicand and multiplier by a number, divides the product by both numbers.

5. Multiplying one factor and dividing the other by the same number, does not alter the product.

6. Adding a number to either factor increases the product by as many times the other factor as there are units in the number added.

7. Subtracting a number from either factor diminishes the product by as many times the other factor as there are units in the number subtracted.

OF DIVISION.

1. Multiplying the dividend multiplies the quotient, and dividing the dividend divides the quotient.

2. Multiplying the divisor divides the quotient, and dividing the divisor multiplies the quotient.

3. Multiplying or dividing both dividend and divisor by the same number does not alter the quotient.

4. Adding a number to the dividend increases the remainder by this number, if the quotient remains the same.

5. Subtracting a number from the dividend diminishes the remainder by this number, if the quotient remains the same.

6. *Adding* any number to the divisor *diminishes* the quotient by as many units as the new divisor is contained times in the product of the quotient by the number added.

7. *Subtracting* any number from the divisor *increases* the quotient by as many units as the new divisor is contained times in the product of the quotient by the number subtracted.

GENERAL LAWS.

1. A change, by addition or subtraction, of any term in addition, produces a *similar* change in the sum.

2. A change in the minuend by addition or subtraction, produces a *similar* change in the difference; but such a change in the subtrahend produces an *opposite* change in the difference.

3. A change in either factor in multiplication, by multiplication or division, produces a *similar* change in the product.

4. A change in the dividend by multiplication or division produces a *similar* change in the quotient; but such a change in the divisor produces an *opposite* change in the quotient.

SECTION III.

SECONDARY OPERATIONS.

138. The **Primary Operations** of Arithmetic are those of synthesis and analysis, including the four fundamental rules.

139. The **Secondary**, or **Derivative Operations** are those which arise from or grow out of the primary operations of synthesis and analysis.

140. The **Secondary Operations** are *Composition*, *Factoring*, *Greatest Common Divisor*, *Least Common Multiple*, *Involution*, and *Evolution.*

COMPOSITION.

141. Composition is the process of forming composite numbers when their factors are given.

142. A **Composite Number** is a number which can be produced by multiplying together two or more numbers, each greater than a unit; as 8, 12, 15, etc.

143. The **Factors** of a composite number are the numbers which, when multiplied together, will produce it; thus 4 and 2 are the factors of 8.

144. A **Prime Number** is one that cannot be produced by multiplying together two or more numbers, each greater than a unit; as 2, 5, 7, 11, etc.

145. A **Power** of a number is a number formed by taking the given number several times as a factor; thus 64 is the third power of 4.

146. The **Second Power** of the number is the composite number formed by using the number twice as a factor. Thus 9 is the second power of 3.

147. The **Third Power** of a number is the composite number formed by using the number three times as a factor. Thus 27 is the third power of 3.

148. The **Symbol** for the power of a number is a small figure, called an *exponent*, placed a little above it at the right; thus, 5^3 denotes the third power of 5, etc.

NOTE.—In the fundamental operations, each *synthetic* process has its corresponding *analytic* process; it follows, therefore, that there should be a synthetic process corresponding to the analytic process of *Factoring*. This process I have called *Composition*. This new generalization, given in my Algebra, has already been approved by several mathematicians.

149. Cases.—The subject is treated under *six cases*. The development of these cases is based upon the following principles:

PRINCIPLES.

1. *Every composite number is equal to the product of its factors.*

2. *A factor of a number is a factor of any number of times that number.*

CASE I.

150. *To form a composite number out of any factors.*

1. Form a composite number out of 3, 4, and 5.

SOLUTION.—A composite number formed out of the factors 3, 4, and 5, is equal to their product, which is 60.

OPERATION.

$3 \times 4 \times 5 = 60$

Rule.—*Take the product of the factors as indicated by the problem.*

Form composite numbers out

2. Of 3, 4, 5, and 7. *Ans.* 420.

3. Of two 2's, two 3's, and 5. *Ans.* 180.

4. Of four 2's, three 5's, and two 7's. *Ans.* 98000.

5. Of five 2's, four 3's, and 9. *Ans.* 23328.

6. Of four 5's, two 7's, two 11's. *Ans.* 3705625.

CASE II.

151. *To form a composite number out of equal factors.*

1. Find the composite number consisting of eight 2's.

SOLUTION.—Multiplying 2 by 2 we have 4, which consists of two 2's; multiplying 4 by 4 we have 16, which consists of *two*+*two*, or *four* 2's; multiplying 16 by 16 we have 256, which consists of *four*+*four*, or *eight* 2's.

OPERATION.

$2 \times 2 = 4$
$4 \times 4 = 16$
$16 \times 16 = 256$

Rule.—*Multiply one power of the factor by another until we have a product which contains the factor the required number of times.*

2. Find the sixth power of 2; of 5; of 6.
Ans. 64; 15625; 46656.

3. Find the twelfth power of 2; of 3; of 4.
Ans. 4096; 531441; 16777216.

4. Form a composite number of four equal factors; of five; of six; of seven.

5. Find the 2d, 3d, 5th, 7th, and 10th powers of 3; of 4; of 5. *Ans.* 59049; 1048576; 9765625, etc.

CASE III.

152. ***To form a composite number out of factors bearing certain relations to each other.***

1. Find a composite number of three factors, the smallest being 4, the second being twice, and the third three times the first.

SOLUTION.—The number will contain 4 used three times as a factor, and also 2 and 3; hence we raise 4 to the third power, which is 64, and multiply the result by the product of 2 and 3, or 6, which gives 384.

OPERATION.
$4^3 = 64$
$64 \times 6 = 384$

Rule.—*Raise the given factor with which the others are compared, to the power indicated, and multiply the result by the product of the factors indicating the relation.*

NOTE.—We may also find each factor and then multiply them together.

2. Form a number of three factors, the first being 3, the second twice, and the third 3 times the first. *Ans.* 162.

3. Find a number whose smallest factor is 5, the second being twice, and the third 6 times as great. *Ans.* 1500.

4. Required a number, one of whose factors is 8, another one-half as large, and the third twice as large. *Ans.* 512.

5. Find a number, two of whose factors are 6, two one-half as large, and two 3 times as large. *Ans.* 104976.

6. Find a number one of whose factors is 8, another 4 more, another half the sum of these two, and another half the difference of the first and second. *Ans.* 1920.

CASE IV.

153. ***To form all the composite numbers possible out of given factors.***

1. Form all the composite numbers possible out of 2, 3, and 5.

SOLUTION.—It will be seen that if we write 1 and 2 with a hyphen between them, and multiply each term by 1 and 3, which may be written in the same way, we shall have all the numbers which can be obtained from the multiplication of these factors; and if we multiply the series by 1 and 5, which may be written as before, we shall have 1, 2, 3, and 5, and all the composite numbers that can be formed from 2, 3, and 5. Omitting 1, 2, 3, and 5, in the last result, we shall have all the composite numbers that can be formed with 2, 3, and 5.

OPERATION.

1-2
1-3
———
1-2-3-6
1-5
———
1-2-3-6-5-10-15-30

Ans. 6, 10, 15, 30

Rule.—I. *Connect* 1 *and the first given factor by a hyphen, and multiply each term by* 1 *and by the second given factor, and this series by* 1 *and by the third factor.*

II. *Proceed in the same way till all the factors are used, and the terms of the last product, omitting one and the given factors, will be the numbers required.*

Form the possible composite numbers

2. Out of 2, 3, 5, and 11. *Ans.* 6, 10, 15, 30, 22, etc.

3. Out of 2, 3, 5, 7, and 11. *Ans.* 6, 10, 15, 14, etc

4. Out of 2, 3, 7, 11, and 13.
Ans. 6, 14, 21, 42, 22, 33, 77, 26, 39, etc.

5. Out of 2, 3, 5, 7, 11, and 13.
Ans. 6, 10, 15, 14, 21, 35, 77, 91, 143, etc.

CASE V.

154. ***To form all the composite numbers possible when some of the factors are alike.***

1. Find the composite numbers which can be formed out of 2, 2, 2, 3, and 3.

SOLUTION.—Since 2 is used three times, the first series will evidently be $1-2-2^2-2^3$, or 1-2-4-8; and since 3 is used twice, the second series will be $1-3-3^2$, or 1-3-9, and the products of these terms, omitting the unit and the factors themselves, will be the composite numbers required.

OPERATION.

1-2-4-8
1-3-9
———
1-2-4-8-3-6-12
24-9-18-36-72

Rule.—I. *Write* 1 *and the successive powers of a factor repeated (the highest power being indicated by the number of times the factor occurs) in a row; under this write* 1 *and the successive powers of another factor, and find the products of the terms of the two series.*

II. *Proceed in a similar manner with the products and the remaining factors; the terms of the last product, omitting* 1 *and the original factors, will be the numbers required.*

Form the possible composite numbers out

2. Of 2, 2, 3, 3, and 3. *Ans.* 4, 12, 36, 108, etc.

3. Of two 2's, two 3's, and 5. *Ans.* 4, 6, 9, 12, etc.

4. Of three 2's, two 3's, and 7. *Ans.* 4, 8, 24, 72, etc.

5. Of four 3's, 5, and 7. *Ans.* 9, 27, 81, 405, etc.

6. Of four 2's, three 3's, two 5's, and 7.

Ans. 4, 8, 16, 9, 27, 210, etc.

CASE VI.

155. ***To find the number of composite numbers that can be formed from given factors.***

1. How many composite numbers can be formed with three 2's and two 3's?

SOLUTION.—2 used *three* times as a factor gives, with unity, a series of *four* terms, and three used *twice* gives a series of *three* terms; hence their product will give a series of 4×3, or 12 terms; and, omitting the unit and 2 and 3, we will have 9 terms. Hence there will be 9 composite numbers.

OPERATION.

1-2-4-8 = 4 terms.
1-3-9 = 3 terms.

$4 \times 3 = 12$
$12 - 3 = 9$

Rule.—*Increase the number of times each factor is used by* 1, *take the product of the results, and diminish this by the number of different factors used plus one.*

How many composite numbers can be formed

2. Out of two 3's and two 5's? *Ans.* 6.

3. Out of four 2's and three 5's? *Ans.* 17.

4. Out of three 2's, four 3's, and three 5's? *Ans.* 76.

5. Out of 3, four 2's, two 5's, 7, and 29? *Ans.* 114.

6. Out of 2, 3^2, 5, 7^2, and 11? *Ans.* 66.

7. Out of 2^2, 3^3, 5^4, 7^5, 11, and 13? *Ans.* 1433.

DIVISIBILITY OF COMPOSITE NUMBERS.

156. Composite Numbers can be divided by the factors which produce them.

157. The **Factors** of many composite numbers may be seen by inspection from the following principles:

PRINCIPLES.

1. *A number is divisible by* 2 *when the right hand term is zero or an even digit.*

For, the number is evidently an even number, and all even numbers are divisible by 2.

2. *A number is divisible by* 3 *when the sum of the digits is divisible by* 3.

This may be shown by trying several numbers, and, seeing that it is true with these, we infer that it is true with all. A rigid demonstration is given in the latter part of the book.

3. *A number is divisible by* 4 *when the two right hand terms are ciphers, or when the number they express is divisible by* 4.

If the two right hand terms are ciphers, the number equals a number of hundreds, and since 100 is divisible by 4, any number of hundreds is divisible by 4.

If the number expressed by the two right hand digits is divisible by 4, the number will consist of a number of *hundreds* plus the number expressed by the two right hand digits (thus $1232 = 1200 + 32$); and since both of these are divisible by 4, their sum, which is the number itself, is divisible by 4.

4. *A number is divisible by* 5 *when its right hand term is* 0 *or* 5.

When the unit figure is 0 the last partial dividend must be 0, 10, 20, 30, or 40, each of which is divisible by 5. When the unit figure is 5, the last partial dividend must be 15, 25, 35, or 45, each of which is divisible by 5.

5. *A number is divisible by* 6 *when it is even, and the sum of the digits is divisible by* 3.

Since the number is even it is divisible by 2, and since the sum of the digits is divisible by 3 the number is divisible by 3, and since it contains both 2 and 3, it will contain their product, 3×2, or 6.

6. *A number is divisible by* 8 *when the three right hand terms are ciphers, or when the number expressed by them is divisible by* 8.

If the three right hand terms are ciphers, the number equals a number of *thousands*, and since 1000 is divisible by 8, any number of thousands is divisible by 8.

If the number expressed by the three right hand digits is divisible by 8, the entire number will consist of a number of *thousands* plus the number expressed by the three right hand digits (thus 17368 = 17000 + 368); and since both of these parts are divisible by 8, their sum, which is the number itself, is divisible by 8.

7. *A number is divisible by* 9 *when the sum of the digits is divisible by* 9.

This may be shown by trying several numbers, and, seeing that it is true with these, we can infer that it is true with all. It may also be rigidly demonstrated.

8. *A number is divisible by* 10 *when the unit figure is* 0.

For, such a number equals a number of *tens*, and any number of tens is divisible by 10, hence the number is divisible by 10.

NOTE.—1. *A number is divisible by* 7 *when the sum of the odd numerical periods, minus the sum of the even numerical periods, is divisible by* 7.

2. *A number is divisible by* 11 *when the difference between the sums of the digits in the odd places and in the even places is divisible by* 11, *or when this difference is* 0.

3. These two principles are rather curious than useful. For their demonstration see the latter part of the book, where will also be found quite a full treatment of the *Properties of Numbers*.

FACTORING.

158. Factoring is the process of finding the factors of composite numbers.

159. The **Factors** of a composite number are the numbers which, when multiplied together, will produce it. Unity and the number itself are not regarded as factors.

160. The **Prime Factors** of a composite number are the prime numbers which, multiplied together, will produce it.

161. A **Root** of a number is ONE of its *several equal* factors; thus 3 is a root of 9, and 4 of 64.

162. The **Second Root** of a number is one of its two equal factors; thus, since $2 \times 2 = 4$, 2 is the 2d root of 4.

163. The **Third Root** of a number is one of its three equal factors; the 4th root is one of its 4 equal factors, etc.

164. The **Symbol** of roots is the radical sign $\surd$; a small figure placed at the left of the sign, called the *index*,

indicates the degree of the root. Thus $\sqrt[2]{4}$, or $\sqrt{4}$, indicates the second root; $\sqrt[3]{27}$, the third root, etc.

165. Cases.—The subject embraces *seven cases.* The development of these cases is based upon the following principles:

PRINCIPLES.

1. *A divisor of a number (excepting unity and the number itself) is a factor of the number.*

2. *A divisor of a factor of a number (excepting unity) is a factor of the number.*

3. *A number is divisible by its prime factors or by any product of them.*

4. *A number is divisible only by its prime factors, or some product of them, or by unity.*

CASE I.

166. ***To resolve a number into its prime factors.***

1. Find the prime factors of 165.

SOLUTION.—Dividing by 3, we find that 3 is a factor of 165 (Prin. 1). Dividing the quotient by 5, we find that 5 and 11 are also factors of 165 (Prin. 2); and since these numbers 3, 5, and 11, are prime, they are the prime factors of 165.

OPERATION.

$$\begin{array}{r} 3)\underline{165} \\ 5)\underline{55} \\ 11 \end{array}$$

Rule.—I. *Divide the given number by any prime number, greater than* 1, *that will exactly divide it.*

II. *Divide the quotient, if composite, in the same manner, and thus continue until the quotient is prime.*

III. *The divisors and last quotient will be the prime factors required.*

Find the prime factors

2. Of 385. *Ans.* 5, 7, 11.

3. Of 1365. *Ans.* 3, 5, 7, 13.

4. Of 1260. *Ans.* 2^2, 3^2, 5, 7.

5. Of 3465. *Ans.* 3^2, 5, 7, 11.

6. Of 39270. *Ans.* 2, 3, 5, 7, 11, 17.

7. Of 7780500. *Ans.* 2^2, 3^2, 5^3, 7, 13, 19.

CASE II.

167. ***To resolve a number into equal factors.***

1. Find the two equal factors of 225.

SOLUTION.—We first resolve the number into its prime factors. Now, since there are *two* 3's, we take *one* 3 for each factor; and since there are *two* 5's, we take *one* 5 for each factor; hence each of the two equal factors is 3×5 or 15; therefore 15 is one of the two equal factors of 225.

OPERATION.

$225 = 3^2 \times 5^2$
$3\times 5 = 15$, *Ans.*

Rule.—I. *Resolve the number into its prime factors.*

II. *Take the continued product of one of each of the two equal factors, when we wish the two equal factors; one of each of the three, for the three equal factors; etc.*

2. Find one of the two equal factors of 256, 576, 5184, 9216, and 20736. *Ans.* 16, 24, 72, 96, 144.

3. Find one of the three equal factors of 216, 512, 1000, 1728, 2744, 3375, 5832. *Ans.* 6, 8, 10, 12, 14, 15, 18.

Find the value of

4.	$\sqrt{225}$.	*Ans.* 15.	**8.**	$\sqrt[4]{50625}$.	*Ans.* 15.
5.	$\sqrt[3]{4096}$.	*Ans.* 16.	**9.**	$\sqrt[5]{14348907}$.	*Ans.* 27.
6.	$\sqrt[4]{4096}$.	*Ans.* 8.	**10.**	$\sqrt[5]{184528125}$.	*Ans.* 45.
7.	$\sqrt[3]{262144}$.	*Ans.* 64.	**11.**	$\sqrt[6]{11390625}$.	*Ans.* 15.

CASE III.

168. ***To resolve a number into factors bearing certain relations to each other.***

1. Resolve 192 into two factors, one of which shall be 3 times the other.

SOLUTION.—By the condition of the problem, 3 times the second factor equals the first. Now, the second factor multiplied by 3 times the second factor equals 3 times the square of the second factor, which equals 192; hence the square of the second factor equals $\frac{1}{3}$ of 192, or 64; if 64 is the square of the second factor, $\sqrt{64}$, or 8, is the second factor, and 8×3, or 24, is the first factor.

OPERATION.

$3)\underline{192}$
$\quad 64$

$\sqrt{64} = 8$, 2d factor.
$3\times 8 = 24$, 1st factor.

Rule.—I. *Divide the given number by the product of the numbers indicating the relation of the factors to the smallest factor, and extract that root of the quotient indicated by the number of factors; the result will be the smallest factor*

II. *Multiply the smallest factor by the numbers indicating the relation of the other factors to it, and the result will be the other factors.*

2. Resolve 4096 into three factors, such that the second shall equal twice the first, and the third twice the second.
Ans. 8, 16, 32.

3. Resolve 2592 into three factors, of which the second is twice the first and the third three times the second.
Ans. 6, 12, 36.

4. Resolve 82944 into four factors, so that the second shall be twice the first, the third twice the second, and the fourth twice the third. *Ans.* 6, 12, 24, 48.

5. Resolve 373248 into four factors, the second being twice the first, the third 3 times the second, and the fourth 4 times the third. *Ans.* 6, 12, 36, 144.

6. The contents of a cistern are 1728 cu. ft., the length being 2 times the breadth, and the breadth 2 times the depth; what are the dimensions? *Ans.* 24 ft.; 12 ft.; 6 ft.

7. A farmer had a bin containing 324 cu. ft., whose length was 3 times the breadth, and breadth twice the depth; required the dimensions. *Ans.* 18 ft.; 6 ft.; 3 ft.

8. A person wished to dig a tank to hold 1024 cu. ft., but having but little ground he was obliged to make the breadth half the length and the length $\frac{1}{4}$ of the depth; what were the dimensions? *Ans.* 4 ft.; 8 ft.; 32 ft.

CASE IV.

169. ***To find all the divisors of a number whose factors are all unequal.***

1. Find all the divisors of 66.

SOLUTION.—The prime factors of 66 are 2, 3, and 11. If we multiply 1-2 by 1-3 we obtain 1, 2, 3, and the product of 2 and 3, and these multiplied by 1-11 will give 1, 2, 3, 11, and all the products that can be formed out of 2, 3, and 11. Hence, according to Prin. 4, these are all the divisors of 66.

OPERATION.

1-2
1-3
———
1-2-36
1-11
———
1-2-3-6-11-22-33-66

Rule.—*Resolve the number into its prime factors, multiply* 1 *and the first factor by* 1 *and the second factor, the*

products by 1 *and the third factor, etc., until all the factors have been used; the result will be the divisors required.*

Find all the divisors

2. Of 105. *Ans.* 1, 3, 5, 7, 15, 21, 35, 105.
3. Of 385. *Ans.* 1, 5, 7, 11, 35, 55, 77, 385.
4. Of 570. *Ans.* 1, 2, 3, 5, 6, 10, 15, 19, 38, etc.
5. Of 1001. *Ans.* 1, 7, 11, 13, 77, 91, 143, 1001.
6. Of 19019. *Ans.* 1, 7, 11, 13, 19, 77, 91, 133, 143, etc.

CASE V.

170. ***To find all the divisors of a number, some of whose factors are equal.***

1. To find all the different divisors of 108.

OPERATION.

$$108 = 2\times 2\times 3\times 3\times 3$$
1-3-9-27
1-2-4

1-3-9-27-2-6-18-54-4-12-36-108

SOLUTION.—We find the factors of 108 are two 2's and three 3's. Since 3 is a factor three times, 1, 3, 3^2, 3^3 is the first series of divisors; and since 2 is a factor twice, 1, 2, 4, is the second series of divisors; and the products of the terms of these two series will give the prime factors and all possible products of them, and therefore all the divisors of the given numbers.

Rule.—I. *Resolve the number into its prime factors, form a series, consisting of* 1 *and the successive powers of one factor, under this write* 1 *and the successive powers of another factor, and take the products of the terms of the two series.*

II. *Proceed in a similar manner with these products and the remaining factors, if any, and the terms of the last product will be all the divisors of the given number.*

Find all the divisors

2. Of 48. *Ans.* 1, 2, 4, 8, 16, 3, 6, 12, 24, 48.
3. Of 72. *Ans.* 1, 2, 4, 8, 3, 6, 9, 12, 24, 18, 36, 72.
4. Of 100. *Ans.* 1, 2, 4, 5, 10, 20, 25, 50, 100.
5. Of 360. *Ans.* 1, 2, 4, 8, 9, 3, 6, 12, 24, 10, etc.
6. Of 810. *Ans.* 1, 2, 3, 5, 9, 27, 10, 45, 90, etc.
7. Of 840. *Ans.* 1, 2, 3, 5, 7, 12, 4, 8, 24, etc.
8. Of 960. *Ans.* 1, 2, 3, 4, 5, 6, 8, 10, 12, 16, etc.

CASE VI.

171. ***To find the number of divisors of a number.***

1. How many divisors has 108?

SOLUTION.—It is evident that 1 with the 1st, 2d, and 3d powers of 3 will give a series of *four* divisors, and 1 with the 1st and 2d powers of 2 will give a series of *three* divisors; hence, their product will give a series of 4×3 or 12 divisors. Hence the product of the exponents of the factors increased by 1, will give the number of divisors.

OPERATION.

$108 = 2^2\times3^3$

$(2+1)\times(3+1)$

$= 12$, *Ans.*

Rule.—*Resolve the number into its prime factors, increase the exponent of each factor by* 1, *and take the product of the results.*

Find the number of divisors

2. Of 72.	*Ans.* 12.	5. Of 840.	*Ans.* 32.
3. Of 360.	*Ans.* 24.	6. Of 2160.	*Ans.* 40.
4. Of 810.	*Ans.* 20.	7. Of 75600.	*Ans.* 120.

CASE VII.

172. ***To find all the divisors common to two or more numbers.***

1. Find the divisors common to 108 and 144.

SOLUTION.—Resolving the numbers into their prime factors, we find that $2^2\times3^2$ is a common factor; hence 1, 2, 4, 3, 9, and the products arising from their combination, as in Case V., are all the common divisors.

OPERATION.

$$\left.\begin{array}{l}108 = 2^2\times3^3\\144 = 2^4\times3^2\end{array}\right\}\therefore 2^2\times3^2 = \text{the common factors.}$$

1-3-9
1-2-4

1-3-9-2-6-18-4-12-36

Rule.—I. *Resolve the numbers into their prime factors.*
II. *Take* 1 *and all the common prime factors, and all the numbers arising from their combination.*

Find the divisors common to

2. 36 and 48. *Ans.* 1, 2, 3, 4, 6, 12.
3. 48, 96, and 120. *Ans.* 1, 2, 3, 4, 6, 8, 12, 24.
4. 480, 720, and 840. *Ans.* 1, 2, 3, 4, 5, 6, 8, etc.
5. 576, 864, 1152, and 1728.
Ans. 1, 2, 3, 4, 6, 8, 9, 12, etc.

THE GREATEST COMMON DIVISOR.

173. A **Divisor** of a number is a number that will exactly divide it; or it is an *exact divisor* of a number.

174. A **Common Divisor** of two or more numbers is a number that exactly divides each of them.

175. The **Greatest Common Divisor** of two or more numbers is the greatest number that exactly divides each of them.

NOTE.—The greatest common divisor may be represented by the initials G. C. D.

PRINCIPLES.

1. *A factor of a number is a divisor of the number, and of any number of times the number.*

2. *A common factor of two or more numbers is a factor of their greatest common divisor.*

3. *The product of all the common prime factors of two or more numbers is their greatest common divisor.*

4. *A common divisor of two numbers is a divisor of their sum and also of their difference.*

DEMONSTRATION.—Take any two numbers, as 18 and 30, of which 6 is a *common divisor.* Now 18 equals *three* times 6, and 30 equals *five* times 6; and their sum is *three* times 6 plus *five* times 6, or *eight* times 6, which contains 6; their difference is *two* times 6, which also contains 6.

CASE I.

176. ***When the numbers are small and can be readily factored.***

177. The **First Method** consists in finding the common factors, and taking their product.

1. Find the greatest common divisor of 66, 132, and 198.

OPERATION.

$$\begin{array}{r|l} 2 & 66-132-198 \\ \hline 3 & 33-\ 66-\ 99 \\ \hline 11 & 11-\ 22-\ 33 \\ \hline & \ 1\ -\ 2\ -\ 3 \end{array}$$

G.C.D. $= 2 \times 3 \times 11 = 66$.

SOLUTION.—We write the numbers one beside another as in the margin. Dividing by 2, we see that 2 is a factor of each number; it is therefore a factor of the G. C. D. (Prin. 2); dividing the quotients by 3, we see that 3 is a factor of each number, and therefore a factor of the G. C. D.; in the same way we see that 11 is a factor of the G. C. D.: now, since the quotients 1, 2, and 3 are prime to each other, 2, 3, and 11 are all the common factors; hence their product, which is 66, is the G. C. D. (Prin. 3.)

CASE VI.

171. ***To find the number of divisors of a number.***

1. How many divisors has 108?

SOLUTION.—It is evident that 1 with the 1st, 2d, and 3d powers of 3 will give a series of *four* divisors, and 1 with the 1st and 2d powers of 2 will give a series of *three* divisors; hence, their product will give a series of 4×3 or 12 divisors. Hence the product of the exponents of the factors increased by 1, will give the number of divisors.	OPERATION. $108=2^2\times3^3$ $(2+1)\times(3+1)$ $=12$, *Ans.*

Rule.—*Resolve the number into its prime factors, increase the exponent of each factor by* 1, *and take the product of the results.*

Find the number of divisors

2. Of 72.	*Ans.* 12.	5. Of 840.	*Ans.* 32.
3. Of 360.	*Ans.* 24.	6. Of 2160.	*Ans.* 40.
4. Of 810.	*Ans.* 20.	7. Of 75600.	*Ans.* 120.

CASE VII.

172. ***To find all the divisors common to two or more numbers.***

1. Find the divisors common to 108 and 144.

SOLUTION.—Resolving the numbers into their prime factors, we find that $2^2\times3^2$ is a common factor; hence 1, 2, 4, 3, 9, and the products arising from their combination, as in Case V., are all the common divisors.	OPERATION. $\left.\begin{array}{l}108=2^2\times3^3\\144=2^4\times3^2\end{array}\right\}\therefore 2^2\times3^2=$ the common factors. 1-3-9 1-2-4 —— 1-3-9-2-6-18-4-12-36

Rule.—I. *Resolve the numbers into their prime factors.*

II. *Take* 1 *and all the common prime factors, and all the numbers arising from their combination.*

Find the divisors common to

2. 36 and 48. *Ans.* 1, 2, 3, 4, 6, 12.
3. 48, 96, and 120. *Ans.* 1, 2, 3, 4, 6, 8, 12, 24.
4. 480, 720, and 840. *Ans.* 1, 2, 3, 4, 5, 6, 8, etc.
5. 576, 864, 1152, and 1728.
Ans. 1, 2, 3, 4, 6, 8, 9, 12, etc.

THE GREATEST COMMON DIVISOR.

173. A **Divisor** of a number is a number that will exactly divide it; or it is an *exact divisor* of a number.

174. A **Common Divisor** of two or more numbers is a number that exactly divides each of them.

175. The **Greatest Common Divisor** of two or more numbers is the greatest number that exactly divides each of them.

NOTE.—The greatest common divisor may be represented by the initials G. C. D.

PRINCIPLES.

1. *A factor of a number is a divisor of the number, and of any number of times the number.*

2. *A common factor of two or more numbers is a factor of their greatest common divisor.*

3. *The product of all the common prime factors of two or more numbers is their greatest common divisor.*

4. *A common divisor of two numbers is a divisor of their sum and also of their difference.*

DEMONSTRATION.—Take any two numbers, as 18 and 30, of which 6 is a *common divisor*. Now 18 equals *three* times 6, and 30 equals *five* times 6; and their sum is *three* times 6 plus *five* times 6, or *eight* times 6, which contains 6; their difference is *two* times 6, which also contains 6.

CASE I.

176. ***When the numbers are small and can be readily factored.***

177. The **First Method** consists in finding the common factors, and taking their product.

1. Find the greatest common divisor of 66, 132, and 198.

OPERATION.

$$\begin{array}{r|l} 2 & 66-132-198 \\ \hline 3 & 33-\ 66-\ 99 \\ \hline 11 & 11-\ 22-\ 33 \\ \hline & \ \ 1\ -\ 2\ -\ 3 \end{array}$$

$$G.C.D. = 2 \times 3 \times 11 = 66.$$

SOLUTION.—We write the numbers one beside another as in the margin. Dividing by 2, we see that 2 is a factor of each number; it is therefore a factor of the G. C. D. (Prin. 2); dividing the quotients by 3, we see that 3 is a factor of each number, and therefore a factor of the G. C. D.; in the same way we see that 11 is a factor of the G. C. D.: now, since the quotients 1, 2, and 3 are prime to each other, 2, 3, and 11 are all the common factors; hence their product, which is 66, is the G. C. D. (Prin. 3.)

Rule.—I. *Write the numbers one beside another, with a vertical line at the left, and divide by any common factor of all the numbers.*

II. *Divide the quotients in the same manner, and thus continue until the quotients have no common factor.*

III. *Take the product of all the divisors; the result will be the greatest common divisor.*

Find the greatest common divisor of

2. 270, 315, and 405. *Ans.* 45.
3. 336, 384, and 528. *Ans.* 48.
4. 504, 616, and 728. *Ans.* 56.
5. 392, 448, and 504. *Ans.* 56.
6. 864, 1008, and 1296. *Ans.* 144.
7. 756, 1008, and 1260. *Ans.* 252.
8. 768, 1152, 1536, and 1920. *Ans.* 384.
9. 3168, 3456, 3744, and 4032. *Ans.* 288.

178. The **Second Method** consists in resolving the numbers into their prime factors, and taking the product of the common factors.

1. Find the greatest common divisor of 66, 132, and 198.

SOLUTION.—Resolving the numbers into their prime factors, we find 2, 3, and 11 are all the factors common to the three numbers; hence their product, which is 66, is the greatest common divisor of these numbers (Prin. 3).

OPERATION.

$$66 = 2 \times 3 \times 11$$
$$132 = 2 \times 2 \times 3 \times 11$$
$$198 = 2 \times 3 \times 3 \times 11$$
$$\text{G. C. D.} = 2 \times 3 \times 11 = 66$$

Rule.—*Resolve the numbers into their prime factors, and take the product of all the common factors.*

Find the greatest common divisor of

2. 120, 210, and 360. *Ans.* 30.
3. 252, 336, and 420. *Ans.* 84.
4. 330, 495, and 660. *Ans.* 165.
5. 204, 306, and 510. *Ans.* 102.
6. 840, 1260, and 1890. *Ans.* 210.
7. 336, 504, 840, and 1008. *Ans.* 168.
8. 364, 728, 910, and 2002. *Ans.* 182.
9. 560, 1008, 3136, and 16016. *Ans.* 112.

CASE II.

179. ***When the numbers are large and cannot be readily factored.***

1. Find the greatest common divisor of 234 and 286.

OPERATION.

```
234)286(1
    234
    ---
     52)234(4
        208
        ---
         26)52(2
            52
            --
```

SOLUTION.—We divide 286 by 234, the divisor 234 by the remainder 52, the divisor 52 by the remainder 26, and have no remainder; then will 26 be the greatest common divisor of 234 and 286. To show this we will prove, 1st, that *the last divisor is a number of times the G. C. D.*, and 2d, that it is *once the G. C. D.*

1st. *The last remainder is a* NUMBER OF TIMES *the G. C. D.* For, since 234 and 286 are each a number of times the G. C. D., their difference 52 is also a number of times the G. C. D. (Prin. 4); and since 234 and 52×4, or 208, are each a number of times the G. C. D., their difference 26 is also a number of times the G. C. D. Hence the last divisor, 26, is a *number of times* the G. C. D.

2d. *The last divisor*, 26, *is* ONCE *the G. C. D.* For, since 26 divides 52, it will divide 52×4, or 208 (Prin. 1), and will also divide $208+26$, or 234 (Prin. 4); and since it divides 52 and 234, it will divide $52+234$, or 286; and now, since 26 divides 234 and 286, and is a *number of times the G. C. D.*, 26 must be *once* the G. C. D.

NOTE.—In the latter part of the solution it may be shown that since the G. C. D. can be neither *greater* nor *less* than 26, it must be 26.

OPERATION.

234	286	1
	234	
208	52	4
26	52	2
	0	

NEW FORM.—In the margin on the right is a more concise form of writing the division. Pupils should be required to adopt it after they become familiar with the common form.

Rule.—I. ***Divide the greater number by the less, the divisor by the remainder, and thus continue to divide the last divisor by the last remainder until there is no remainder; the last divisor will be the greatest common divisor.***

II. ***If there are more than two numbers, find the greatest common divisor of two of them, then of that divisor and one of the other numbers, etc.; the last greatest common divisor will be the divisor required.***

NOTES.—1. In finding the greatest common divisor of more than two numbers, begin with the smallest two.

2. A factor of any number not found in the others, may be rejected, since it is not a factor of the G. C. D.

3. An obvious common factor can be set aside as forming a factor of the G. C. D.; the G. C. D. of the other factors multiplied by this common factor will give the G. C. D. required.

Find the greatest common divisor of

2. 230 and 322. *Ans.* 46.

3. 1829 and 2419. *Ans.* 59.

4. 3139 and 4307. *Ans.* 73.

5. 4183 and 6497. *Ans.* 89.

6. 256, 480, and 1296. *Ans.* 16.

7. 2041, 8476, and 9477. *Ans.* 13.

8. 292, 1095, and 2044. *Ans.* 73.

9. 7011, 11193, and 16113. *Ans.* 123.

10. 217473 and 309363. *Ans.* 3063.

11. 1389548 and 2247404. *Ans.* 4468.

PRACTICAL PROBLEMS.

1. John Smith has a four-sided field whose sides are 256, 292, 384, and 400 feet respectively; what is the length of the rails used to fence it, if they are all of equal length and the longest that can be used? *Ans.* 4 ft.

2. A farmer wishes to put 364 bushels of oats, 455 bushels of corn and 637 bushels of wheat into bins of uniform size without mixing the different grains; what is the capacity of the largest bin that may be used, all being filled, and how many are required? *Ans.* 91 bushels; 16 bins.

3. Four drovers, A, B, C, and D, have \$584, \$657, \$803, and \$876 respectively; now suppose they should pay such a price per head for cattle as would exactly use each man's money, what is the highest price per head they could give, and how many would each buy? *Ans.* 8; 9; 11; 12.

4. A gentleman has three large tracts of land, containing 870 acres, 1479 acres, and 1740 acres, which he wishes to divide into small farms of equal size; what is the largest number of acres which can be given to each farm, and how many farms will there be? *Ans.* 87 acres; 47 farms.

5. There is a triangular field whose sides are 288, 450, and 390 ft. respectively; how many rails will it require to fence it, if the fence is 5 rails high, and what must be the length of the rails if they lap over 1 foot?

Ans. Length of rail, 7 ft.; number, 940.

ABBREVIATED METHOD.

180. The **Abbreviated Method** of Greatest Common Divisor is both interesting and practical. It is a logical outgrowth of the method of explanation we have given.

1. Find the greatest common divisor of 32 and 116.

Solution.—We take 4 times 32, which is 128, and subtract 116 from it, since we will thus obtain a smaller remainder and hence be nearer *once* the G. C. D. than if we subtract 3 times 32 from 116. We then take 3 times 12, or 36, and subtract 32 from it, since we will thus obtain a smaller remainder than if we subtract 2 times 12 from 32, and hence be nearer *once* the G. C. D., etc.

ABBREVIATED METHOD.

32	116	4
	128	
36	12	3
4	12	3
	0	

COMMON METHOD.

32	116	3
	96	
20	20	1
12	12	1
8	8	1
4	8	2
	0	

Notes.—1. In the margin is a solution of the same problem by the common method, showing that in this problem we save two divisions by the abbreviated method.

2. The method is applicable wherever it will give a smaller remainder than the ordinary method, and this can readily be determined by inspection.

Rule.—*Proceed as by the ordinary method, finding the difference between the dividend and such a multiple of the divisor as will give the smallest remainder.*

Find the greatest common divisor of

2. 693 and 4004. *Ans.* 77.
3. 615 and 3000. *Ans.* 15.
4. 195649 and 330479. *Ans.* 97.
5. 1015439 and 1994507. *Ans.* 983.
6. 1816667 and 3411167. *Ans.* 1063.

LEAST COMMON MULTIPLE.

181. A **Multiple** of a number is one or more times the number; thus, 4 times 5, or 20, is a multiple of 5.

182. A **Common Multiple** of two or more numbers is a number which is a multiple of each of them; thus, 36 is a common multiple of 3, 6, and 9.

183. The **Least Common Multiple** of two or more numbers is the least number which is a multiple of each of them; *thus, 18 is* the least common multiple of 3, 6, and 9

NOTE.—We may also define thus: a multiple of a number is another number which will exactly contain the former; or, a multiple of a number is a number of which the former is an exact divisor, etc. The least common multiple may be represented by the initials L. C. M.

PRINCIPLES.

1. *A multiple of a number is exactly divisible by that number.*

2. *A multiple of a number must contain all the prime factors of that number.*

3. *A common multiple of two or more numbers must contain all the prime factors of each of those numbers.*

4. *The least common multiple of two or more numbers must contain all the prime factors of each number, and no other factors.*

CASE I.

184. ***When the numbers are small and can be readily factored.***

185. The **First Method** consists in resolving the numbers into their prime factors, and taking the product of all the different factors.

1. Find the least common multiple of 20, 30, and 70.

OPERATION.

$$20 = 2\times2\times5$$
$$30 = 2\times3\times5$$
$$70 = 2\times5\times7$$
$$2\times2\times5\times3\times7 = 420.$$

SOLUTION.—We first resolve the numbers into their prime factors. A multiple of 20 must contain the factors of 20, or 2, 2, 5; a multiple of 30 must contain the factors of 30, or 2, 3, 5; a multiple of 70 must contain the factors of 70, or 2, 5, 7; hence the least common multiple of 20, 30, and 70 must contain all these different factors and no others; therefore $2\times2\times5\times3\times7$, or 420, is the L. C. M. of 20, 30, and 70 (Prin. 4).

Rule.—I. *Resolve the numbers into their prime factors.*

II. *Take the product of all the different factors, using each factor the greatest number of times it occurs in either number.*

NOTE.—Any numbers which are divisors of the others may be omitted, since the multiple of the other numbers will be a multiple of these.

Find the least common multiple of

2. 25, 30, and 42. *Ans.* 1050.

3. 11, 32, and 40. *Ans.* 1760.

4. 56, 72, and 96. *Ans.* 2016.

5. 72, 84, and 108.	*Ans.* 1512.
6. 30, 60, 84, and 144.	*Ans.* 5040.
7. 33, 55, 66, 77, and 140.	*Ans.* 4620.
8. 21, 56, 63, 114, and 171.	*Ans.* 9576.
9. 36, 57, 60, 231, and 330.	*Ans.* 263340.

186. The Second Method consists in taking out the prime factors of the least common multiple, and finding their product.

1. Find the least common multiple of 24, 30, and 70.

OPERATION.

$$\begin{array}{r|l} 2 & 24\text{-}30\text{-}70 \\ \hline 3 & 12\text{-}15\text{-}35 \\ \hline 5 & 4\text{-}5\text{-}35 \\ \hline & 4\text{-}1\text{-}7 \end{array}$$

$$2 \times 3 \times 5 \times 4 \times 7 = 840$$

SOLUTION.—Placing the numbers one beside another, and dividing by 2, we find that 2 is a factor of each of them; it is therefore a factor of the L. C. M. (Prin. 4); dividing the quotients by 3, we find that 3 is a factor of some of the numbers; it is therefore a factor of the least common multiple (Prin. 4); dividing the next quotients by 5, we find that 5 is a factor of some of the numbers; it is therefore a factor of the L. C. M.; and the quotients having no other common factors, we see that all the different factors of the given numbers are 2, 3, 5, 4, and 7; hence their product, which is 840, is the L. C. M. required. Hence the following

Rule.—I. *Write the numbers one beside another, divide by any prime number that will exactly divide two or more, and write the quotients and undivided numbers beneath.*

II. *Divide the quotients in the same manner, and thus continue until no two numbers in the lowest line have a common factor.*

III. *Take the product of the divisors and final quotients; the result will be the least common multiple required.*

Find the least common multiple of

2. 12, 18, 24, and 27.	*Ans.* 216.
3. 22, 33, 55, and 66.	*Ans.* 330.
4. 14, 19, 38, and 57.	*Ans.* 798.
5. 64, 84, 120, and 216.	*Ans.* 60480
6. 1, 2, 3, 4, 5, 6, 7, and 8.	*Ans.* 840.
7. 18, 36, 126, 40, and 48.	*Ans.* 5040.
8. 13, 37, 7, 91, and 11.	*Ans.* 37037.
9. 96, 126, 180, and 252.	*Ans.* 10080.
10. 15, 25, 45, 75, 135, and 209	*Ans.* 141075.

CASE II.

187. ***When the numbers are large and cannot be readily factored.***

1. Find the least common multiple of 45 and 72.

SOLUTION.—The greatest common divisor of these numbers is 9; 45 equals 5 times 9 and 72 equals 8 times 9; hence the L. C. M., as found in the first method, is $5 \times 9 \times 8$, which equals $45 \times \frac{72}{9}$; and which we see is *the first number multiplied by the second divided by their greatest common divisor*. From the result of this operation we may derive the following rule:	OPERATION. $45 = 5 \times 9$ $72 = 8 \times 9$ L. C. M. $= 5 \times 9 \times 8$ $= 45 \times \frac{72}{9}$

Rule.—I. *Find the greatest common divisor of two numbers, divide one number by it, and multiply the other number by the quotient.*

II. *When there are more than two numbers, find the least common multiple of two of the numbers, and then of this number and the third number, etc.*

Find the least common multiple of

2. 1110 and 777. *Ans.* 7770.
3. 4087 and 4757. *Ans.* 290177.
4. 9797 and 10403. *Ans.* 1009091.
5. 9523 and 11663. *Ans.* 1038007.
6. 29606 and 35894. *Ans.* 4056022.
7. 94106 and 202484. *Ans.* 42724124.
8. 9144407 and 10347059. *Ans.* 31154994649.
9. 4343, 6363, and 7373. *Ans.* 19973457.
10. 56056, 99099, 777777. *Ans.* 205333128.

PRACTICAL PROBLEMS.

1. What is the least sum of money with which I could purchase either pigs at \$5, sheep at \$7, cows at \$40, or horses at \$75? *Ans.* \$4200.

2. A can dig 5 rods of ditch in a day, B 8 rods, C 12 rods, and D 15 rods; how many rods will it require to give an exact number of days' work for each? *Ans.* 120 rods.

3. The circumferences of the driving wheels of four locomotives are 12, 18, 20, and 21 feet respectively; what is the shortest distance in which each wheel can make an exact number of revolutions? *Ans.* 1260 ft.

4. A, B, C, and D start from the same point; A goes a mile in 15 minutes, B in 18 minutes, C in 21 minutes, and D in 25 minutes; how far can each travel, and all return to the starting point at the same time? *Ans.* A, 210; etc.

5. There is an island 120 miles in circumference, and A, B, C, and D start to travel around it; A goes 12 miles a day, B 15, C 20, and D 24; in what time would they all come together at the starting point? *Ans.* 120 days.

6. The circumferences of the wheels of a carriage are 13 and 16 feet respectively; a nail on the tire of each was on top when the carriage started; how far will the carriage have gone when the nails shall be uppermost for the 450th time, there being 5280 feet in a mile? *Ans.* 17 mi. 3840 ft.

ABBREVIATED METHOD.

188. The **Abbreviated Method** of Least Common Multiple here given will be found useful in practice.

1. Find the least common multiple of 45, 48, 80, and 120

OPERATION.

45-48-80-|120

3-2-|2

3-1

SOLUTION.—Having written the numbers in a line as in the last method, we cut off 120, the right hand number. Now, the least common multiple must consist of 120, multiplied by those factors of the other numbers which are not found in 120; dividing 45 by 15, the greatest divisor common to it and 120, we obtain 3; dividing 48 by 24, the greatest divisor common to it and 120, we have 2; dividing 80 in the same way, we have 2. As the factors thus obtained are not prime to each other, we cut off 2, and divide by the greatest divisors common to it and 3 and 2 respectively, when we have 3 and 2 as the only factors of the L. C. M. not contained in 120. Multiplying 120 by these factors, we have 720 for the L. C. M. Hence the following

Rule.—I. *Write the numbers one beside another, cut off any convenient number, generally the largest, and draw a line beneath them.*

II. *Divide each remaining number by the greatest divisor common to it and the number cut off. If the factors thus obtained are not prime to each other, cut off one of them and proceed as before until all the factors are prime to each other.*

III. *Multiply the numbers cut off and the last row of factors together for the least common multiple*

NOTES.—1. If one number is contained in any of the others, it may be omitted.

2. If the number cut off is found to be prime to the others in the same line, cut off another and proceed as before, reserving the first as a factor of the L. C. M.

Find the least common multiple of

2. 30, 80, 120, and 135. *Ans.* 2160.

3. 77, 91, 143, and 165. *Ans.* 15015.

4. 93, 132, 232, and 319. *Ans.* 237336.

GENERAL PRINCIPLES

OF GREATEST COMMON DIVISOR AND LEAST COMMON MULTIPLES.

189. These **General Principles** express the relations between the greatest common divisor and the least common multiple.

1. *The greatest common divisor of two or more numbers is a divisor of their least common multiple.*

2. *The product of two numbers divided by their greatest common divisor equals their least common multiple.*

3. *The product of the relatively prime parts of two or more numbers multiplied by the G. C. D. equals the L. C. M.*

4. *The quotient of the L. C. M. of two or more numbers, divided by their G. C. D., equals the product of the factors not common.*

5. *The prime factors not common may be found by resolving the quotient of the L. C. M. divided by the G. C. D. into its prime factors.*

6. *The G. C. D., multiplied by each of the factors not common, will give numbers having the same G. C. D. and L. C. M.*

NOTE.—The pupil may be required to illustrate these principles.

PRACTICAL PROBLEMS.

1. The L. C. M. of 6 and 8 and a number prime to each of them is 120; what is the third number?

SOLUTION.—120 contains all the factors of 6, of 8, and of the 3d number; hence all the factors of 120 not found in 6 and 8 constitute the third number. The only factor is 5, therefore 5 is the number required.

OPERATION.

$$120 = 2^3 \times 3 \times 5$$
$$6 = 2 \times 3$$
$$8 = 2 \times 2 \times 2$$
$$\therefore 5 = \text{the number}$$

2. The L. C. M. of 8, 12, and 45, and another number prime to each, is 2520; required the number. *Ans.* 7.

3. The G. C. D. of two numbers is 5, and their L. C. M. is 30; what are the numbers? *Ans.* 10, 15, or 5 and 30.

4. The L. C. M. of 6, 9, 10, and a fourth number, is 630; what is the smallest number that it may be? *Ans.* 7.

5. The G. C. D. of two numbers is 12, and their L. C. M is 72; required the numbers. *Ans.* 24, 36.

6. The G. C. D. of three numbers of two factors each is 7, and their L. C. M. is 210; required the numbers.
Ans. 14, 21, 35.

7. What two numbers between 13 and 78 have the latter for their L. C. M. and the former for their G. C. D.?
Ans. 26, 39.

8. What three numbers of two factors each between 17 and 510 have the former for their G. C. D. and the latter for their L. C. M.? *Ans.* 34, 51, 85.

9. Find a number between 209 and 247 which has with each of them the same G. C. D. that they have with each other. *Ans.* 228.

10. Find 3 numbers between 161 and 1265 which have the same L. C. M. as these numbers. *Ans.* 253, 385, 805.

11. Find 3 numbers between 119 and 187 which have the same G. C. D. as these numbers. *Ans.* 136, 153, 170.

12. Required three numbers between 119 and 374 which have with these numbers the same L. C. M. as the numbers themselves. *Ans.* 154, 187, 238.

13. The G. C. D. of four composite numbers of two factors each is 11, and their L. C. M. is 2310; what are the numbers? *Ans.* 22, 33, 55, 77

14. Required all the numbers whose G. C. D. is 45 and L. C M. is 4680.
Ans. 45, 90, 180, 360, 585, 1170, 2340, 4680

CANCELLATION.

190. Cancellation is the process of abbreviating arithmetical operations by rejecting equal factors from both dividend and divisor.

191. The **Symbol** of Cancellation is an oblique line drawn across a figure; as $\cancel{4}$, $\cancel{5}$, $\cancel{6}$, etc.

PRINCIPLES.

1. *The cancelling of a factor from any number divides the number by that factor.*

2. *The cancelling of a factor from both dividend and divisor will not change the quotient.*

1. Divide $21\times24\times75$ by 14×36.

OPERATION.

$$\frac{\overset{3}{\cancel{21}}\times\overset{\cancel{2}}{\cancel{24}}\times\overset{25}{\cancel{75}}}{\underset{\cancel{2}}{\cancel{14}}\times\underset{\cancel{3}}{\cancel{36}}}=75$$

Solution.—We cancel the common factor 12 from 24 and 36, writing 2, the other factor of 24, above 24, and 3, the other factor of 36, below 36; then cancel the common factor 7 from 21 and 14, writing 3, the other factor of 21, above 21, and 2, the other factor of 14, below 14; the common factor 2 is then cancelled, and since 3 is contained in 75, we cancel the 3 and 75, writing 25 above 75. Multiplying together 3 and 25, the remaining factors, we have 75.

Rule.—I. *Cancel the common factors from the dividend and divisor.*

II. *Then divide the product of the remaining factors of the dividend by the product of the remaining factors of the divisor.*

Notes.—1. The unit 1 takes the place of a cancelled factor, but need not be written, except in the dividend of the quotient, when there are no other factors of the dividend.

2. A factor in one term will cancel two or more factors in the other term, when their product is equal to the former.

2. Divide $24\times9\times10$ by $6\times4\times5$. *Ans.* 18.

3. Divide $42\times18\times60\times4$ by $7\times24\times8\times2$. *Ans.* $67\frac{1}{2}$.

4. Divide $5\times16\times81\times63$ by $8\times7\times9\times45$. *Ans.* 18.

5. Divide $100\times33\times250$ by 125×150. *Ans.* 44.

6. Divide $225\times65\times320$ by $26\times150\times16$. *Ans.* 75.

7. Divide $16\times40\times60\times28$ by $80\times24\times7$. *Ans.* 80.

8. Divide $231\times95\times384\times150$ by $24\times38\times21\times112$. *Ans.* $589\frac{2}{7}$.

9. Divide $432 \times 529 \times 441$ by $27 \times 23 \times 7 \times 9$.

Ans. 2576.

10. Divide $9801 \times 2025 \times 2401$ by $891 \times 45 \times 77$.

Ans. 15435.

PRACTICAL PROBLEMS.

1. How many yards of alpaca, at 48 cents a yard, can be obtained for 36 bushels of corn at 84 cents a bushel?

SOLUTION.—If one bushel of corn is worth 84 cents, 36 bushels are worth 36×84 cents; for 36×84 cents at 48 cents a yard, we can get as many yards of alpaca as 48 is contained times in 36×84, which we find by cancellation to be 63.

OPERATION.

$$\frac{\overset{3}{\cancel{36}} \times \overset{21}{\cancel{84}}}{\underset{\cancel{4}}{\cancel{48}}} = 63 \text{ Ans.}$$

2. How many barrels of pork, at $16 a barrel, can be obtained for 64 tons of hay, at $23 a ton? *Ans.* 92.

3. A merchant sold 18 hhd. of molasses, each containing 75 gal., at 64 cents a gal., and received in payment a number of chests of tea, each containing 24 pounds, at 90 cents a pound; how many chests were there? *Ans.* 40.

4. Multiply 45 by 6 times 25 and divide by 91; multiply the quotient by 13 times 63 and divide by 81; multiply this result by 12 times 19 and divide by 6 times 95. *Ans.* 300.

5. A dealer exchanged Minnesota extra flour, at $9.50 per barrel, for 19 cases of children's shoes, each containing 60 pairs, at $1.25 a pair; how many barrels of flour were exchanged? *Ans.* 150 barrels.

6. A commission merchant sold 21 bales of "middling upland" cotton, each containing 400 pounds, at 16 cents a pound, and received in payment 16 hogsheads of molasses, containing 120 gallons each; what was the cost of the molasses per gallon? *Ans.* 70 cents.

7. A grocer bought 7 chests of souchong tea, containing 24 pounds each, at $1.05 per pound; how many firkins of butter, at 35 cents a pound, will be required to pay for it, each firkin containing 56 pounds? *Ans.* 9 firkins.

SECTION IV.

COMMON FRACTIONS.

192. A **Fraction** is a number of the equal parts of a *unit;* as 3 *fourths.*

193. A **Fractional Unit** is *one* of the *equal parts* of the Unit. A *Fraction* is a number of *fractional units.*

194. **Similar Fractional Units** are those which are alike; as 2 *fourths,* 3 *fourths.*

195. **Dissimilar Fractional Units** are those which are unlike; as 3 *fourths,* 4 *fifths.*

196. **Fractions** are divided into two classes; *common fractions* and *decimal fractions.*

197. A **Common Fraction** is one in which the unit is divided into *any number* of equal parts.

198. A **Decimal Fraction** is a number of the decimal divisions of the unit.

NOTES.—1. Units are distinguished as *Integral units* and *Fractional units.* The word *Unit,* without any qualifying word, means the *Integral unit.* When the term *fraction* is used without any qualifying word, the *common fraction* is meant.

2. A fraction implies three things: 1st, a *thing* to be divided; 2d, equal *parts* of the thing; and 3d, the *number of parts* taken—that is, the *integral unit,* the *fractional unit* and its *relation* to the *integral* unit, and the *number* of fractional units taken.

3. The primary conception of a fraction is that it is *a number of equal parts of a unit.* It may, however, be regarded as *a number of equal parts* of *one* thing, or *one equal part* of a *number of things.* Thus, *four fifths* may be regarded as *four-fifths* of *one* or *one-fifth* of *four.*

199. A **Common Fraction** is expressed by two numbers, one written above the other, with a line between them. Thus, $\frac{4}{5}$ expresses 4 fifths.

200. The **Denominator** denotes the number of equal parts into which the unit is divided; it is written below the line.

201. The **Numerator** denotes the number of equal parts which are taken; it is written above the line.

202. The **Terms** of a fraction, called respectively the *Numerator* and the *Denominator*, are the two numbers by which it is expressed.

CLASSES OF COMMON FRACTIONS.

203. Common Fractions consist of three principal classes; namely, *Simple, Compound,* and *Complex.*

204. A **Simple Fraction** is a fraction having a single integral numerator and denominator; as, $\frac{2}{3}$, $\frac{4}{5}$.

205. A **Proper Fraction** is a simple fraction whose value is less than a unit; as, $\frac{2}{3}$, $\frac{3}{4}$.

206. An **Improper Fraction** is a simple fraction whose value is equal to or greater than a unit; as, $\frac{5}{5}$, $\frac{7}{6}$, $\frac{12}{8}$, etc.

207. A **Compound Fraction** is a fraction of a fraction; as, $\frac{2}{3}$ of $\frac{3}{4}$, $\frac{4}{5}$ of $\frac{5}{6}$ of $\frac{7}{8}$, etc.

208. A **Complex Fraction** is one whose numerator or denominator, or both, are fractional; as, $\frac{4}{\frac{2}{3}}$, $\frac{\frac{3}{4}}{5}$, $\frac{\frac{2}{3} \text{ of } \frac{3}{4}}{\frac{4}{5} \text{ of } \frac{5}{6}}$.

209. A **Mixed Number** consists of an integer and a fraction; as, $2\frac{1}{3}$, $7\frac{3}{5}$, etc.

210. An **Integer** may be expressed fractionally by writing 1 under it as a denominator; as, $6=\frac{6}{1}$.

211. The **Reciprocal** of a quantity is a unit divided by that quantity; thus the reciprocal of 5 is $\frac{1}{5}$.

NOTES.—1. A fraction means primarily a *part*, hence only a *proper fraction* is properly a fraction. The *improper* fraction is not properly a fraction according to the primary signification of the term.

2. The *complex fraction* is not properly a fraction, according to the definition of a fraction, or the functions ascribed to the terms. Thus, if the denominator is $\frac{2}{3}$ it indicates that the unit is divided into $\frac{2}{3}$ equal parts, which is impossible.

PRINCIPLES OF THE TERMS.

212. The **Principles** of the terms state the use and relation of the terms of a common fraction.

1. *The numerator expresses the number of fractional units taken.*

2. *The denominator expresses*

   a. The number of equal parts into which the unit is divided.

b. The number of fractional units which equal the unit

c. The kind and denomination of the fractional units.

3. *When the numerator of a fraction is equal to the denominator, the fraction is equal to* 1.

4. *When the numerator of a fraction is less than the denominator, the fraction is less than* 1.

5. *When the numerator of a fraction is greater than the denominator, the fraction is greater than* 1.

NUMERATION AND NOTATION.

213. Numeration of Fractions is the art of reading a fraction when expressed by figures.

Rule.—*Read the number of fractional units expressed by the numerator, and give them the name indicated by the denominator.*

Read the following fractions:

1. $\frac{5}{6}$, $\frac{7}{8}$.	3. $\frac{11}{12}$, $\frac{14}{15}$.	5. $\frac{19}{23}$, $\frac{125}{347}$.
2. $\frac{8}{9}$, $\frac{9}{10}$.	4. $\frac{17}{21}$, $\frac{13}{22}$.	6. $\frac{123}{500}$, $\frac{6795}{21341}$.

214. Notation of Fractions is the art of expressing fractions by means of figures.

Rule.—*Write the number of fractional units, draw a line beneath, under which write the number which indicates the kind of fractional units.*

Write the following fractions:

1. Two-thirds.	5. Nineteen twenty-seconds.
2. Three-fourths.	6. Fifty-three hundredths.
3. Ten-twelfths.	7. Fifty three-thousandths.
4. Five twenty-firsts.	8. Forty-five ten-millionths.

ANALYSIS OF FRACTIONS.

215. To **Analyze** a fraction is to explain what is expressed by the fractional notation.

1. Analyze the fraction $\frac{4}{5}$.

SOLUTION.—In the fraction $\frac{4}{5}$, the denominator 5 indicates that the unit is divided into 5 equal parts, and the numerator 4 denotes that 4 of these parts are taken.

Analyze the following fractions:

2. $\frac{5}{6}$.	4. $\frac{11}{12}$.	6. $\frac{17}{21}$
3. $\frac{8}{9}$.	5. $\frac{15}{16}$.	7. $\frac{25}{37}$.

CASES OF FRACTIONS.

216. The **Number of Cases** of common fractions is eight They are as follows:

1. Reduction.
2. Addition.
3. Subtraction.
4. Multiplication.
5. Division.
6. Relation of Fractions.
7. Greatest Common Divisor.
8. Least Common Multiple.

METHODS OF TREATMENT.

217. There are **Two Methods** of treating common fractions, which may be distinguished as the *Inductive* and *Deductive Methods.*

218. By the **Inductive Method** we solve all the different cases by analysis, and derive the *rules* or *methods of operation* from these analyses by *inference* or *induction.*

219. By the **Deductive Method** we first establish a few general principles, and then derive the *rules* or *methods of operation* from these general principles.

NOTE.—These two methods are entirely distinct in principle and form, and demand attention. We have given both methods, and teachers may use either or both, as they choose. The first solution given under each case is by the inductive method, the second solution is by the deductive method.

PRINCIPLES OF FRACTIONS.

1. *Multiplying the numerator of a fraction by any number multiplies the value of the fraction by that number.*

If we multiply the numerator of any fraction by any number, as 5, the resulting fraction will express 5 times as many fractional units, each of the same size as before, hence the value of the fraction is 5 times as great. Therefore, etc.

2. *Dividing the numerator of a fraction by any number divides the value of the fraction by that number.*

If we divide the numerator of a fraction by any number, as 4, the resulting fraction will express $\frac{1}{4}$ as many fractional units, each of the same size as before, hence the value of the fraction is divided by 4.

3. *Multiplying the denominator of a fraction by any number divides the value of the fraction by that number.*

Since the denominator denotes the number of equal parts into which the unit is divided, if we multiply the denominator of a fraction by any number, as 5, the unit will be divided into 5 times as many equal parts, hence each fractional unit will be $\frac{1}{5}$ as large as before, and the same number of fractional units being taken, the value of the fraction is $\frac{1}{5}$ as great.

4. *Dividing the denominator of a fraction by any number multiplies the value of the fraction by that number.*

Since the denominator denotes the number of equal parts into which the unit is divided, if we divide it by any number, as 6, the unit will be divided into $\frac{1}{6}$ as many equal parts, hence each fractional unit will be 6 times as large as before, and the same number of fractional units being taken, the value of the fraction will be 6 times as great.

5. *Multiplying both numerator and denominator of a fraction by the same number does not change its value.*

Since multiplying the numerator multiplies the value of the fraction, and multiplying the denominator divides the value of the fraction, multiplying both numerator and denominator both multiplies and divides the value of the fraction by the same number, and hence does not change its value.

6. *Dividing both numerator and denominator of a fraction by the same number does not change its value.*

Since dividing the numerator divides the value of the fraction, and dividing the denominator multiplies the value, dividing both numerator and denominator both divides and multiplies the value of the fraction, and hence does not change its value.

7. *A fraction is equal to the quotient of its numerator divided by its denominator.*

For the fraction $\frac{4}{5}$ is the same as 4 times $\frac{1}{5}$; but 4 times $\frac{1}{5}$ is equal to $\frac{1}{5}$ of 4; and $\frac{1}{5}$ of 4 is equal to 4 divided into 5 equal parts; and to divide a number into 5 equal parts we must divide it by 5; hence $\frac{4}{5}$ is equal to 4 divided by 5; and since this is general, therefore the principle is correct.

NOTE.—Authors usually assume this principle as true, but it is clear that it is not an immediate inference from the explanation that the denominator denotes the number of equal parts into which the unit is divided, and the numerator expresses the number of equal parts taken.

220. These principles may be embodied in one general law as follows:

General Principle.—*A change in the* NUMERATOR *by multiplication or division produces a* SIMILAR *change in the value of the fraction, but such a change in the* DENOMINATOR *produces an* OPPOSITE *change in the value of the fraction.*

REDUCTION OF FRACTIONS.

221. The **Reduction of Fractions** is the process of changing their form without altering their value.

222. There are **Six Cases** of reduction:

1st. Numbers to fractions.	4th. To lower terms.
2d. Fractions to numbers.	5th. Compound to simple.
3d. To higher terms.	6th. Complex to simple.

NOTE.—Reducing to a *Common Denominator* and a *Common Numerator* are included in these six cases.

CASE I.

223. *To reduce whole or mixed numbers to improper fractions.*

1. How many fifths in $7\frac{3}{5}$?

SOLUTION.—In *one* there are 5 *fifths*, and in 7 there are 7 times 5 fifths, or 35 fifths, which added to 3 fifths equals 38 fifths. Therefore $7\frac{3}{5}=\frac{38}{5}$.

OPERATION.

$$\begin{array}{r} 7\frac{3}{5} \\ 5 \\ \hline \frac{38}{5} \end{array}$$

SOLUTION 2D.—7 equals $\frac{7}{1}$, and multiplying both terms by 5, we have $\frac{7}{1}=\frac{35}{5}$ (Prin. 5); and $\frac{35}{5}+\frac{3}{5}=\frac{38}{5}$. Therefore $7\frac{3}{5}=\frac{38}{5}$.

OPERATION.

$$7=\frac{7}{1}=\frac{35}{5}$$
$$\frac{35}{5}+\frac{3}{5}=\frac{38}{5}.$$

Rule.—*Multiply the whole number by the denominator of the fraction, add the numerator to the product, and write the denominator under the sum.*

Reduce the following to improper fractions:

2. $12\frac{4}{5}$.	*Ans.* $\frac{64}{5}$.	8. $126\frac{5}{11}$.	*Ans.* $\frac{1391}{11}$
3. $23\frac{5}{6}$.	*Ans.* $\frac{143}{6}$.	9. $365\frac{12}{13}$.	*Ans.* $\frac{4757}{13}$.
4. $27\frac{4}{7}$.	*Ans.* $\frac{193}{7}$.	10. $483\frac{11}{42}$.	*Ans.* $\frac{20297}{42}$.
5. $46\frac{5}{8}$.	*Ans.* $\frac{373}{8}$.	11. $763\frac{29}{54}$.	*Ans.* $\frac{41231}{54}$.
6. $96\frac{7}{9}$.	*Ans.* $\frac{871}{9}$.	12. $25\frac{91}{128}$.	*Ans.* $\frac{3291}{128}$.
7. 25 to sixths.	*Ans.* $\frac{150}{6}$.	13. $42\frac{126}{307}$.	*Ans.* $\frac{13020}{307}$

CASE II.

224. *To reduce improper fractions to whole or mixed numbers.*

1. How many units in $\frac{25}{7}$?

SOLUTION.—In *one* there are $\frac{7}{7}$, hence in $\frac{25}{7}$ there are as many ones as 7 is contained times in 25, which are $3\frac{4}{7}$. Therefore $\frac{25}{7}$ equals $3\frac{4}{7}$

OPERATION

$$\frac{25}{7}=3\frac{4}{7}.$$

SOLUTION 2D.—Since by Prin. 6, dividing both terms by the same number does not change the value of the fraction, by dividing both terms by 7, we have $\frac{3\frac{4}{7}}{1}$ or $3\frac{4}{7}$.

Rule.—*Divide the numerator by the denominator and the quotient will be the whole or mixed number.*

Reduce to whole or mixed numbers,

2. $\frac{28}{7}$.	*Ans.* 4.	7. $\frac{297}{16}$.	*Ans.* $18\frac{9}{16}$.
3. $\frac{56}{8}$.	*Ans.* 7.	8. $\frac{897}{25}$.	*Ans.* $35\frac{22}{25}$.
4. $\frac{49}{11}$	*Ans.* $4\frac{5}{11}$.	9. $\frac{7213}{136}$.	*Ans.* $53\frac{5}{136}$.
5. $\frac{76}{13}$.	*Ans.* $5\frac{11}{13}$.	10. $\frac{2149}{257}$.	*Ans.* $8\frac{93}{257}$.
6. $\frac{372}{14}$.	*Ans.* $26\frac{8}{14}$.	11. $\frac{87654}{3168}$.	*Ans.* $27\frac{2258}{3168}$.

CASE III.

225. *To reduce fractions to higher terms.*

226. Reducing a Fraction to higher terms is the process of reducing it to an equivalent fraction having a greater numerator and denominator.

1. How many twentieths in $\frac{4}{5}$?

SOLUTION.—In *one* there are $\frac{20}{20}$, and in $\frac{1}{5}$ there are $\frac{1}{5}$ of $\frac{20}{20}$, which are $\frac{4}{20}$, and in $\frac{4}{5}$ there are 4 times $\frac{4}{20}$, which are $\frac{16}{20}$; therefore $\frac{4}{5}=\frac{16}{20}$.

SOLUTION 2D.—Since multiplying both numerator and denominator of a fraction by the same number does not alter its value (Prin. 5), we muliply both terms by 4, which gives us $\frac{4}{5}=\frac{16}{20}$.

OPERATION.

$$\frac{4}{5}=\frac{4\times4}{5\times4}=\frac{16}{20}.$$

Rule.—*Multiply both numerator and denominator by the number which will give the required denominator.*

2. Reduce $\frac{1}{3}$, $\frac{3}{4}$, and $\frac{5}{6}$ to twelfths. *Ans.* $\frac{4}{12}$, $\frac{9}{12}$, $\frac{10}{12}$
3. Reduce $\frac{3}{4}$, $\frac{4}{5}$, and $\frac{7}{10}$ to twentieths. *Ans.* $\frac{15}{20}$, $\frac{16}{20}$, $\frac{14}{20}$
4. Reduce $\frac{3}{4}$, $\frac{5}{6}$, and $\frac{7}{8}$ to twenty-fourths. *Ans.* $\frac{18}{24}$, $\frac{20}{24}$, $\frac{21}{24}$.
5. Reduce $\frac{3}{4}$, $\frac{5}{6}$, and $\frac{9}{8}$ to forty-eighths. *Ans.* $\frac{36}{48}$, $\frac{40}{48}$, $\frac{54}{48}$.
6. Reduce $\frac{2}{3}$, $\frac{6}{7}$, and $\frac{8}{9}$ to sixty-thirds. *Ans.* $\frac{42}{63}$, $\frac{54}{63}$, $\frac{56}{63}$.
7. Reduce $\frac{4}{5}$, $\frac{5}{6}$, $\frac{7}{8}$, $\frac{9}{10}$, $\frac{11}{12}$ to 240ths.

Ans. $\frac{192}{240}$, $\frac{200}{240}$, $\frac{210}{240}$, $\frac{216}{240}$, $\frac{220}{240}$.

CASE IV.

227. *To reduce fractions to lower terms.*

228. Reducing a fraction to lower terms is the process of reducing it to an equivalent fraction having a smaller numerator and denominator.

Principle.—*A fraction is in its lowest terms when the numerator and denominator are prime to each other.*

1. Reduce $\frac{16}{20}$ to its lowest terms.

SOLUTION.—*One* equals $\frac{20}{20}$, and $\frac{1}{5}$ equals $\frac{4}{20}$; since $\frac{4}{20}$ equals *one*-fifth, $\frac{16}{20}$ equals as many fifths as 4 is contained times in 16, which is 4; hence $\frac{16}{20}$ equals $\frac{4}{5}$. Therefore, etc.

SOLUTION 2D.—Since dividing both terms of a fraction by the same number does not change its value (Prin. 6), we may reduce $\frac{16}{20}$ to lower terms by dividing both numerator and denominator by 4, which gives $\frac{16}{20}$, equal to $\frac{4}{5}$; and since 4 and 5 are prime to each other, the fraction is in its lowest terms. Therefore, etc.

OPERATION.

$4)\frac{16}{20} = \frac{4}{5}$ *Ans.*

Rule I.—*Divide both terms successively by their common factors.*

Rule II.—*Divide both terms by their greatest common divisor.*

Reduce the following fractions to lowest terms.

2. $\frac{72}{90}$.	*Ans.* $\frac{4}{5}$.	7. $\frac{1512}{1680}$.	*Ans.* $\frac{9}{10}$.
3. $\frac{105}{126}$.	*Ans.* $\frac{5}{6}$.	8. $\frac{4620}{5082}$.	*Ans.* $\frac{10}{11}$.
4. $\frac{294}{336}$.	*Ans.* $\frac{7}{8}$.	9. $\frac{9977}{10884}$.	*Ans.* $\frac{11}{12}$.
5. $\frac{336}{378}$.	*Ans.* $\frac{8}{9}$.	10. $\frac{11652}{12623}$.	*Ans.* $\frac{12}{13}$.
6. $\frac{630}{735}$.	*Ans.* $\frac{6}{7}$.	11. $\frac{35191}{37898}$.	*Ans.* $\frac{13}{14}$.

CASE V.

229. ***To reduce compound fractions to simple ones.***

1. What is $\frac{2}{3}$ of $\frac{4}{5}$?

SOLUTION.—$\frac{1}{3}$ of $\frac{1}{5}$ is *one* of the *three* equal parts into which $\frac{1}{5}$ may be divided. If each fifth is divided into three equal parts, 5 fifths, or the unit, will be divided into 5 times 3 or 15 equal parts; hence each part is $\frac{1}{15}$ of a unit. Since $\frac{1}{3}$ of $\frac{1}{5}$ is $\frac{1}{15}$, $\frac{1}{3}$ of $\frac{4}{5}$ is 4 times $\frac{1}{15}$, or $\frac{4}{15}$, and $\frac{2}{3}$ of $\frac{4}{5}$ is 2 times $\frac{4}{15}$ or $\frac{8}{15}$. Therefore $\frac{2}{3}$ of $\frac{4}{5}$ is $\frac{8}{15}$.

OPERATION.

$$\frac{2}{3} \text{ of } \frac{4}{5} = \frac{2 \times 4}{3 \times 5} = \frac{8}{15} \textit{ Ans.}$$

SOLUTION 2D.—$\frac{1}{3}$ of $\frac{4}{5}$ equals $\frac{4}{15}$ (Prin. 3), and since $\frac{1}{3}$ of $\frac{4}{5} = \frac{4}{15}$, $\frac{2}{3}$ of $\frac{4}{5}$ equals 2 times $\frac{4}{15}$, which, by Prin. 1, equals $\frac{8}{15}$.

Rule.—*Multiply the numerators together and the denominators together, cancelling the factors common to both terms.*

NOTES.—1. The problem given above may be solved thus: $\frac{4}{5} = \frac{12}{15}$, and $\frac{2}{3}$ of $\frac{12}{15} = \frac{8}{15}$. But this merely *obtains* the results, without showing the *reason* for the operation.

2. Reduce whole or mixed numbers to fractions before commencing the reduction to a simple fraction. To reduce complex fractions to simple ones, see Art. 249.

What is the value of

2. $\frac{3}{4}$ of $\frac{7}{9}$? *Ans.* $\frac{7}{12}$.
3. $\frac{4}{5}$ of $\frac{15}{17}$? *Ans.* $\frac{12}{17}$.
4. $\frac{7}{8}$ of $\frac{24}{37}$? *Ans.* $\frac{21}{37}$.
5. $\frac{12}{13}$ of $\frac{26}{29}$? *Ans.* $\frac{24}{29}$.
6. $\frac{14}{15}$ of $\frac{57}{62}$? *Ans.* $\frac{133}{155}$.
7. $\frac{21}{25}$ of $\frac{85}{91}$? *Ans.* $\frac{51}{65}$.
8. $\frac{3}{7}$ of $\frac{42}{75}$ of $\frac{65}{84}$? *Ans.* $\frac{13}{70}$
9. $\frac{4}{5}$ of $\frac{12}{35}$ of $\frac{21}{80}$? *Ans.* $\frac{24}{125}$.
10. $\frac{7}{8}$ of $\frac{42}{45}$ of $\frac{36}{21}$? *Ans.* $1\frac{2}{5}$.
11. $\frac{6}{7}$ of $\frac{13}{15}$ of $\frac{28}{39}$? *Ans.* $\frac{8}{15}$.
12. $\frac{8}{9}$ of $\frac{11}{12}$ of $\frac{36}{55}$? *Ans.* $\frac{8}{15}$.
13. $\frac{18}{25}$ of $\frac{35}{27}$ of $\frac{117}{140}$? *Ans.* $\frac{39}{50}$.
14. $\frac{6}{7}$ of $\frac{4}{11}$ of $\frac{57}{8}$ of $1\frac{29}{48}$? *Ans.* $3\frac{9}{16}$.
15. $\frac{5}{9}$ of $\frac{17}{18}$ of $\frac{35}{57}$ of $\frac{27}{55}$ of $\frac{266}{561}$ of $13\frac{82}{245}$? *Ans.* 1.
16. $1\frac{89}{115}$ of $6\frac{4}{11}$ of $\frac{38}{190}$ of $1\frac{3}{7}$ of $29\frac{1}{17}$ of $6\frac{21}{44}$ of $2\frac{1}{126}$? *Ans.* 1080.

COMMON DENOMINATOR.

230. A Common Denominator is a denominator common to several fractions.

Principle.—*A common denominator of several fractions is a common multiple of their denominators.*

1. Reduce $\frac{3}{5}$, $\frac{6}{7}$, and $\frac{7}{8}$ to a common denominator.

OPERATION.

$\frac{3}{5}, \frac{6}{7}, \frac{7}{8} = \frac{168}{280}, \frac{240}{280}, \frac{245}{280}$.

SOLUTION.—Since the product of the denominators of the fractions is a common multiple of their denominators, $5 \times 7 \times 8$, which equals 280, will be the common denominator. Then multiplying both terms of $\frac{3}{5}$ by 7 and 8, we have $\frac{3}{5} = \frac{168}{280}$ (Prin. 5). Multiplying both terms of $\frac{6}{7}$ by 5 and 8 we have $\frac{6}{7} = \frac{240}{280}$, etc. Hence the following

Rule.—*Multiply both terms of each fraction by the denominators of the other fractions.*

Reduce to a common denominator

2. $\frac{3}{4}$, $\frac{5}{6}$, and $\frac{5}{8}$. *Ans.* $\frac{144}{192}$, $\frac{160}{192}$, $\frac{120}{192}$.
3. $\frac{3}{5}$, $\frac{4}{9}$, and $\frac{7}{8}$. *Ans.* $\frac{216}{360}$, $\frac{160}{360}$, $\frac{315}{360}$.
4. $\frac{5}{8}$, $\frac{7}{6}$, and $\frac{8}{9}$. *Ans.* $\frac{270}{432}$, $\frac{504}{432}$, $\frac{384}{432}$.
5. $\frac{7}{8}$, $\frac{8}{9}$, and $\frac{9}{10}$. *Ans.* $\frac{630}{720}$, $\frac{640}{720}$, $\frac{648}{720}$.
6. $2\frac{1}{7}$, $3\frac{7}{9}$, and $8\frac{5}{6}$. *Ans.* $\frac{810}{378}$, $\frac{1428}{378}$, $\frac{3339}{378}$.
7. $\frac{11}{12}$, $\frac{13}{14}$, and $\frac{14}{15}$ *Ans.* $\frac{2310}{2520}$, $\frac{2340}{2520}$, $\frac{2352}{2520}$.
8. $\frac{2}{3}$, $\frac{3}{4}$, $\frac{5}{6}$, $\frac{7}{8}$, and $\frac{8}{9}$. *Ans.* $\frac{3456}{5184}$, $\frac{3888}{5184}$, $\frac{4320}{5184}$, $\frac{4536}{5184}$, $\frac{4608}{5184}$.
9. $2\frac{7}{12}$, $5\frac{5}{8}$, $3\frac{17}{18}$, and $\frac{5}{6}$ of $3\frac{9}{10}$.
Ans. $2\frac{4032}{6912}$, $5\frac{4320}{6912}$, $3\frac{6528}{6912}$, $3\frac{1728}{6912}$.

LEAST COMMON DENOMINATOR.

231. The **Least Common Denominator** of several fractions is the smallest denominator to which all can be reduced.

Principle.—*The least common denominator of several fractions is the least common multiple of their denominators.*

1. Reduce $\frac{5}{6}$, $\frac{7}{8}$, and $\frac{8}{9}$ to their least common denominator.

SOLUTION.—We find the least common multiple of the denominator to be 72, hence 72 is the least common denominator. Dividing 72 by 6, the denominator of $\frac{5}{6}$, we find we must multiply 6 by 12 to produce 72; hence multiplying both terms of $\frac{5}{6}$ by 12, we have $\frac{5}{6}=\frac{60}{72}$ (Prin. 5). Dividing 72 by 8, the denominator of $\frac{7}{8}$, we find we must multiply 8 by 9 to produce 72; hence multiplying both terms of $\frac{7}{8}$ by 9, we have $\frac{7}{8}=\frac{63}{72}$, etc.

OPERATION.

L. C. M. = 72

$$\frac{5}{6}=\frac{5\times12}{6\times12}=\frac{60}{72}$$

$$\frac{7}{8}=\frac{7\times9}{8\times9}=\frac{63}{72}$$

$$\frac{8}{9}=\frac{8\times8}{9\times8}=\frac{64}{72}$$

Rule.—I. *Find the least common multiple of the denominators, for the least common denominator.*

II. *Divide the least common denominator by the denominator of each fraction, and multiply both terms by the quotient.*

NOTES.—1. Reduce compound fractions to simple ones, mixed numbers to improper fractions, and all to their lowest terms, before finding the least common denominator.

2. When several fractions are reduced to a least common denominator, their numerators and the common denominator will be relatively prime.

Reduce to their least common denominators

2. $\frac{7}{8}$, $\frac{8}{9}$, and $\frac{13}{15}$. *Ans.* $\frac{315}{360}$, $\frac{320}{360}$, $\frac{312}{360}$.

3. $\frac{15}{16}$, $\frac{16}{18}$, and $\frac{24}{26}$. *Ans.* $\frac{1755}{1872}$, $\frac{1664}{1872}$, $\frac{1728}{1872}$.

4. $\frac{13}{14}$, $\frac{19}{21}$, and $\frac{27}{36}$. *Ans.* $\frac{78}{84}$, $\frac{76}{84}$, $\frac{63}{84}$.

5. $\frac{18}{26}$, $7\frac{5}{8}$, and $13\frac{4}{5}$. *Ans.* $\frac{360}{520}$, $\frac{3965}{520}$, $\frac{7176}{520}$.

6. $\frac{11}{26}$, $\frac{17}{39}$, and $\frac{41}{75}$. *Ans.* $\frac{825}{1950}$, $\frac{850}{1950}$, $\frac{1066}{1950}$.

7. $\frac{16}{51}$, $\frac{25}{68}$, and $\frac{28}{85}$. *Ans.* $\frac{320}{1020}$, $\frac{375}{1020}$, $\frac{336}{1020}$.

8. $\frac{24}{35}$, $\frac{3}{8}$ of $2\frac{5}{6}$, and $\frac{13}{14}$ of $12\frac{2}{3}$. *Ans.* $\frac{1152}{1680}$, $\frac{1785}{1680}$, $\frac{19760}{1680}$.

9. $\frac{3}{7}$ of $\frac{8}{9}$, $\frac{6}{5}$ of $4\frac{7}{8}$, and $\frac{7}{11}$ of $5\frac{7}{12}$.

Ans. $\frac{1760}{4620}$, $\frac{27027}{4620}$, $\frac{16415}{4620}$.

10. $\frac{5}{8}$ of $2\frac{1}{3}$, $\frac{8}{9}$ of $4\frac{1}{4}$, and $4\frac{2}{9}$ of $3\frac{7}{8}$.

Ans. $\frac{105}{72}$, $\frac{272}{72}$, $\frac{1178}{72}$.

11. $\frac{113}{633}$, $\frac{151}{844}$, $\frac{313}{1055}$, and $\frac{401}{1266}$.

Ans. $\frac{2260}{12660}$, $\frac{2265}{12660}$, $\frac{3756}{12660}$, $\frac{4010}{12660}$.

12. $\frac{241}{1203}$, $\frac{251}{1604}$, $\frac{271}{2005}$, and $\frac{381}{2406}$.

Ans. $\frac{4820}{24060}$, $\frac{3765}{24060}$, $\frac{3252}{24060}$, $\frac{3810}{24060}$.

13. $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$, $\frac{6}{7}$, $\frac{7}{8}$, $\frac{8}{9}$, $\frac{9}{10}$, $\frac{10}{11}$, $\frac{11}{12}$.

Ans. $\frac{13860}{27720}$, $\frac{18480}{27720}$, $\frac{20790}{27720}$, $\frac{22176}{27720}$, etc.

COMMON NUMERATOR.

232. Fractions have a **Common Numerator** when they have the same number for a numerator.

Principle.—*A common numerator of several fractions is a common multiple of their numerators.*

1. Reduce $\frac{3}{5}$, $\frac{4}{7}$, and $\frac{5}{9}$ to a common numerator.

OPERATION.

$\frac{3}{5}, \frac{4}{7}, \frac{5}{9} =$

$\frac{60}{100}, \frac{60}{105}, \frac{60}{108}$.

SOLUTION.—Since the product of the numerators is a common multiple of the numerators, $3 \times 4 \times 5$, which equals 60, will be the common numerator. Then multiplying both terms of $\frac{3}{5}$ by 4 and 5, we have $\frac{3}{5} = \frac{60}{100}$ (Prin. 5). Multiplying both terms of $\frac{4}{7}$ by 3 and 5, we have $\frac{4}{7} = \frac{60}{105}$, etc. Hence the following

Rule.—*Multiply both terms of each fraction by the numerators of the other fractions.*

Reduce to a common numerator

2. $\frac{3}{5}$, $\frac{5}{6}$, and $\frac{7}{8}$. *Ans.* $\frac{105}{175}$, $\frac{105}{126}$, $\frac{105}{120}$.

3. $\frac{4}{5}$, $\frac{6}{7}$, and $\frac{7}{9}$. *Ans.* $\frac{168}{210}$, $\frac{168}{196}$, $\frac{168}{216}$.

4. $\frac{7}{8}$, $\frac{8}{9}$, and $\frac{9}{100}$. *Ans.* $\frac{504}{576}$, $\frac{504}{567}$, $\frac{504}{5600}$.

5. $\frac{11}{12}$, $\frac{13}{14}$, and $\frac{14}{15}$. *Ans.* $\frac{2002}{2184}$, $\frac{2002}{2156}$, $\frac{2002}{2145}$.

6. $\frac{15}{16}$, $\frac{17}{18}$, and $\frac{18}{19}$. *Ans.* $\frac{4590}{4896}$, $\frac{4590}{4860}$, $\frac{4590}{4845}$

LEAST COMMON NUMERATOR.

233. The **Least Common Numerator** of several fractions is the smallest numerator to which all can be reduced

Principle.—*The least common numerator of several fractions is the least common multiple of their numerators.*

1. Reduce $\frac{3}{4}$, $\frac{4}{5}$, and $\frac{6}{7}$ to their least common numerator.

OPERATION.

L. C. M. = 12.

$$\frac{3}{4} = \frac{3 \times 4}{4 \times 4} = \frac{12}{16}$$

$$\frac{4}{5} = \frac{4 \times 3}{5 \times 3} = \frac{12}{15}$$

$$\frac{6}{7} = \frac{6 \times 2}{7 \times 2} = \frac{12}{14}$$

SOLUTION.—The least common multiple of the numerators we find is 12, hence 12 is the least common numerator. Dividing 12 by 3, we find we must multiply the numerator of $\frac{3}{4}$ by 4 to produce 12, hence multiplying both terms of $\frac{3}{4}$ by 4, we have $\frac{3}{4} = \frac{12}{16}$. Dividing 12 by 4, we find we must multiply the numerator of $\frac{4}{5}$ by 3 to produce 12, hence multiplying both terms of $\frac{4}{5}$ by 3 we have $\frac{4}{5} = \frac{12}{15}$, etc. Hence the following

Rule.—I. *Find the least common multiple of the numerators for the least common numerator.*

II. *Divide the least common numerator by the numerator of each fraction, and multiply both terms by the quotient.*

Reduce the following to their least common numerator:

2. $\frac{4}{5}$, $\frac{5}{6}$, and $\frac{6}{7}$. *Ans.* $\frac{60}{75}$, $\frac{60}{72}$, $\frac{60}{70}$.
3. $\frac{3}{5}$, $\frac{6}{7}$, and $\frac{8}{9}$ *Ans.* $\frac{24}{40}$, $\frac{24}{28}$, $\frac{24}{27}$.
4. $\frac{5}{6}$, $\frac{8}{9}$, and $\frac{10}{11}$. *Ans.* $\frac{40}{48}$, $\frac{40}{45}$, $\frac{40}{44}$
5. $\frac{8}{11}$, $\frac{12}{13}$, and $\frac{28}{31}$. *Ans.* $\frac{168}{231}$, $\frac{168}{182}$, $\frac{168}{186}$.
6. $\frac{12}{13}$, $\frac{16}{17}$, and $\frac{20}{21}$. *Ans.* $\frac{240}{260}$, $\frac{240}{255}$, $\frac{240}{252}$.

ADDITION OF FRACTIONS.

234. Addition of Fractions is the process of finding the sum of two or more fractions.

PRINCIPLES

1. *To add two or more fractions, they must express similar fractional units.*

2. *To add two or more fractions, they must be reduced to a common denominator*

1. What is the sum of $\frac{5}{6}$, $\frac{7}{8}$, and $\frac{8}{9}$?

SOLUTION.—Reducing the fractions to a common denominator, that they may express similar fractional units, we have $\frac{5}{6}=\frac{60}{72}$, $\frac{7}{8}=\frac{63}{72}$, and $\frac{8}{9}=\frac{64}{72}$; and 60 *seventy-seconds*, plus 63 *seventy-seconds*, plus 64 *seventy-seconds* equals 187 *seventy-seconds*, which reduced equals $2\frac{43}{72}$. Hence the following

OPERATION.

$$\frac{5}{6}+\frac{7}{8}+\frac{8}{9}=$$
$$\frac{60}{72}+\frac{63}{72}+\frac{64}{72}=\frac{187}{72}=2\frac{43}{72}$$

Rule.—*Reduce the fractions to a common denominator, then add the numerators and write the sum over the common denominator.*

NOTES.—1. Reduce compound fractions to simple ones, and reduce each fraction and the sum to lowest terms.

2. To add mixed numbers, add the integers and fractions separately, and then unite their sums.

Find the sum of

2. $\frac{1}{4}$, $\frac{1}{5}$, and $\frac{1}{6}$. *Ans.* $\frac{37}{60}$.
3. $\frac{2}{3}$, $\frac{2}{9}$, $\frac{11}{15}$, and $\frac{1}{10}$. *Ans.* $1\frac{13}{18}$.
4. $\frac{4}{5}$, $\frac{7}{9}$, $\frac{5}{12}$, and $\frac{7}{8}$. *Ans.* $2\frac{313}{360}$
5. $\frac{19}{20}$, $\frac{3}{40}$, $\frac{1}{90}$, $\frac{5}{6}$, and $\frac{7}{10}$. *Ans.* $2\frac{41}{72}$
6. $\frac{2}{3}$ of $\frac{3}{4}$, $\frac{5}{6}$ of $\frac{19}{20}$, $\frac{3}{10}$ of $\frac{17}{18}$, and $\frac{5}{6}$. Ans. $2\frac{49}{120}$

7. $4\frac{1}{2}$, $6\frac{1}{3}$, $8\frac{1}{4}$, and $9\frac{1}{16}$. *Ans.* $28\frac{7}{48}$.

8. $21\frac{1}{9}$, $33\frac{1}{8}$, $62\frac{1}{10}$, and $75\frac{1}{12}$. *Ans.* $191\frac{151}{360}$.

9. $5\frac{2}{3}$, $6\frac{3}{4}$, $7\frac{4}{5}$, $8\frac{5}{6}$, and $9\frac{6}{7}$. *Ans.* $38\frac{127}{140}$.

10. $\frac{2}{3}$, $\frac{8}{9}$, $\frac{11}{12}$, $\frac{14}{15}$, $\frac{17}{18}$, $\frac{35}{36}$, and $\frac{111}{180}$. *Ans.* $5\frac{169}{180}$.

11. $\frac{5}{7}$, $\frac{2}{3}$ of $\frac{7}{8}$, $\frac{6}{7}$ of $\frac{8}{9}$, $\frac{7}{8}$ of $\frac{13}{14}$, and $\frac{17}{60}$ of $\frac{25}{51}$. *Ans.* $3\frac{11}{1008}$.

12. $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$, $\frac{6}{7}$, $\frac{7}{8}$, $\frac{8}{9}$, $\frac{9}{10}$, $\frac{10}{11}$, and $\frac{11}{12}$. *Ans.* $8\frac{24859}{27720}$.

SUBTRACTION OF FRACTIONS.

235. Subtraction of Fractions is the process of finding the difference between two fractions.

PRINCIPLES.

1. *To subtract one fraction from another they must express similar fractional units.*

2. *To subtract one fraction from another they must be reduced to a common denominator.*

1. What is the difference between $\frac{5}{8}$ and $\frac{8}{9}$?

OPERATION.

$\frac{8}{9}-\frac{5}{8}=$

$\frac{64}{72}-\frac{45}{72}=\frac{19}{72}$

SOLUTION.—Reducing the fractions to a common denominator that they may express similar fractional units, we have $\frac{8}{9}=\frac{64}{72}$ and $\frac{5}{8}=\frac{45}{72}$; 64 *seventy-seconds* minus 45 *seventy-seconds* equals 19 *seventy-seconds*. Hence the following

Rule.—*Reduce the fractions to a common denominator, take the difference of the numerators and write it over the common denominator.*

NOTE.—Reduce compound fractions to simple ones, and reduce each fraction and the difference to lowest terms.

Find the value of

2. $\frac{6}{7}-\frac{4}{5}$. *Ans.* $\frac{2}{35}$.

3. $\frac{41}{99}-\frac{4}{11}$. *Ans.* $\frac{5}{99}$.

4. $\frac{4}{7}-\frac{11}{31}$. *Ans.* $\frac{47}{217}$.

5. $\frac{109}{105}-\frac{19}{35}$. *Ans.* $\frac{52}{105}$.

6. $\frac{17}{18}-\frac{49}{57}$. *Ans.* $\frac{29}{342}$.

7. $\frac{5}{91}-\frac{5}{1001}$. *Ans.* $\frac{50}{1001}$.

8. $\frac{2}{3}$ of $\frac{4}{5}-\frac{1}{2}$ of $\frac{3}{4}$. *Ans.* $\frac{19}{120}$.

9. $\frac{8}{9}$ of $\frac{18}{19}-\frac{41}{50}$ of $\frac{45}{57}$. *Ans.* $\frac{37}{190}$.

10. $\frac{15}{16}-\frac{3}{7}$ of $\frac{49}{50}$ of $\frac{16}{21}$. *Ans.* $\frac{247}{400}$.

11. $(\frac{3}{2}+\frac{5}{4})-(\frac{2}{3}+\frac{4}{5})$. *Ans.* $1\frac{17}{60}$.

12. $\frac{55}{56}-\frac{91}{100}$ of $\frac{50}{77}$ of $\frac{11}{13}$. *Ans.* $\frac{27}{56}$.

13. $(\frac{5}{4}+\frac{9}{8}+\frac{16}{15})-(\frac{1}{4}+\frac{1}{8}+\frac{1}{15})$. *Ans.* 3.

14. Find the difference between $15\frac{5}{9}$ and $12\frac{8}{9}$.

SOLUTION.—We cannot take $\frac{8}{9}$ from $\frac{5}{9}$, so we take 1 from 15, which equals $\frac{9}{9}$, and this added to $\frac{5}{9}=\frac{14}{9}$; $\frac{14}{9}-\frac{8}{9}=\frac{6}{9}$; $14-12=2$; hence $15\frac{5}{9}-12\frac{8}{9}=2\frac{6}{9}$ or $2\frac{2}{3}$.

15. $14\frac{1}{6}-11\frac{5}{6}$.	*Ans.* $2\frac{1}{3}$.	**17.** $35\frac{3}{4}-26\frac{11}{12}$.	*Ans.* $8\frac{5}{6}$.
16. $15\frac{3}{8}-12\frac{7}{8}$.	*Ans.* $2\frac{1}{2}$.	**18.** $42\frac{3}{5}-33\frac{4}{7}$.	*Ans.* $9\frac{1}{35}$.

19. $75\frac{3}{7}-46\frac{5}{9}$. *Ans.* $28\frac{55}{63}$.

20. $(3\frac{1}{8}+5\frac{1}{9})-(2\frac{1}{5}+3\frac{1}{6})$. *Ans.* $2\frac{313}{360}$.

21. $(16\frac{1}{3}+12\frac{1}{2})-(5\frac{2}{3}+6\frac{17}{20})$. *Ans.* $16\frac{19}{60}$.

22. $(56\frac{1}{4}+83\frac{1}{3})-(37\frac{1}{2}+49\frac{1}{7})$. *Ans.* $52\frac{79}{84}$.

PRACTICAL PROBLEMS

IN ADDITION AND SUBTRACTION.

1. A merchant sold a customer $22\frac{1}{2}$ yd. of silk, $3\frac{1}{4}$ yd. of paper muslin, $1\frac{1}{8}$ yd. of silesia, $5\frac{3}{4}$ yd. of cambric, and $5\frac{1}{3}$ yd. of ruffling; how many yards were sold? *Ans.* $37\frac{23}{24}$.

2. A lady in shopping makes a bill of \$$37\frac{3}{8}$ for silk, \$$12\frac{1}{4}$ for merino, \$$2\frac{1}{8}$ for calico and \$$1\frac{3}{4}$ for braid, sewing-silk, etc.; she gave the merchant a 100 dollar bill in payment; what change should she receive? *Ans.* \$$46\frac{1}{2}$.

3. A person engages 4 tons of coal (2000 lb.); but when delivered, he finds the first ton falls short $22\frac{1}{2}$ lb., the second $23\frac{5}{8}$ lb., the third $19\frac{7}{12}$ lb., and the fourth $26\frac{3}{4}$ lb.; how much coal should he pay for? *Ans.* $7907\frac{13}{24}$ lb.

4. From a barrel of vinegar containing $56\frac{1}{4}$ gallons, there were drawn at one time $21\frac{1}{2}$ gallons; at another time $13\frac{1}{3}$ gallons, of which $3\frac{1}{8}$ gallons were returned to the barrel; at a third time $15\frac{5}{9}$ gallons were drawn; and at a fourth time $19\frac{1}{10}$ gallons were poured in; how many gallons were in the barrel finally? *Ans.* $28\frac{31}{360}$ gallons.

SPECIAL SOLUTIONS

IN ADDITION AND SUBTRACTION OF FRACTIONS.

CASE I.

236. ***To add or subtract two fractions whose numerators are a unit.***

1. Find the sum and the difference of $\frac{1}{3}$ and $\frac{1}{5}$.

SOLUTION.—Reducing to a common denominator and adding, we see that we have the sum of the denominators divided by their product.

If we subtract, we shall have the difference of the denominators divided by their product. Hence we have the following

OPERATION.

$$\frac{1}{3}+\frac{1}{5}=\frac{5}{15}+\frac{3}{15}=\frac{5+3}{15}=\frac{8}{15}\ \textit{Ans.}$$

$$\frac{5-3}{15}=\frac{2}{15}\ \textit{Ans.}$$

Rule.—*Take the sum or difference of the two denominators and write it over the product of the denominators.*

What is the value of

2. $\frac{1}{4}+\frac{1}{7}$.	*Ans.* $\frac{11}{28}$.	6. $\frac{1}{5}-\frac{1}{8}$.	*Ans.* $\frac{3}{40}$.
3. $\frac{1}{8}+\frac{1}{9}$.	*Ans.* $\frac{17}{72}$.	7. $\frac{1}{7}-\frac{1}{9}$.	*Ans.* $\frac{2}{63}$.
4. $\frac{1}{15}+\frac{1}{17}$.	*Ans.* $\frac{32}{255}$.	8. $\frac{1}{10}-\frac{1}{12}$.	*Ans* $\frac{1}{60}$.
5. $\frac{1}{25}+\frac{1}{34}$	*Ans.* $\frac{59}{850}$.	9. $\frac{1}{24}-\frac{1}{56}$.	*Ans.* $\frac{1}{42}$.

CASE II.

237. ***To add or subtract two fractions having a common numerator.***

1. Find the sum and difference of $\frac{3}{4}$ and $\frac{3}{5}$.

SOLUTION.—Reducing to a common denominator, and adding, we see we have 3 times the sum of the denominators divided by their product.

If we subtract we shall have three times the difference of the denominators divided by their product. Hence the

OPERATION.

$$\frac{3}{4}+\frac{3}{5}=\frac{3(5+4)}{5\times4}$$
$$=\frac{27}{20}.\ \textit{Ans.}$$

Rule.—*Take the sum or difference of the denominators, multiply it by the common numerator, and write the product over the product of the denominators.*

What is the value of

2. $\frac{3}{5}+\frac{3}{7}$.	*Ans.* $\frac{36}{35}$.	6. $\frac{3}{5}-\frac{3}{7}$.	*Ans.* $\frac{6}{35}$.
3. $\frac{4}{5}+\frac{4}{9}$.	*Ans.* $\frac{56}{45}$.	7. $\frac{4}{5}-\frac{4}{9}$.	*Ans.* $\frac{16}{45}$.
4. $\frac{6}{11}+\frac{6}{13}$.	*Ans.* $\frac{144}{143}$.	8. $\frac{6}{11}-\frac{6}{13}$.	*Ans.* $\frac{12}{143}$.
5. $\frac{12}{13}+\frac{12}{15}$.	*Ans.* $\frac{112}{65}$.	9. $\frac{12}{13}-\frac{12}{15}$.	*Ans.* $\frac{8}{65}$.

CASE III.

238. ***To add or subtract two fractions by reducing to a common numerator.***

Rule.—*Reduce to a common numerator and proceed as in Case II.*

What is the value of

1. $\frac{2}{3}+\frac{3}{4}$.	*Ans.* $1\frac{5}{12}$.	6. $\frac{3}{4}-\frac{2}{3}$.	*Ans.* $\frac{1}{12}$.
2. $\frac{5}{6}+\frac{7}{8}$.	*Ans.* $1\frac{17}{24}$.	7. $\frac{6}{7}-\frac{4}{5}$.	*Ans.* $\frac{2}{35}$.
3. $\frac{4}{5}+\frac{6}{7}$.	*Ans.* $1\frac{23}{35}$.	8. $\frac{7}{8}-\frac{5}{6}$.	*Ans.* $\frac{1}{24}$.
4. $\frac{6}{7}+\frac{8}{9}$.	*Ans.* $1\frac{47}{63}$.	9. $\frac{8}{9}-\frac{6}{7}$.	*Ans.* $\frac{2}{63}$.
5. $\frac{8}{11}+\frac{16}{19}$.	*Ans.* $1\frac{119}{209}$.	10. $\frac{24}{23}-\frac{12}{19}$.	*Ans.* $\frac{180}{437}$.

NOTE.—It will be noticed that this is sometimes less work than reducing to a common denominator.

MULTIPLICATION OF FRACTIONS

239. Multiplication of Fractions is the process of finding a product when one or both factors are fractions.

240. There are **Three Cases**: 1st. A fraction by an integer; 2d. An integer by a fraction; 3d. A fraction by a fraction.

CASE I.

241. ***To multiply a fraction by an integer.***

1. Multiply $\frac{13}{18}$ by 6.

SOLUTION.—Six times 13 eighteenths equals 78 eighteenths ($\frac{78}{18}$), which equals $\frac{13}{3}$, or $4\frac{1}{3}$; or 6 times $\frac{1}{18}=\frac{6}{18}$, or $\frac{1}{3}$, and if 6 times $\frac{1}{18}$ equals $\frac{1}{3}$, 6 times $\frac{13}{18}$ equals 13 times $\frac{1}{3}$, which is $\frac{13}{3}$, or $4\frac{1}{3}$. Therefore, etc.

SOLUTION 2D.—Multiplying the numerator (Prin. 1), we have 6 times $\frac{13}{18}$ equals $\frac{78}{18}$; or, dividing the denominator (Prin. 4), we have 6 times $\frac{13}{18}=\frac{13}{3}$, or $4\frac{1}{3}$.

OPERATION.

$$\frac{13}{18}\times 6=\frac{78}{18}=\frac{13}{3}$$
$$\frac{13}{18}\times 6=\frac{13}{3}=4\frac{1}{3}$$

Rule.—*To multiply a fraction by an integer, multiply the numerator, or divide the denominator.*

Multiply

2. $\frac{19}{64}$ by 8. *Ans.* $2\frac{3}{8}$.
3. $\frac{57}{60}$ by 12. *Ans.* $11\frac{2}{5}$.
4. $\frac{73}{84}$ by 16. *Ans.* $13\frac{19}{21}$.
5. $\frac{91}{96}$ by 36 *Ans.* $34\frac{1}{8}$.
6. $\frac{111}{224}$ by 64. *Ans.* $31\frac{5}{7}$.
7. $18\frac{17}{20}$ by 15. *Ans.* $282\frac{3}{4}$
8. $37\frac{61}{72}$ by 18. *Ans.* $681\frac{1}{4}$.
9. $46\frac{67}{81}$ by 27. *Ans.* $1264\frac{1}{3}$.
10. $\frac{3121}{3456}$ by 96. *Ans.* $86\frac{25}{36}$.
11. $\frac{4141}{5184}$ by 576. *Ans.* $460\frac{1}{9}$.

CASE II.

242. ***To multiply an integer by a fraction.***

1. Multiply 17 by $\frac{5}{6}$; also by $8\frac{5}{6}$.

SOLUTION.—17 multiplied by $\frac{5}{6}$ equals $\frac{1}{6}$ of 5 times 17; 5 times 17 are 85, and $\frac{1}{6}$ of 85 is $\frac{85}{6}$, or $14\frac{1}{6}$. Therefore etc.

OPERATION.

$$17\times\frac{5}{6}=\frac{85}{6}=14\frac{1}{6}$$

SOLUTION.—Multiplying 17 by 5 and dividing by 6, we have $14\frac{1}{6}$; and then multiplying 17 by 8 we have 136; adding 136 to $14\frac{1}{6}$ we have $150\frac{1}{6}$. Therefore, etc.

OPERATION.

$$\begin{array}{r} 17 \\ \underline{8\frac{5}{6}} \\ 6)\underline{85} \\ 14\frac{1}{6} \\ \underline{136} \\ 150\frac{1}{6} \end{array}$$

NOTE.—This method of multiplying by a mixed number is more convenient than the one usually presented.

Rule.—*Multiply the integer by the numerator of the multiplier, and divide the product by the denominator; or divide first and then multiply.*

Multiply

2. 28 by $\frac{5}{7}$. *Ans.* 20.
3. 32 by $\frac{7}{8}$. *Ans.* 28.
4. 81 by $\frac{8}{9}$. *Ans.* 72.
5. 96 by $3\frac{5}{6}$. *Ans.* 368.
6. 99 by $7\frac{6}{7}$. *Ans.* $777\frac{6}{7}$.
7. 256 by $8\frac{7}{8}$. *Ans.* 2272.
8. 144 by $6\frac{3}{5}$. *Ans.* $950\frac{2}{5}$.
9. 273 by $\frac{80}{91}$. *Ans.* 240.
10. 288 by $\frac{25}{26}$. *Ans.* $276\frac{12}{13}$.
11. 852 by $\frac{35}{36}$. *Ans.* $828\frac{1}{3}$.
12. 768 by $9\frac{47}{48}$. *Ans.* 7664.
13. 720 by $8\frac{71}{72}$. *Ans.* 6470.
14. 960 by $27\frac{80}{81}$. *Ans.* $26868\frac{4}{27}$.
15. 819 by $36\frac{90}{91}$. *Ans.* 30294.

CASE III.

243. ***To multiply a fraction by a fraction.***

1. What is the product of $\frac{7}{8}$ by $\frac{5}{6}$?

SOLUTION.—$\frac{7}{8}$ multiplied by *one* equals $\frac{7}{8}$, hence $\frac{7}{8}$ multiplied by $\frac{1}{6}$ equals $\frac{1}{6}$ of $\frac{7}{8}$, which is $\frac{7}{48}$, and $\frac{7}{8}$ multiplied by $\frac{5}{6}$ equals 5 times $\frac{7}{48}$, which are $\frac{35}{48}$. Therefore, etc.

OPERATION.

$$\frac{7}{8}\times\frac{5}{6}=\frac{35}{48}$$

Rule.—*Multiply the numerators together and the denominators together, cancelling common factors.*

NOTE.—Practically this case is the same as finding a fractional part of a fraction, but theoretically the two cases are entirely distinct.

Find the product of

2. $\frac{36}{35}$ by $\frac{7}{8}$. *Ans.* $\frac{9}{10}$.
3. $\frac{49}{50}$ by $\frac{15}{14}$. *Ans.* $\frac{21}{20}$.
4. $\frac{121}{144}$ by $\frac{12}{22}$. *Ans.* $\frac{11}{24}$.
5. $9\frac{3}{4}$ by $5\frac{1}{3}$. *Ans.* 52.
6. $13\frac{1}{4}$ by $7\frac{1}{5}$. *Ans.* $95\frac{2}{5}$.
7. $101\frac{7}{9}$ by $16\frac{1}{3}$. *Ans.* $1662\frac{10}{27}$.
8. $315\frac{3}{10}$ by $19\frac{5}{9}$. *Ans.* $6165\frac{13}{15}$.
9. $95\frac{5}{7}$ by $17\frac{1}{2}$. *Ans.* 1675.
10. $36\frac{90}{91}$ by $27\frac{80}{81}$. *Ans.* $1035\frac{193}{819}$.
11. $\frac{2}{3}$ of $\frac{4}{5}$ by $\frac{9}{16}$ of $\frac{32}{51}$. *Ans.* $\frac{16}{85}$.
12. $\frac{4}{7}$ of $\frac{21}{32}$ by $\frac{75}{49}$ of $\frac{98}{99}$. *Ans.* $\frac{25}{44}$.
13. $\frac{5}{11}$ of $\frac{16}{25}$ by $\frac{88}{91}$ of $\frac{125}{192}$. *Ans.* $\frac{50}{273}$.
14. $\frac{19}{20}$ of $\frac{7}{11}$ of $6\frac{7}{19}$ by $\frac{21}{32}$ of $\frac{56}{57}$ of $19\frac{3}{23}$. *Ans.* $47\frac{425}{874}$.
15. $\frac{6}{19}$ of $\frac{41}{42}$ of $8\frac{1}{7}$ by $\frac{17}{18}$ of $\frac{36}{51}$ of $41\frac{18}{117}$. *Ans.* $68\frac{554}{687}$
16. $\frac{60}{69}$ of $81\frac{7}{19}$ by $\frac{57}{58}$ of $\frac{23}{25}$ of $67\frac{61}{279}$. *Ans.* $4300\frac{1868}{13485}$.
17. $4\frac{4}{5}\times5\frac{1}{7}\times\frac{4}{11}\times\frac{5}{54}\times2\frac{14}{15}\times13\frac{1}{8}$. *Ans.* 32.
18. $4\frac{2}{11}\times\frac{63}{65}$ of $3\frac{3}{7}\times\frac{13}{144}$ of $3\frac{8}{9}\times\frac{15}{28}$ of $18\frac{6}{7}$. *Ans.* 60.

PRACTICAL PROBLEMS

IN MULTIPLICATION OF FRACTIONS.

Required the cost of

1. $9\frac{1}{2}$ yards of cloth at \$$4\frac{1}{5}$ a yard. *Ans.* \$$39\frac{9}{10}$.
2. $8\frac{3}{5}$ quarts of nuts at $8\frac{3}{4}$ cts. a quart. *Ans.* \$$0.75\frac{1}{4}$.
3. $25\frac{1}{4}$ yards of muslin at $25\frac{1}{2}$ cts. a yard. *Ans.* \$$6.43\frac{7}{8}$.
4. $16\frac{2}{3}$ pounds of sugar at $7\frac{1}{4}$ cts. a pound. *Ans.* \$$1.20\frac{5}{6}$.
5. $28\frac{1}{2}$ barrels of sugar at \$$17\frac{3}{5}$ a bar. *Ans.* \$$501\frac{3}{5}$.
6. $46\frac{1}{5}$ cords of wood at \$$5\frac{3}{8}$ a cord. *Ans.* \$$248\frac{13}{40}$.
7. $38\frac{3}{4}$ tons of hay at \$$14\frac{3}{5}$ a ton. *Ans.* \$$565\frac{3}{4}$.
8. $53\frac{1}{2}$ tons of coal at \$$8\frac{3}{4}$ a ton. *Ans.* \$$468\frac{1}{8}$.
9. $5\frac{3}{4}$ dozens of eggs at $12\frac{1}{2}$ cts. a dozen. *Ans.* \$$0.71\frac{7}{8}$.
10. $96\frac{2}{3}$ lb. of cotton at $16\frac{1}{2}$ cts. a pound. *Ans.* \$15.95.
11. $85\frac{4}{5}$ lb. of meat at $18\frac{3}{4}$ cts. a pound. *Ans.* \$$16.08\frac{3}{4}$.
12. $19\frac{3}{4}$ barrels of sugar at \$$21\frac{4}{5}$ a barrel. *Ans.* \$$430\frac{11}{20}$.
13. What is the value of $\frac{28}{125}\times\frac{46}{84}\times\frac{325}{690}\times\frac{1728}{1872}$? *Ans.* $\frac{4}{75}$.
14. Of $\frac{561}{784}\times\frac{497}{720}\times\frac{396}{891}\times\frac{576}{2601}$? *Ans.* $\frac{781}{16065}$.
15. Of $\frac{2304}{6561}\times\frac{1296}{9025}\times\frac{5625}{9801}\times\frac{9999}{10000}$? *Ans.* $\frac{25856}{898475}$.
16. Of $\frac{6912}{9409}\times\frac{1261}{1296}\times\frac{9797}{10000}\times\frac{7500}{7777}\times\frac{462}{24}$? *Ans.* 13.
17. Of $(\frac{7}{8})^2\times(\frac{3}{4})^3\times(\frac{2}{3})^4$? *Ans.* $\frac{49}{768}$
18. Of $(\frac{9}{16})^2\times(\frac{7}{15})^3\times\frac{87}{91}$? *Ans.* $\frac{12789}{416000}$.
19. Of $(\frac{7}{12})^3\times(\frac{19}{20})^2\times(\frac{5}{19})^4\times(\frac{4}{7})^5\times(5\frac{1}{4})^4$? *Ans.* $\frac{3675}{92416}$.
20. Of $\overline{(\frac{1}{2}+\frac{9}{17})\times\frac{17}{18}}\times(\frac{20}{21}-\frac{9}{56})$? *Ans.* $\frac{665}{864}$.
21. Of $(\frac{10}{11}+\frac{15}{19})\times(\frac{21}{22}-\frac{35}{48})\times\frac{627}{710}\times\frac{44}{49}$? *Ans.* $\frac{17}{56}$.
22. Of $101\frac{7}{9}-(22\frac{3}{10}-19\frac{5}{9})+16\frac{1}{3}\times\frac{19}{30}$? *Ans.* $109\frac{17}{45}$.
23. $(16\frac{1}{2}+27\frac{7}{9})-(56\frac{2}{3}-49\frac{2}{9})+95\frac{3}{7}\times 7\frac{1}{9}$? *Ans.* $715\frac{55}{126}$.

24. When St. Louis flour is worth \$$6\frac{2}{5}$ a barrel, how much will $26\frac{5}{8}$ barrels cost? *Ans.* \$$170\frac{2}{5}$.

25. Bought 10 dozen O. H. Swedes carpet tacks, amounting to \$20.40, with a deduction first of $\frac{1}{10}$ and after of $\frac{3}{20}$; what was the bill? *Ans.* \$$15.60\frac{3}{5}$.

26. A bill of books amounts to \$$596\frac{3}{4}$, but I get $\frac{1}{3}$ off for wholesale and $\frac{3}{50}$ for cash; what do I pay? *Ans.* \$$373\frac{289}{300}$.

27. A farmer bought some sheep for \$$75\frac{1}{4}$, and then sold 7 of them at \$$7\frac{1}{2}$ apiece, and traded the rest for half a barrel of sugar at \$20 a barrel, and a barrel of mackerel at \$15 a barrel; what was his gain? Ans. \$$2\frac{1}{4}$.

DIVISION OF FRACTIONS.

244. Division of Fractions is the process of dividing when one or both terms are fractional.

245. There are **Three Cases:**

1st. A fraction by an integer; 2d. An integer by a fraction; 3d. A fraction by a fraction.

CASE I.

246. *To divide a fraction by an integer.*

1. Divide $\frac{12}{13}$ by 6, also by 7.

SOLUTION.—$\frac{12}{13}$ divided by *one* equals $\frac{12}{13}$, hence $\frac{12}{13}$ divided by 6 equals $\frac{1}{6}$ of $\frac{12}{13}$, which is $\frac{2}{13}$; $\frac{12}{13}$ divided by 7 equals $\frac{1}{7}$ of $\frac{12}{13}$, or $\frac{12}{91}$.

OPERATION.
$$\frac{12}{13} \div 6 = \frac{2}{13}$$

SOLUTION 2D.—Dividing the numerator by 6, we have $\frac{12}{13} \div 6$ equals $\frac{2}{13}$ (Prin. 2); multiplying the denominator by 7, we have $\frac{12}{13} \div 7$ equals $\frac{12}{91}$ (Prin. 3).

OPERATION.
$$\frac{12}{13} \div 7 = \frac{12}{91}$$

2. Divide $627\frac{3}{4}$ by 6.

SOLUTION.—Dividing 627 by 6 we have 104 and a remainder of 3; 3 equals $\frac{12}{4}$, which, added to $\frac{3}{4}$, equals $\frac{15}{4}$; $\frac{15}{4}$ divided by 6 equals $\frac{5}{8}$; hence the quotient is $104\frac{5}{8}$.

OPERATION.
$$6)\underline{627\tfrac{3}{4}}$$
$$104\tfrac{5}{8}$$

Rule.—*Divide the numerator or multiply the denominator of the dividend by the divisor.*

NOTE.—Reduce a mixed number to a fraction, or divide the integer, unite the remainder with the fraction and divide the result.

Divide

2. $\frac{15}{17}$ by 5. *Ans.* $\frac{3}{17}$.
3. $\frac{28}{29}$ by 7. *Ans.* $\frac{4}{29}$.
4. $\frac{36}{39}$ by 8. *Ans.* $\frac{3}{26}$.
5. $\frac{52}{57}$ by 28. *Ans.* $\frac{13}{399}$.
6. $325\frac{2}{3}$ by 6. *Ans.* $54\frac{5}{18}$.
7. $152\frac{2}{3}$ by 8. *Ans.* $19\frac{1}{12}$.
8. $263\frac{7}{8}$ by 12. *Ans.* $21\frac{95}{96}$.
9. $492\frac{12}{13}$ by 15. *Ans.* $32\frac{56}{65}$.
10. $709\frac{15}{17}$ by 16. *Ans.* $44\frac{25}{68}$.
11. $1220\frac{19}{31}$ by 105. *Ans.* $11\frac{678}{1085}$.
12. $1467\frac{55}{73}$ by 40. *Ans.* $36\frac{1013}{1460}$.
13. $3146\frac{75}{81}$ by 60. *Ans.* $52\frac{727}{1620}$.

CASE II.

247. *To divide an integer by a fraction.*

1. Divide 12 by the fraction $\frac{6}{7}$.

SOLUTION.—12 divided by *one* equals 12, hence 12 divided by $\frac{1}{7}$ equals 7 times 12, and 12 divided by $\frac{6}{7}$ equals $\frac{1}{6}$ of 7 times 12, which is $\frac{7}{6}$ times 12, which *equals 14. Hence the following*

OPERATION.
$$12 \div \frac{6}{7} =$$
$$12 \times \frac{7}{6} = 14$$

Rule I.—*Multiply the dividend by the denominator of the fraction, and divide the product by the numerator.*

Rule II.—*Invert the divisor and proceed as in multiplication of fractions.*

Divide

2. 12 by $\frac{3}{4}$.	*Ans.* 16.	7. 228 by $7\frac{3}{5}$.	*Ans.* 30
3. 21 by $\frac{7}{8}$.	*Ans.* 24.	8. 801 by $9\frac{8}{9}$.	*Ans.* 81
4. 36 by $\frac{8}{9}$.	*Ans.* $40\frac{1}{2}$.	9. 1269 by $13\frac{12}{13}$.	*Ans.* $91\frac{26}{181}$.
5. 46 by $3\frac{2}{7}$.	*Ans.* 14.	10. 3070 by $23\frac{15}{16}$.	*Ans.* $128\frac{96}{383}$.
6. 159 by $5\frac{8}{9}$.	*Ans.* 27.	11. 7029 by $46\frac{21}{25}$.	*Ans.* $150\frac{75}{1171}$.

CASE III.

248. ***To divide a fraction by a fraction.***

1. Divide $\frac{8}{9}$ by $\frac{5}{6}$.

SOLUTION.—$\frac{8}{9}$ divided by *one* equals $\frac{8}{9}$, hence $\frac{8}{9}$ divided by $\frac{1}{6}$ equals 6 times $\frac{8}{9}$, and $\frac{8}{9}$ divided by $\frac{5}{6}$ equals $\frac{1}{5}$ of 6 times $\frac{8}{9}$, which is $\frac{6}{5}$ times $\frac{8}{9}$, which equals $\frac{16}{15}$.

OPERATION.

$$\frac{8}{9} \div \frac{5}{6} = \frac{8}{9} \times \frac{6}{5} = \frac{16}{15}.$$

SOLUTION 2D.—$\frac{8}{9}$ equals $\frac{48}{54}$, $\frac{5}{6}$ equals $\frac{45}{54}$, and $\frac{48}{54}$ divided by $\frac{45}{54}$ equals 48 divided by 45, which equals $\frac{48}{45}$ or $\frac{16}{15}$.

OPERATION.

$$\frac{8}{9} \div \frac{5}{6} = \frac{48}{54} \div \frac{45}{54} = 48 \div 45 = \frac{48}{45} = \frac{16}{15}$$

Rule I.—*Multiply the dividend by the denominator of the divisor, and divide by the numerator.*

Rule II.—*Invert the divisor and proceed as in multiplication of fractions.*

NOTES.—1. Reduce mixed numbers to simple fractions. When the divisor is a compound fraction, invert each term and multiply, cancelling when possible.

2. "Why do we invert the divisor?" The analysis *requires* it, or *dictates* it, as is shown in first solution. This reply is better than "for convenience."

Divide

2. $\frac{15}{16}$ by $\frac{5}{32}$.	*Ans.* 6.	7. $7\frac{9}{11}$ by $14\frac{9}{10}$.	*Ans.* $\frac{860}{1639}$
3. $\frac{19}{22}$ by $\frac{57}{132}$.	*Ans.* 2.	8. $25\frac{3}{19}$ by $11\frac{16}{27}$.	*Ans.* $2\frac{1012}{5947}$
4. $\frac{71}{72}$ by $\frac{35}{36}$.	*Ans.* $1\frac{1}{70}$.	9. $54\frac{1}{4}$ by $13\frac{13}{16}$.	*Ans.* $3\frac{205}{221}$.
5. $\frac{23}{24}$ by $\frac{67}{108}$.	*Ans.* $1\frac{73}{134}$.	10. $258\frac{7}{8}$ by $63\frac{7}{20}$.	*Ans.* $4\frac{219}{2534}$.
6. $5\frac{2}{7}$ by $3\frac{2}{5}$.	*Ans.* $1\frac{66}{119}$.	11. $\frac{6}{13}$ of $\frac{14}{15}$ by $\frac{5}{9}$ of $\frac{7}{8}$.	*Ans.* $\frac{288}{325}$.

12. $\frac{5}{11}$ of $\frac{17}{21}$ by $\frac{2}{21}$ of $\frac{22}{29}$. *Ans.* $5\frac{45}{484}$.

13. $\frac{16}{19}$ of $\frac{18}{23}$ by $\frac{7}{20}$ of $3\frac{1}{57}$. *Ans.* $\frac{4320}{6923}$

14. $\frac{2}{3}$ of $\frac{4}{5}$ of $\frac{7}{8}$ by $\frac{5}{6}$ of $\frac{14}{15}$ of $\frac{20}{21}$. Ans. $\frac{63}{100}$.

15. $\frac{3}{4}$ *of* $\frac{6}{7}$ *of* $\frac{8}{9}$ by $\frac{7}{8}$ of $\frac{9}{10}$ of $\frac{11}{14}$ of $\frac{16}{27}$. Ans. $1\frac{43}{77}$.

PRACTICAL PROBLEMS

IN DIVISION OF FRACTIONS.

1. If $7\frac{3}{5}$ yards of cloth cost $\$47\frac{1}{2}$, what is the price per yard? *Ans.* $\$6\frac{1}{4}$.

2. If $18\frac{1}{7}$ tons of hay cost $\$285\frac{3}{4}$, what is the price per ton? *Ans.* $\$15\frac{3}{4}$.

3. If $23\frac{3}{7}$ bags of coffee cost $410, what is the price per bag? *Ans.* $\$17\frac{1}{2}$.

4. If $17\frac{1}{5}$ barrels of sugar cost $\$238\frac{13}{20}$, what is the price per barrel? *Ans.* $\$13\frac{7}{8}$.

5. If $37\frac{1}{2}$ tons of coal cost $\$253\frac{1}{8}$, what is the price per ton? *Ans.* $\$6\frac{3}{4}$.

6. If $96\frac{15}{16}$ acres of land cost $\$6761\frac{25}{64}$, what is the price per acre? *Ans.* $\$69\frac{3}{4}$.

7. If $86\frac{7}{8}$ cords of wood cost $\$680\frac{25}{48}$, what is the price per cord? *Ans.* $\$7\frac{5}{6}$.

8. What is the value of $\frac{25}{57}\times\frac{42}{65}\times\frac{19}{20}\div\frac{7}{13}$? *Ans.* $\frac{1}{2}$.

9. Of $\frac{396}{792}\times\frac{561}{720}\times\frac{4096}{5499}\div\frac{340}{428}$? *Ans.* $\frac{352}{975}$.

10. Of $\frac{11781}{9801}\times\frac{9900}{10201}\div(\frac{510}{909}\times\frac{6561}{8787})$? *Ans.* $1\frac{403}{2187}$.

11. Of $(\frac{68}{78}+2\frac{5}{11})\div7\frac{1}{2}-3\frac{1}{3}\times8\frac{7}{9}+52\frac{1}{11}$. *Ans.* $52\frac{50856}{282425}$.

12. Of $(2\frac{1}{2}\times2\frac{1}{3}+\frac{5}{9}\text{ of }\frac{7}{16})\times(\frac{8}{9})^3\div(7\frac{7}{9}-3\frac{1}{2}\times\frac{49}{50})$? *Ans.* $\frac{400000}{407511}$.

13. Find the value of $\overline{54\frac{1}{4}-17\frac{7}{10}}\times(6\frac{4}{11}+3\frac{11}{15}-2\frac{1}{3})\div\overline{276\frac{1}{10}-8\frac{1}{15}}$. *Ans.* $1\frac{71}{1210}$.

14. If $\frac{5}{8}$ of $3\frac{3}{7}$ gallons of kerosene cost $\frac{1}{13}$ of $9\frac{3}{4}$, what will 1 gallon cost? *Ans.* $\$\frac{7}{20}$.

15. The product of two numbers is 156, and one of them is $13\frac{13}{16}$; what is the other? *Ans.* $11\frac{5}{17}$.

16. What number is that which being multiplied by $\frac{5}{7}$ of $\frac{49}{55}$ of $9\frac{1}{4}$, will produce $17\frac{7}{20}$? *Ans.* $2\frac{1227}{1295}$.

17. If 4 men plow $20\frac{2}{3}$ acres in 5 days, how much at the same rate would 6 men plow in $7\frac{1}{5}$ days? *Ans.* $44\frac{16}{25}$.

18. Mr. Shaw bought $87\frac{1}{2}$ bushels of corn for $\$109\frac{3}{8}$, and sold $\frac{2}{3}$ of the quantity to Mr. Landis at a profit of $\$\frac{3}{4}$ per bushel; what did he receive for the part sold? *Ans.* $\$116\frac{2}{3}$.

REDUCTION OF COMPLEX FRACTIONS.

249. The **Reduction** of Complex Fractions is the process of changing them to simple fractions.

NOTE.—A complex fraction is not really a fraction, according to the definition of a fraction. It is rather a complex fractional expression of one fraction divided by another.

1. Reduce $\dfrac{\frac{4}{5}}{\frac{8}{9}}$ to a simple fraction.

SOLUTION 1ST.—This fraction means that $\frac{4}{5}$ is to be divided by $\frac{8}{9}$, and inverting the divisor and multiplying, we have $\frac{4}{5} \times \frac{9}{8}$, which equals $\frac{9}{10}$.

OPERATION.

$$\frac{\frac{4}{5}}{\frac{8}{9}} = \frac{4}{5} \div \frac{8}{9} = \frac{4}{5} \times \frac{9}{8} = \frac{9}{10}.$$

SOLUTION 2D.—Multiplying both terms of the complex fraction by 45, the least common multiple of the denominators of the terms, and reducing the resulting fraction to its lowest terms, we have $\frac{9}{10}$.

OPERATION.

$$\frac{\frac{4}{5}}{\frac{8}{9}} = \frac{\frac{4}{5} \times 45}{\frac{8}{9} \times 45} = \frac{36}{40} = \frac{9}{10}.$$

Rule I.—*Multiply the numerator of the complex fraction by its denominator inverted.*

Rule II.—*Multiply both terms of the complex fraction by the least common multiple of the denominators.*

Reduce to simple fractions

2. $\dfrac{\frac{2}{7}}{\frac{2}{9}}$. *Ans.* $1\frac{2}{7}$.

3. $\dfrac{\frac{5}{13}}{\frac{9}{11}}$. *Ans.* $\frac{55}{117}$.

4. $\dfrac{3\frac{1}{2}}{4\frac{1}{3}}$. *Ans.* $\frac{21}{26}$.

5. $\dfrac{6\frac{9}{10}}{9\frac{15}{16}}$. *Ans.* $\frac{184}{265}$.

6. $\dfrac{16\frac{2}{3}}{33\frac{1}{3}}$. *Ans.* $\frac{1}{2}$.

7. $\dfrac{56\frac{1}{4}}{99}$. *Ans.* $\frac{25}{44}$.

8. $\dfrac{\frac{1}{2}-\frac{3}{7}}{\frac{4}{5}-\frac{2}{3}}$. *Ans.* $\frac{15}{28}$.

9. $\dfrac{\frac{3}{11}+\frac{3}{16}}{\frac{3}{22}+\frac{3}{8}}$. *Ans.* $\frac{9}{10}$.

10. $\dfrac{\frac{5}{7}}{\frac{6}{13}}+\dfrac{\frac{4}{7}}{\frac{6}{19}}$. *Ans.* $3\frac{5}{14}$.

11. $\dfrac{5\frac{3}{10}}{1\frac{1}{2}}-\dfrac{1\frac{3}{5}}{\frac{15}{16}}$. *Ans.* $1\frac{62}{75}$.

12. $\dfrac{12\frac{1}{2}}{9\frac{4}{19}}\times\dfrac{17\frac{7}{9}}{7\frac{3}{5}}$. *Ans.* $3\frac{11}{63}$.

13. $\dfrac{49\frac{7}{11}}{33\frac{1}{3}}\div\dfrac{57\frac{1}{11}}{16\frac{2}{3}}$. *Ans.* $\frac{273}{628}$.

14. If 12 be the numerator and $\dfrac{\frac{5}{4}}{2\frac{7}{11}}$ the denominator of a complex fraction, what is its value? *Ans.* $25\frac{17}{55}$.

RELATION OF NUMBERS.

250. The **Relation of Numbers** is their relative value as compared with one another.

NOTE.—This subject is equivalent to Ratio, but is presented here as affording an excellent illustration of the analysis of numbers. The treatment of the subject under Ratio is *demonstrative* rather than *analytic*.

CASE I.

251. *To find the relation of an integer to an integer.*

1. What is the relation of 29 to 7?

SOLUTION.—*One* is $\frac{1}{7}$ of 7, and if 1 is $\frac{1}{7}$ of 7, 29 is 29 times $\frac{1}{7}$ of 7, which are $\frac{29}{7}$, or $4\frac{1}{7}$ times 7. Therefore, 29 is $4\frac{1}{7}$ times 7. Hence we have the following

Rule.—*Divide the number which you compare, by the number with which it is compared.*

NOTE.—The rule is the same for each case, and need not be repeated.

What is the relation of

2. 84 to 7? *Ans.* 12.
3. 138 to 23? *Ans.* 6.
4. 288 to 729? *Ans.* $\frac{32}{81}$
5. 216 to 6561? *Ans.* $\frac{8}{243}$

6. At the rate of 15 apples for 12 cents, what will 45 apples cost?

SOLUTION.—If 15 apples cost 12 cents, 45 apples, which are $\frac{45}{15}$, or 3 times 15 apples, will cost 3 times 12 cents, or 36 cents.

7. If 12 oranges cost 15 cents, what will 84 oranges cost? *Ans.* \$1.05.

8. If 18 pounds of tea cost \$16.20, what will 108 pounds cost? *Ans.* \$97.20.

9. A man, having a farm containing 80 acres, sold 56 acres; what part of his farm remained? *Ans.* $\frac{3}{10}$.

10. A pole whose height was 80 feet, was broken off by the wind 48 feet from the top; what part of the pole was left standing? *Ans.* $\frac{2}{5}$.

11. A has 18 cows, B 24 cows, and C 28 cows; if each obtains $\frac{1}{2}$ of the others, what part of A's will equal B's and C's respectively? *Ans.* B's, $\frac{23}{26}$; C's, $\frac{21}{26}$.

CASE II.

252. *To find the relation of a fraction to a number.*

1. The fraction $\frac{7}{8}$ is what part of 9?

SOLUTION.—*One* is $\frac{1}{9}$ of 9, and if 1 is $\frac{1}{9}$ of 9, $\frac{1}{8}$ is $\frac{1}{8}$ of $\frac{1}{9}$, which is $\frac{1}{72}$, of 9, and $\frac{7}{8}$ is 7 times $\frac{1}{72}$, or $\frac{7}{72}$ of 9.

What is the relation of

2. $\frac{6}{11}$ to 24?	*Ans.* $\frac{1}{44}$.	5. $\frac{41}{49}$ to 12?	*Ans.* $\frac{41}{588}$.
3. $\frac{19}{20}$ to 95?	*Ans.* $\frac{1}{100}$.	6. $\frac{98}{117}$ to 14?	*Ans.* $\frac{7}{117}$.
4. $\frac{16}{17}$ to 112?	*Ans.* $\frac{1}{119}$.	7. $\frac{105}{121}$ to 35?	*Ans.* $\frac{3}{121}$.

8. If I give away $\frac{3}{4}$ of \$15, what part is that of my brother's money, if he has \$35? *Ans.* $\frac{9}{28}$.

CASE III.

253. ***To find the relation of a number to a fraction.***

1. What is the relation of 7 to $\frac{5}{6}$?

SOLUTION.—$\frac{1}{6}$ is $\frac{1}{5}$ of $\frac{5}{6}$, and if $\frac{1}{6}$ is $\frac{1}{5}$ of $\frac{5}{6}$, $\frac{6}{6}$ or 1 is 6 times $\frac{1}{5}$ or $\frac{6}{5}$ of $\frac{5}{6}$, and 7 is 7 times $\frac{6}{5}$, or $\frac{42}{5}$, which equals $8\frac{2}{5}$. Therefore, etc.

What is the relation of

2. 96 to $\frac{16}{17}$?	*Ans.* 102.	6. 175 to $3\frac{1}{8}$?	*Ans.* 56.
3. 216 to $\frac{36}{37}$?	*Ans.* 222.	7. 961 to $6\frac{1}{5}$?	*Ans.* 155.
4. 529 to $2\frac{3}{10}$?	*Ans.* 230.	8. 1872 to $7\frac{1}{5}$?	*Ans.* 260.
5. 729 to $\frac{27}{40}$?	*Ans.* 1080.	9. 1020 to $\frac{4}{11}$ of $12\frac{4}{7}$?	*Ans.* $223\frac{1}{8}$.

10. A has 56 bushels of wheat, and B $\frac{5}{6}$ as many, +4 bushels; how many times B's number equals A's?

Ans. $1\frac{2}{19}$.

CASE IV.

254. ***To find the relation of a fraction to a fraction.***

1. What part of $\frac{15}{16}$ is $\frac{5}{8}$?

SOLUTION.—$\frac{1}{16}$ is $\frac{1}{15}$ of $\frac{15}{16}$, and $\frac{16}{16}$, or *one*, is 16 times $\frac{1}{15}$, which equals $\frac{16}{15}$ of $\frac{15}{16}$. If 1 equals $\frac{16}{15}$ of $\frac{15}{16}$, $\frac{1}{8}$ equals $\frac{1}{8}$ of $\frac{16}{15}$, and $\frac{5}{8}$ equals 5 times $\frac{1}{8}$ of $\frac{16}{15}$, which equals $\frac{5}{8}$ of $\frac{16}{15}$, or $\frac{2}{3}$.

What is the relation of

2. $\frac{14}{15}$ to $\frac{42}{75}$?	*Ans.* $1\frac{2}{3}$.	5. $9\frac{10}{13}$ to $12\frac{9}{26}$?	*Ans.* $\frac{254}{321}$.
3. $\frac{57}{64}$ to $\frac{19}{24}$?	*Ans.* $1\frac{1}{8}$.	6. $17\frac{15}{29}$ to $35\frac{10}{87}$?	*Ans.* $\frac{1524}{3055}$.
4. $\frac{29}{32}$ to $\frac{59}{64}$?	*Ans.* $\frac{58}{59}$.	7. $87\frac{16}{21}$ to $29\frac{17}{63}$?	*Ans.* $2\frac{1841}{1844}$.

8. James, having $\frac{6}{13}$ of a peck of walnuts, sold $\frac{5}{11}$ of what he had; what part of $\frac{6}{13}$ of a peck remained? *Ans.* $\frac{6}{11}$.

9. A merchant sold $\frac{1}{5}$ of $\frac{5}{8}$ of his stock in a month; what part of $\frac{5}{8}$ of his stock remained? *Ans.* $1\frac{2}{5}$.

10. A and B had each $\frac{15}{16}$ of a ton of hay; A sold B $\frac{1}{8}$ of what he had; what part of B's equals A's? *Ans.* $\frac{7}{9}$.

GREATEST COMMON DIVISOR.

255. The **Greatest Common Divisor** of two or more fractions is the greatest fraction that will exactly divide each of them.

PRINCIPLES.

1. *A fraction is a divisor of a given fraction when its numerator is a divisor of the given numerator, and its denominator is a multiple of the given denominator.*

To divide by a fraction we divide by its numerator and multiply by its denominator; hence, to obtain an integral quotient, the numerator of the divisor must divide the given numerator, and the denominator of the divisor must contain the given denominator. Illustrate with $\frac{8}{9}$ and $\frac{4}{27}$.

2. *A common divisor of several fractions is a fraction whose numerator is a common divisor of their numerators, and whose denominator is a common multiple of their denominators.*

3. *The greatest common divisor of several fractions is a fraction whose numerator is the greatest common divisor of their numerators, and whose denominator is the least common multiple of their denominators.*

1. Find the greatest common divisor of $\frac{3}{4}$, $\frac{6}{7}$, $\frac{9}{14}$.

OPERATION.

$\frac{3}{4} - \frac{6}{7} - \frac{9}{14}$.
$3 =$ G. C. D. of Num's
$28 =$ L. C. M. of Den's.
$\therefore$ G. C. D $= \frac{3}{28}$, *Ans.*

SOLUTION.—To be a divisor of each of these fractions the numerator must divide each of the given numerators, and the denominator must contain each of the given denominators; hence, the *greatest* common divisor must be a fraction whose numerator is the *greatest* common divisor of the given numerators, and the denominator the *least* common multiple of the given denominators. The greatest common divisor of the numerators is 3, and the least common multiple of the denominators is 28; hence the greatest common divisor of the given fractions is $\frac{3}{28}$.

Rule.—*Reduce the given fractions to simple ones in their lowest terms; then find the G. C. D. of the numerators and divide it by the L. C. M. of the denominators.*

Find the greatest common divisor of

2. $\frac{3}{11}$, $\frac{7}{20}$, $\frac{14}{15}$. *Ans.* $\frac{1}{660}$.

3. $\frac{5}{9}$, $\frac{20}{81}$, $\frac{15}{2}$. *Ans.* $\frac{5}{162}$.

4. $3\frac{3}{4}$, $7\frac{1}{2}$, $\frac{27}{128}$. *Ans.* $\frac{3}{128}$.

5. $5\frac{6}{11}$, $7\frac{19}{20}$, $19\frac{3}{40}$. *Ans.* $\frac{1}{440}$.

6. $\frac{2}{3}$ of $\frac{4}{5}$, $\frac{4}{15}$ of $\frac{17}{18}$, $\frac{5}{9}$ of $\frac{4}{25}$. *Ans.* $\frac{2}{135}$.

7. $\dfrac{6\frac{1}{2}}{3\frac{1}{3}}$, $\dfrac{7\frac{3}{4}}{6\frac{1}{8}}$, $\dfrac{16\frac{2}{3}}{50}$, $\dfrac{66\frac{2}{3}}{62\frac{1}{2}}$. *Ans.* $\frac{1}{2940}$

8. A farmer has $51\frac{1}{4}$ bushels of russets, $71\frac{3}{4}$ bushels of rambos, $143\frac{1}{2}$ bushels of seek-no-farthers, and $35\frac{7}{8}$ bushels of pearmains; required the largest bins of equal size which can be filled, each kind being kept by itself; also the number of bins. *Ans.* $5\frac{1}{8}$ bushels; 59 bins.

9. Mr. Johnson has four fields in the outskirts of a growing Western city, containing respectively $6\frac{3}{5}$ acres, $7\frac{7}{10}$ acres, $10\frac{1}{2}$ acres, $8\frac{2}{5}$ acres, which he wishes to divide into the largest possible house-lots of equal size; what will be the size and number of the lots? *Ans.* $\frac{1}{10}$ of an acre; 332 lots.

10. Mr. Smith has a field whose sides are $335\frac{1}{4}$ feet, $397\frac{1}{8}$ feet, $322\frac{5}{6}$ feet, and $235\frac{11}{12}$ feet. He wishes to build a fence round it, 5 rails high, the rails overlapping $\frac{5}{6}$ of a foot; what is the longest rail that can be used, and how many rails will be required? *Ans.* $13\frac{1}{4}$ ft.; 520 rails.

LEAST COMMON MULTIPLE.

256. The **Least Common Multiple** of two or more fractions is the least number that will exactly contain each of them.

PRINCIPLES.

1. *A multiple of a fraction is a fraction whose numerator is a multiple of the given numerator, and whose denominator is a divisor of the given denominator.*

To divide by a fraction, we divide by its numerator and multiply by its denominator; hence, to give an integral quotient, when we divide a multiple by a fraction, the numerator of the multiple must contain the numerator of the fraction, and the denominator of the multiple must divide the denominator of the fraction. Illustrate with $\frac{8}{5}$, a multiple of $\frac{4}{25}$.

2. *A common multiple of several fractions is a fraction whose numerator is a common multiple of their numerators, and whose denominator is a common divisor of their denominators.*

3. *The least common multiple of several fractions is a fraction whose numerator is the least common multiple of their numerators, and whose denominator is the greatest common divisor of their denominators.*

For, that common multiple is the smallest which has the smallest numerator and the largest denominator.

1. Find the least common multiple of $\frac{7}{8}$, $\frac{35}{32}$, $\frac{21}{40}$.

SOLUTION.—To be a multiple of each of these fractions, the numerator must contain each of the given numerators, and the denominator divide each of the given denominators; hence the *least* common multiple must be a fraction whose numerator is the least common multiple of the given numerators, and whose denominator is the greatest common divisor of the given denominators. The least common multiple of the numerators we find to be 105, and the greatest common divisor of the denominators is 8; hence $\frac{105}{8}$, or $13\frac{1}{8}$, is the least common multiple of the given fractions.

OPERATION.

$\frac{7}{8} - \frac{35}{32} - \frac{21}{40}$
L. C. M. of Num. = 105
G. C. D. of Den. = 8
$\therefore$ L. C. M. $= \frac{105}{8} = 13\frac{1}{8}$

Rule.—*Reduce the fractions to simple ones in their lowest terms; then find the L. C. M. of the numerators, and divide it by the G. C. D. of the denominators.*

Find the least common multiple of

2. $\frac{6}{7}$, $\frac{5}{14}$, $\frac{4}{21}$. *Ans.* $8\frac{4}{7}$.	5. $6\frac{2}{21}$, $7\frac{11}{35}$, $11\frac{11}{63}$. *Ans.* $402\frac{2}{7}$.
3. $\frac{7}{11}$, $\frac{2}{33}$, $\frac{20}{77}$. *Ans.* $12\frac{8}{11}$.	6. $8\frac{4}{9}$, $5\frac{5}{18}$, $16\frac{5}{27}$. *Ans.* $971\frac{1}{9}$.
4. $\frac{24}{35}$, $\frac{36}{25}$, $\frac{8}{15}$, $\frac{7}{20}$. *Ans.* $100\frac{4}{5}$.	7. $6\frac{3}{5}$, $7\frac{7}{10}$, $9\frac{7}{20}$, $10\frac{3}{25}$. *Ans.* $18064\frac{1}{5}$.

8. The Earth, Mars, and Saturn were in conjunction December, 1875; when will they be again in conjunction at the same point of their orbits, the period of revolution of Mars being $1\frac{8}{9}$ years and of Saturn $29\frac{1}{2}$ years? *Ans.* In 1003 yr.

9. A man has a square lot which he wishes to fence, and has rails of four different lengths, namely, $12\frac{3}{8}$ feet, $12\frac{1}{2}$ feet, $13\frac{1}{2}$ feet, and $13\frac{3}{4}$ feet, and not enough of either to fence any two sides of the lot; what was the smallest possible side of the lot? *Ans.* $3712\frac{1}{2}$ ft.

10. A, B, and C start at the same place and travel round an island, A making the circuit in $\frac{2}{3}$ of a day, B in $\frac{4}{9}$ of a day, and C in $\frac{5}{6}$ of a day; in how many days will they meet at the starting place, and how many times will each have gone round the island? *Ans.* $6\frac{2}{3}$ days.

11. A, B, C, and D start at the same place and travel round an island 72 miles in circumference, A traveling $2\frac{1}{2}$ miles an hour, B $3\frac{1}{3}$ miles an hour, C $3\frac{3}{4}$ miles an hour, D $4\frac{2}{7}$ miles an hour; how many days before they meet at the starting place, if they travel 10 hours a day, and how far will each have traveled? *Ans.* $120\frac{24}{25}$ days; A, 42 times, etc.

MISCELLANEOUS PROBLEMS.

1. If $6\frac{3}{7}$ barrels of flour cost $\$51\frac{3}{7}$, what will $4\frac{3}{8}$ barrels cost? *Ans.* \$35.

2. If $9\frac{3}{8}$ pounds of sugar cost \$2.25, what will $12\frac{3}{4}$ pounds cost? *Ans.* \$3.06.

3. If $5\frac{3}{8}$ tons of hay cost $\$28\frac{2}{3}$, how many tons will $\$85\frac{1}{3}$ buy? *Ans.* 16 tons.

4. The sum of two fractions is $\frac{219}{365}$, and one is $\frac{123}{219}$; what is the other? *Ans.* $\frac{14}{365}$.

5. The difference of two fractions is $\frac{111}{211}$, and the greater is $\frac{1684}{1899}$; what is the less? *Ans.* $\frac{685}{1899}$.

6. The multiplicand is $\frac{23}{18}$, and product $\frac{113}{54}$; required the multiplier. *Ans.* $1\frac{44}{69}$.

7. The divisor is $\frac{88}{315}$, and quotient $\frac{252}{143}$; what is the dividend? *Ans.* $\frac{32}{65}$.

8. The dividend is $\frac{70}{182}$, and quotient $\frac{84}{99}$; required the divisor. *Ans.* $\frac{5}{8}$.

9. Divide the fraction $\frac{17}{18}$ into two parts, one of which is $2\frac{1}{2}$ times the other. *Ans.* $\frac{17}{63}$; $\frac{85}{126}$.

10. The sum of two fractions is $\frac{8}{9}$, and difference $\frac{1}{7}$; required the fractions. *Ans.* $\frac{65}{126}$; $\frac{47}{126}$.

11. One-half of the sum of two fractions is $\frac{393}{440}$, and twice the difference is $\frac{6}{270}$; required the fractions.

Ans. $\frac{3559}{3960}$; $\frac{703}{792}$.

12. What is the value of $(\frac{3}{4}+\frac{5}{6}-\frac{7}{8})\times(\frac{4}{5}+\frac{6}{8}-\frac{9}{10})$, divided by $3\frac{2}{5}$? *Ans.* $\frac{13}{96}$.

13. What is the value of $(5\frac{7}{8}-\frac{3}{5}+2\frac{7}{10})\div(3\frac{1}{2}-1\frac{3}{5}+2\frac{1}{4})$ multiplied by $2\frac{1}{2}$ divided by $1\frac{3}{4}$? *Ans.* $2\frac{433}{581}$.

14. Divide $\frac{3}{4}$ of $\frac{8}{11}$ of $7\frac{3}{7}$ by $\frac{5}{9}$ of $\frac{18}{19}$ of $5\frac{4}{7}$. *Ans.* $1\frac{21}{55}$.

15. Multiply $\frac{8}{11}$ of $\dfrac{5\frac{1}{2}}{16}$ of $\dfrac{31\frac{1}{4}}{281\frac{1}{2}}$ by $\frac{3}{7}$ of $56\frac{3}{10}$ of $54\frac{6}{11}$.

Ans. $36\frac{81}{154}$.

16. Subtract $\frac{1}{2}$ of $\dfrac{4\frac{1}{11}}{17\frac{1}{7}}$ from $\frac{3}{10}$ of $\dfrac{19\frac{3}{7}}{5\frac{2}{3}}$. *Ans.* $\frac{5601}{6160}$.

17. Add $\dfrac{3\frac{1}{2}}{7}$ of $41\frac{1}{4}$ of $3\frac{1}{15}$, $\dfrac{27\frac{1}{4}}{3\frac{3}{4}}$ of $\frac{4}{11}$ of $\frac{21}{109}$, and $\dfrac{67\frac{1}{2}}{15}$.

Ans. $68\frac{57}{220}$.

18. Find the value of $\left(\frac{\frac{1}{7}+\frac{2}{11}}{8\frac{1}{3}}+\frac{7\frac{7}{9}}{6\frac{5}{12}}\right)\div 4\times\frac{5}{8\frac{5}{9}}$. *Ans.* $\frac{289}{546}$

19. Find the value of $\frac{1}{4}$ of $\frac{1-\frac{1}{7}}{2}\times\frac{2-\frac{1}{7}}{3}\times\frac{3-\frac{1}{7}}{4}\times\frac{4-\frac{1}{7}}{5}$.

Ans. $\frac{351}{9604}$.

20. Find the value of $\frac{5\frac{1}{9}\times 5\frac{1}{9}\times 5\frac{1}{9}\times 5\frac{1}{9}-1}{5\frac{1}{9}\times 5\frac{1}{9}\times 5\frac{1}{9}-1}+\frac{26}{869468}$.

Ans. $5\frac{1962}{13801}$.

21. Find the value of $\frac{2-\frac{1}{2}}{4}\times\frac{(3\frac{3}{5})^2}{7}+\overline{(2+\frac{3}{11})\div 3+\frac{10}{11}}+\frac{2\frac{2}{3}}{5\frac{2}{15}}$.

Ans. $2\frac{10169}{11550}$.

22. Find the value of $\left(\frac{2\frac{7}{11}}{3\frac{17}{33}}+\frac{4\frac{1}{5}}{7\frac{14}{25}}-\frac{5\frac{11}{20}}{62\frac{7}{16}}\right)\times 4\,\frac{\frac{3\frac{11}{17}}{4\frac{3}{7}}}{8}\div\frac{\frac{1}{20}+\frac{28}{49}-\frac{1}{10}}{4\frac{3}{7}\times 5\frac{1}{2}\div 200\frac{10}{17}}$.

Ans. 3.

23. What is the sum of $\frac{1}{2}$, $\frac{2}{3}$, $\frac{4}{5}$, $\frac{5}{6}$, $\frac{7}{8}$, $\frac{9}{10}$, $\frac{11}{12}$, $\frac{13}{14}$, $\frac{15}{16}$, and $\frac{17}{18}$? *Ans.* $8\frac{1528}{5040}$.

24. A man sold $\frac{2}{5}$ of $\frac{10}{11}$ of his bank stock in a month; how many fifths of $\frac{10}{11}$ remained? *Ans.* $3\frac{1}{2}$ fifths.

25. If I pay \$0.62$\frac{1}{2}$ a cord for sawing wood 4 feet long into 3 pieces, how much more should I pay for sawing wood 8 feet long into pieces of the same length? *Ans.* \$0.15$\frac{5}{8}$.

26. A dry goods merchant bought silk for \$613$\frac{1}{8}$ at \$1$\frac{7}{8}$ a yard, and sold $\frac{1}{8}$ of the quantity bought at a profit of $\frac{2}{3}$ of a dollar a yard; what did he receive for the part sold?

Ans. \$277$\frac{1}{24}$.

27. A bought of B 17$\frac{1}{2}$ tons of hay at \$11$\frac{2}{3}$ a ton, and of C 22$\frac{1}{2}$ tons at \$12$\frac{1}{4}$ a ton, and then sold D 15 tons at \$13$\frac{1}{3}$ a ton, and the remainder E bought at \$13$\frac{3}{8}$ a ton; what was A's gain? *Ans.* \$54$\frac{7}{12}$.

28. Required the least number of yards of velvet, allowing 1 yard for waste, that can be cut up without loss into either bonnets or hats, one style of bonnet requiring 1$\frac{1}{4}$ yd., and another $\frac{7}{8}$ yd., and a hat requiring $\frac{5}{6}$ yd. *Ans.* 36 yd.

29. A grocer bought 25 barrels of apples at \$4$\frac{2}{3}$ a barrel; he sold Mr. Smith $\frac{2}{5}$ of them at \$5$\frac{1}{4}$, but finding they were

beginning to spoil, and wishing to get rid of them, he sold the remainder to Mr. Brown at \$4 a barrel; what did he gain or lose by the whole transaction? *Ans.* Lost \$$4\frac{1}{8}$.

30. Samuel Jackson agreed to work for a farmer a year, receiving as wages \$300 and a suit of clothes. Having worked 8 months, his employer sold his farm, and Jackson received as his pay \$$186\frac{2}{3}$ and the clothes; what was the value of the suit? *Ans.* \$40.

31. Three men start at the same time to walk around a circular race-course 80 rods in circumference, the first walking 26 rods, the second 35 rods, and the third 50 rods a minute; when are they first together after starting, and how far from the starting point? *Ans.* $26\frac{2}{3}$ min.; $53\frac{1}{3}$ rods.

32. A steamboat starts from Memphis, Tenn., to go up the Missouri River to a point $1011\frac{1}{7}$ miles from the starting place. Her rate is $10\frac{1}{5}$ miles an hour for $12\frac{1}{2}$ hours a day, anchoring at night for fear of snags; but when the voyage is half completed, the anchor is lost, and she then drifts back every night at the rate of $1\frac{3}{4}$ miles per hour; how many days did the voyage require? *Ans.* $8\frac{1718}{2380}$ days.

33. In a piece of machinery there are 3 wheels, A, B, and C, each measuring $11\frac{2}{3}$ feet in circumference, their axles being in a straight line. If these wheels begin to revolve, A at the rate of $6\frac{1}{2}$ feet in a second, B $7\frac{1}{4}$ feet, and C $9\frac{2}{3}$ feet, how long before the given points will again be in a straight line, and how many revolutions will each wheel have made?

Ans. 140 sec.; A, 78 rev.; B, 87; C, 116.

34. Three men were employed to plow a field; the first plowed a furrow in $17\frac{1}{7}$ minutes, the second in $23\frac{4}{7}$ minutes, and the third in $26\frac{1}{14}$ minutes, and it so happened that they all came to the end of their furrow at the same moment for the first time when the work was finished. How long did they work, how many furrows did they plow, and how much should each receive, if \$65.10 was paid for the work?

Ans. $4714\frac{2}{7}$ min.; 651; 1st, \$27.50; 2d, \$20; 3d, \$17.60.

SECTION V.

DECIMAL FRACTIONS.

257. A Decimal Fraction is a number of the decimal divisions of a unit

258. A Decimal Division of a unit is a *tenth*, a *hundredth*, a *thousandth*, etc. A decimal fraction is thus a number of tenths, hundredths, etc.

259. A Decimal Fraction is usually expressed by placing a point before the numerator and omitting the denominator; thus .5 expresses $\frac{5}{10}$.

260. The **Symbol** of a decimal is the period, called the *decimal point*, or *separatrix*. It indicates the decimal and *separates* decimals and integers.

261. The places at the right of the decimal point are called *decimal places*. The first place to the right of the point is *tenths*, the second place is *hundredths*, etc.

262. This method of expressing decimal fractions arises from the decimal scale used for integers by continuing it to the right of units.

Thus, since *tens* is 1 tenth of *hundreds*, and *units* 1 tenth of tens, if we write a figure to the right of units it will express 1 tenth of units or *tenths;* two places to the right, 1 tenth of tenths or *hundredths*, etc.

263. This beautiful law, as applied to the expression of integers and decimal fractions, is exhibited in the following

DECIMAL NOTATION TABLE.

Hund. Millions.	Ten Millions.	Millions.	Hund.-Thous'ds.	Ten-Thousands.	Thousands.	Hundreds.	Tens.	Units.	*Decimal Point.*	Tenths.	Hundredths.	Thousandths.	Ten Thous'ths.	Hund-Thous.	Millionths.	Ten-Millionths.	Hund.-Mill'ths.	Billionths.
3	3	3	3	3	3	3	3	3	.	3	3	3	3	3	3	3	3	3
9th.	8th.	7th.	6th.	5th.	4th	3d.	2d.	1st.		1st.	2d.	3d.	4th.	5th.	6th.	7th.	8th.	9th.

$\underbrace{\text{9th. … 1st.}}_{\textit{Integers.}}$ $\qquad$ $\underbrace{\text{1st. … 9th.}}_{\text{Decimals.}}$

264. A **Decimal** is a fraction expressed by the decimal notation; thus .5 is a *decimal,* while $\frac{5}{10}$ is a *decimal fraction.*

265. A **Pure Decimal** is one which consists of decimal figures only; as, .25 and .345.

266. A **Mixed Decimal** is one which consists of an integer and a decimal; as, 6.75.

267. A **Complex Decimal** is one which contains a common fraction at the right of the decimal; as, .34$\frac{2}{7}$.

268. A **Terminate Decimal** is one which ends; an *Interminate* is one which does not end.

NOTES.—1. Decimals may originate by passing from common fractions to decimals, or by an extension of the decimal scale to the right of units.

2. Decimal fractions appear to have been first used by Regiomontanus, about the year 1464. The first treatise upon the subject was written by Stevinus, published in 1585.

3. The decimal point, Dr. Peacock thinks, was introduced by Napier, the inventor of logarithms, in 1617; though De Morgan says that Richard Witt made as near an approach to it as Napier.

PRINCIPLES OF DECIMAL NOTATION.

1. *Moving the decimal point one place to the right, multiplies the decimal by* 10*; two places, multiplies by* 100, *etc.*

For, if the point be moved one place to the right, each figure will express ten times as much as before, hence the whole decimal will be ten times as great; etc.

2. *Moving the decimal point one place to the left, divides the decimal by* 10*; two places, divides by* 100, *etc.*

For, if the point be moved one place to the left, each figure will express 1 tenth of its previous value, hence the whole decimal will be only 1 tenth as great; etc.

3. *Placing a cipher between the decimal point and the decimal, divides the decimal by* 10.

For, this moves each figure one place to the right in the scale, in which case they express 1 tenth as much as before, and hence the decimal is only 1 tenth as great.

4. *Annexing ciphers to the right of a decimal, does not change its value.*

For, each figure retains the same place as before, and hence expresses the same value as before, and consequently the value of the decimal is unchanged.

EXAMPLES IN NUMERATION.

1. Read the decimal .685.

Solution.—This expresses 6 tenths, 8 hundredths, and 5 thousandths, or since 6 tenths equals 600 thousandths, and 8 hundredths equals 80 thousandths, and all united equal 685 thousandths, it may also be read 685 thousandths; hence the following rules:

Rule I.—*Begin at tenths, and read the terms in order towards the right, giving each its proper denomination.*

Rule II.—*Read the decimal as a whole number, and give it the denomination of the last term at the right.*

Note.—In the second method we may determine the denominator by numerating *from* the decimal point, and the numerator by numerating *towards* the decimal point.

Read the following decimals:

2. .73.	**6.** 7.039.	**10.** 146.0302056.
3. .24$\frac{1}{2}$.	**7.** 8.1367.	**11.** 376.10070354.
4. 3.70$\frac{2}{8}$.	**8.** 7.0308$\frac{3}{4}$.	**12.** 487.000081035.
5. 2.00$\frac{4}{5}$.	**9.** 9.1007$\frac{5}{6}$.	**13.** 586.0004003256.

EXAMPLES IN NOTATION.

1. Express 45 thousandths in the form of a decimal.

Solution 1st.—45 thousandths equals 40 thousandths plus 5 thousandths, or 4 hundredths and 5 thousandths; hence we write the 5 in the third or thousandths place, the 4 in the second or hundredths place, and fill the vacant tenths place with a cipher, and we have .045.

OPERATION.
45 thousandths
=.045. *Ans.*

Solution 2d.—We write the 45 and then, since the last figure must stand in the third or thousandths place, the denomination being thousandths, write a cipher before the 4 and place the decimal point before it, and we have .045.

Rule I.—*Place the decimal point, and then write each term so that it may express its proper denomination, using ciphers when necessary.*

Rule II.—*Write the numerator, and then place the decimal point so that the right hand term shall be of the same denomination as the decimal.*

Express the following in decimal form:

2. Four hundred and seventy-five thousandths.

3. Seven thousand four hundred *and sixty-five ten-thousandths.*

4. 9 tenths, 8 thousandths, and 7 millionths.

5. Five thousand and one millionths.

6. Six hundred, and seven hundredths.

7. Fifty thousand seven hundred and six millionths.

8. 9 thousandths, 6 hundred-thousandths, and two hundred-millionths.

9 One hundred and one thousand one hundred and one ten-millionths.

10. Two hundred and forty thousand, four hundred and six thousandths.

11. Six hundred and fifty-seven thousand, 4 hundred and forty-eight and seven-ninths millionths.

12. Nine hundred and twenty-six million, 4 thousand and 7 hundred millionths.

13. Four thousand and thirty-nine tenths.

14. Fifty-six million, and fifty-six millionths.

15. Four thousand and two and one-fifth hundredths.

16: 6 ten-thousandths, 5 millionths, and $3\frac{17}{25}$ billionths.

Express the following fractions and mixed numbers decimally:

15. $\frac{848}{10}$, $\frac{743}{100}$, $\frac{45654}{1000}$.

16. $\frac{2}{10000}$, $\frac{46}{100000}$.

17. $24\frac{475}{1000}$, $4\frac{1}{100}$.

18. $\frac{63\frac{7}{8}}{100}$, $\frac{42\frac{3}{7}}{1000}$.

19. $\frac{438\frac{1}{4}}{10}$, $\frac{7\frac{1}{8}}{100000}$.

20. $\frac{643\frac{4}{7}}{10000}$, $\frac{48750068}{1000000000}$.

21. $\frac{4782465}{1000}$, $945\frac{825}{10000000}$.

22. $\frac{42}{10000000}$, $\frac{401\frac{3}{13}}{100000000}$.

REDUCTION OF DECIMALS.

269. The **Reduction of Decimals** consists of three cases, as follows:

1st. To reduce decimals to common fractions.

2d. To reduce common fractions to decimals.

3d. To reduce decimals to a common denominator.

CASE I.

270. *To reduce a decimal to a common fraction.*

1. Reduce .75 and also $.16\frac{2}{3}$ to a common fraction.

SOLUTION.—.75 expressed in the form of a common fraction equals $\frac{75}{100}$, which reduced to its lowest terms, becomes $\frac{3}{4}$.

OPERATION.

$.75 = \frac{75}{100} = \frac{3}{4}$, *Ans.*

SOLUTION.—$.16\frac{2}{3}$ is $16\frac{2}{3}$ hundredths, which, by writing the denominator, becomes $\frac{16\frac{2}{3}}{100}$ which equals $\frac{\frac{50}{3}}{100}$, or $\frac{50}{300}$, which, reduced to its lowest *terms*, equals $\frac{1}{6}$. Hence the following

OPERATION.

$16\frac{2}{3} = \frac{16\frac{2}{3}}{100} = \frac{\frac{50}{3}}{100}$
$= \frac{50}{300} = \frac{1}{6}$, *Ans.*

Rule.—*Write the denominator under the decimal, omitting the decimal point, and reduce the common fraction to its lowest terms.*

Reduce the following to common fractions:

2. .125.	*Ans.* $\frac{1}{8}$.	12. .83$\frac{1}{3}$.	*Ans.* $\frac{5}{6}$.
3. .3125.	*Ans.* $\frac{5}{16}$.	13. .93$\frac{3}{4}$.	*Ans.* $\frac{15}{16}$.
4. .73125.	*Ans.* $\frac{117}{160}$.	14. .08$\frac{1}{3}$.	*Ans.* $\frac{1}{12}$.
5. 7.375.	*Ans.* $7\frac{3}{8}$.	15. .06$\frac{2}{3}$.	*Ans.* $\frac{1}{15}$.
6. 5.008.	*Ans.* $5\frac{1}{125}$.	16. 2.06$\frac{1}{4}$.	*Ans.* $2\frac{1}{16}$.
7. 7.555$\frac{5}{9}$.	*Ans.* $7\frac{5}{9}$.	17. 3.43$\frac{3}{4}$.	*Ans.* $3\frac{7}{16}$.
8. 8.25625.	*Ans.* $8\frac{41}{160}$.	18. 4.00$\frac{2}{3}$.	*Ans.* $4\frac{1}{150}$.
9. 7.46875.	*Ans.* $7\frac{15}{32}$.	19. 5.00$\frac{5}{6}$.	*Ans.* $5\frac{1}{120}$.
10. 9.65625.	*Ans.* $9\frac{21}{32}$.	20. 6.10$\frac{2}{3}$.	*Ans.* $6\frac{8}{75}$.
11. 14.75325.	*Ans.* $14\frac{3013}{4000}$.	21. 7.060$\frac{5}{9}$.	*Ans.* $7\frac{91}{1500}$.

CASE II.

271. ***To reduce a common fraction to a decimal.***

1. Reduce $\frac{7}{8}$ to a decimal.

SOLUTION.—$\frac{7}{8}$ equals $\frac{1}{8}$ of 7. 7 equals 70 tenths; $\frac{1}{8}$ of 70 tenths is 8 tenths and 6 tenths remaining: 6 tenths equal 60 hundredths; $\frac{1}{8}$ of 60 hundredths is 7 hundredths and 4 hundredths remaining: 4 hundredths equal 40 thousandths; $\frac{1}{8}$ of 40 thousandths is 5 thousandths. Therefore $\frac{7}{8}$ equals .875.

OPERATION.

$\frac{7}{8} = \frac{1}{8}$ of $7 =$

$8\,)\,7.000$

$.875$

Rule.—I. *Annex ciphers to the numerator and divide by the denominator.*

II. *Point off as many decimal places in the quotient as there are ciphers used.*

NOTES.—1. In many cases the division will not terminate; the common fraction cannot then be exactly expressed by a decimal. Such decimals are called *interminate* or *infinite* decimals.

2. The symbol + annexed to a decimal, indicates that it contains other decimal terms. The symbol — annexed to a decimal indicates that the last decimal term is increased by 1. This is often done when the next term is greater than 5.

Reduce the following common fractions to decimals:

2. $\frac{15}{16}$.	*Ans.* .9375.	8. $\frac{74}{97}$.	*Ans.* .7628866—
3. $\frac{15}{32}$.	*Ans.* .46875.	9. $\frac{201}{256}$.	*Ans.* .78515625.
4. $\frac{21}{32}$.	*Ans.* .65625.	10. $\frac{125}{1024}$.	*Ans.* .1220703125.
5. $\frac{51}{64}$.	*Ans.* .796875.	11. $\frac{1}{15}$.	*Ans.* .06$\frac{2}{3}$.
6. $\frac{11}{29}$.	*Ans.* 3793103+.	12. $\frac{1}{120}$.	*Ans.* .0083$\frac{1}{3}$.
7. $\frac{41}{47}$.	*Ans.* .8723404+.	13. $\frac{1}{150}$.	*Ans.* 006$\frac{2}{3}$.

14. $\frac{91}{1500}$. *Ans.* .060$\frac{2}{3}$.
15. 7.0$\frac{4}{25}$. *Ans.* 7.016.
16. 1.20$\frac{7}{45}$. *Ans.* 1.201$\frac{5}{9}$.
17. $\frac{307}{1728}$. *Ans.* .177662$\frac{1}{27}$.
18. 3.00$\frac{1}{160}$. *Ans.* 3.0000625.
19. 5.00$\frac{1}{64}$. *Ans.* 5.00015625.
20. 5.30$\frac{7}{5}$. *Ans.* 5.314.
21. 7.301$\frac{28}{8}$. *Ans.* 7.303875.
22. 6.30$\frac{25}{16}$. *Ans.* 6.315625.
23. 7.300$\frac{28}{8}$. *Ans.* 7.302875.

CASE III.

272. ***To reduce decimals to a common denominator.***

1. Reduce .4, .25, and .875 to a common denominator

SOLUTION.—For the decimals to have a common denominator, they must occupy the same number of decimal places; .875 occupies three places, expressing thousandths; hence each of the other decimals must occupy three places that they may express thousandths. .25 equals .250 and .4 equals .400.

OPERATION.

.875
.250
.400

Rule.—*Annex ciphers to the simple decimals and expand the complex ones so as to make each decimal occupy the same number of decimal places.*

NOTE.—Decimals will be reduced to their *least common denominator* when they are reduced to the same number of places as the decimal which occupies the greatest number of places.

Reduce the following to their least common denominator:

2. .25, .025, .37. *Ans.* .250, .025, .370.
3. .523, 4.36, and 5.0315. *Ans.* .5230, 4.3600, 5.0315.
4. $\frac{3}{8}$, .4036, and 5.0$\frac{16}{25}$. *Ans.* .3750, .4036, 5.0640.
5. .375, $\frac{16}{25}$, and $\frac{17}{32}$. *Ans.* .37500, .64000, .53125.
6. .8135, $\frac{81}{160}$, 5.03$\frac{2}{5}$, and $\frac{17}{64}$.
   Ans. .813500, .506250, 5.034000, .265625.
7. .45302, $\frac{99}{200}$, .015$\frac{1}{4}$, and 2.00$\frac{1}{320}$.
   Ans. .45302000, .49500000, .01525000, 2.00003125.
8. 101.01$\frac{3}{4}$, 42$\frac{3}{16}$, $\frac{9}{1600}$, and $\frac{4}{5}$.
   Ans. 101.017500, 42.187500, .005625, .800000.
9. 75119$\frac{3}{80}$, $\frac{7}{8}$, 101.0175, and .005625.
   Ans. 75119.037500, .875000, 101.017500, .005625.
10. .00097656, $\frac{15}{32}$, .125.
   Ans. .00097656, .46875000, .12500000.
11. .30$\frac{1001}{148000}$, 4.008, 5.78$\frac{10}{32}$, .29167.
   Ans. .300070, 4.008000, 5.783125, .291670

ADDITION OF DECIMALS.

273. Addition of Decimals is the process of finding the sum of two or more decimals.

1. What is the sum of 45.37, 56.508, 75.45, and 86.497?

SOLUTION.—We write the numbers so that terms of the same order shall stand in the same column, and begin at the right to add. 7 thousandths plus 8 thousandths are 15 thousandths, which equals 1 hundredth and 5 thousandths; we write the 5 thousandths, and add the 1 hundredth to the next column: 1 and 9 are 10 and 5 are 15 and 7 are 22 hundredths, which equals 2 tenths and 2 hundredths; we write the 2 hundredths and add the tenths to the next column, etc.

OPERATION.

$$\begin{array}{r} 45.37 \\ 56.508 \\ 75.45 \\ 86.497 \\ \hline 263.825 \end{array}$$

Rule.—I. *Write the number so that terms of the same order stand in the same column.*

II. *Add as in whole numbers, and place the decimal point between the units and tenths of the sum.*

NOTE.—When there are complex decimals, reduce all the decimals to a common denominator before adding.

2. Add 12.34, 432.015, 302.23, .00025. *Ans.* 746.58525.

3. Add 137.4263, 3426.01, 412.003, 3.0005. *Ans.* 3978.4398.

4. Add 6340.205, .000632, 4.73, .00325, .99935. *Ans.* 6345.938232.

5. Add 4.25, $\frac{3}{8}$, 463.2504, 5.0$\frac{16}{25}$, .4036. *Ans.* 473.343.

6. Add .000432, 400.25, 72.00$\frac{16}{25}$, $\frac{17}{32}$, 4.32502. *Ans.* 477.113102.

7. Add 500.0006, $\frac{81}{160}$, 5.03$\frac{2}{5}$, .7654, .001. *Ans.* 506.30725.

8. Add .4532, 7.00$\frac{3}{4}$, 1005.700$\frac{14}{25}$, $\frac{5}{8}$, .000$\frac{1}{5}$. *Ans.* 1013.78646.

9. Add 60$\frac{1001}{148000}$, 50.305, 6850.275, $\frac{16}{25}$, .0000$\frac{4}{25}$. *Ans.* 6961.227016.

10. Add .432758, .2$\frac{1}{2}$, .29999997, .00000003. *Ans.* .982758.

11. Add .22$\frac{2}{9}$+.3333$\frac{1}{3}$+.444444$\frac{4}{9}$. *Ans.* 1.

12. Find the sum of 2 decimal units of the 2d order, 2$\frac{1}{2}$ of the 3d order, 4$\frac{1}{5}$ of the 4th, 3$\frac{1}{8}$ of the 5th, 5$\frac{1}{16}$ of the 6th, and 9$\frac{3}{8}$ of the 7th order. *Ans.* .02295725.

SUBTRACTION OF DECIMALS.

274. Subtraction of Decimals is the process of finding the difference between two decimals.

1. From 853.275 subtract 578.437.

OPERATION.
853.275
578.437
274.838

SOLUTION.—We write the numbers so that terms of the same order stand in the same column, and begin at the right to subtract. We cannot subtract 7 thousandths from 5 thousandths, hence we add 10 thousandths to 5 thousandths, which equals 15 thousandths; 7 thousandths from 15 thousandths leaves 8 thousandths, which we write in the order of thousandths: since we have added 10 thousandths, or 1 hundredth, to the minuend, we must add 1 hundredth to the subtrahend; 1 hundredth and 3 hundredths are 4 hundredths; 4 hundredths from 7 hundredths leaves 3 hundredths, etc.

Rule.—I. *Write the subtrahend under the minuend, so that terms of the same order stand in the same column.*

II. *Subtract as in whole numbers, and place the decimal point between the units and tenths of the remainder.*

NOTE.—When there are complex decimals, reduce to a common denominator before subtracting.

2. From 406.375 take 237.00462. *Ans.* 169.37038.
3. From 3462.0004 take 2430.997. *Ans.* 1031.0034.
4. From 1.0003246 take .074532. *Ans.* .9257926.
5. From $22\frac{12}{25}$ take $14.04\frac{2}{5}$. *Ans.* 8.436.
6. From $70.43\frac{3}{4}$ take $\frac{21}{40}$. *Ans.* 69.9125.
7. From 600.4207 take $.346\frac{18}{50}$. *Ans.* 600.07444.
8. From $\frac{991}{100}$ take $.3333\frac{2}{5}$. *Ans.* 9.57666.
9. From $\frac{7}{32}$ take .003125. *Ans.* .215625.
10. From 26 take $15.99999\frac{1}{9}$. *Ans.* $10.00000\frac{8}{9}$.
11. From $3\frac{1}{3}$ hundredths take $3\frac{1}{3}$ hundred-millionths.
Ans. $.033333\dot{3}$.

12. From seven thousand and seventeen millionths take .0004125. *Ans.* .0066045.

13. From six hundred, and forty-five billionths take six hundred and forty-five billionths. *Ans.* 599.9999994.

14. From 9 tenths 4 thousandths and 6 hundred-thousandths take $113\frac{3}{4}$ millionths. *Ans.* .90394625.

15. After subtracting $7\frac{5}{32}$ millionths from $5\frac{3}{4}$ thousandths, how much must be added to the remainder to make $2\frac{7}{8}$ hundredths? *Ans.* .02300715625.

MULTIPLICATION OF DECIMALS.

275. Multiplication of Decimals is the process of finding the product, when one or both factors are decimals.

1. Multiply 4.23 by .36.

OPERATION.

```
  4.23
   .36
  ----
  2538
 1269
------
1.5228
```

SOLUTION 1ST.—4.23 multiplied by 36 equals 152.28; and multiplied by 36 *hundredths* the product is 1 hundredth as great, which by removing the decimal point two places to the left, becomes 1.5228. Hence 4.23 multiplied by .36 equals 1.5228.

SOLUTION 2D.—$4.23 \times .36 = \frac{423}{100} \times \frac{36}{100} = \frac{15228}{10000} = \frac{1}{10000} \times 15228 = 1.5228$. From either of these solutions we derive the following

Rule.—*Multiply as in whole numbers, and from the right of the product point off as many decimal places as there are in both factors, prefixing ciphers when necessary.*

NOTE.—In complex decimals, we may expand, or multiply by using the common fraction, or even reduce to a common fraction before multiplying.

2. Multiply 108.0158 by 21.216. *Ans.* 2291.6632128.
3. Multiply 4.1418 by .000492. *Ans.* .0020377656.
4. Multiply $64.66\frac{2}{3}$ by .00018. *Ans.* .01164.
5. Multiply 27 hundredths by $.4\frac{1}{5}$. *Ans.* .1134.
6. Multiply 42.075 by $13.33\frac{1}{3}$. *Ans.* 561.
7. Multiply $.06\frac{3}{4}$ by .0625. *Ans.* .00421875.
8. Multiply 36 units by 36 tenths. *Ans.* 129.6.
9. Multiply $4\frac{1}{2}$ hundredths by 24 hundreds. *Ans.* 108.
10. Multiply 360 hundredths by 50 tenths. *Ans.* 18.
11. Multiply .2002 by 8.008. *Ans.* 1.6032016.
12. Multiply $63.11\frac{1}{4}$ by $4.44\frac{4}{9}$. *Ans.* 280.5.
13. Multiply $72.6\frac{7}{20}$ by 4800. *Ans.* 348648.
14. Multiply $13.207\frac{14}{25}$ by 124000. *Ans.* 1637737.44.

CONTRACTIONS IN MULTIPLICATION OF DECIMALS.

276. In multiplying decimals, when the product is required to only a certain number of decimal places, the process may be shortened by contracting each partial product to the required number of decimal places.

1. Multiply 4.78567 by 3.14159, retaining four decimal places in the product.

Solution.—Since multiplying any term of a number by a number of units, gives a product of the same order as the term multiplied, we place 3, the units figure of the multiplier, under the fourth decimal figure of the multiplicand; since tenths multiplied by thousandths give ten-thousandths, we place 1, the tenths figure of the multiplier, under the third decimal figure of the multiplicand, and since hundredths multiplied by hundredths give ten-thousandths, we place 4, the hundredths figure of the multiplier, under the second decimal place of the multiplicand, and continuing in this manner, we finally have the multiplier written in an inverted order. Multiplying 4.7856 by 3 units, we have 14.3568, and adding .0002, which is carried from the product of 3 by 7, the rejected term of the multiplicand, we have 14.3570, the first partial product; multiplying 4.785 by 1 tenth, and adding .0001, (since .00006, the product of 1 by 6, the rejected term of the multiplicand, is nearer 1 ten-thousandth than 1 hundred-thousandth,) we have .4786 as the second partial product; multiplying 4.78 by 4 hundredths, and adding from the product of the rejected term, we have .1914, and we so continue until all the terms of the multiplier have been used; 9 hundred-thousandths, the last figure of the multiplier, must be multiplied by tens to produce ten-thousandths, but since there are no tens in the multiplicand, the only product resulting from 9 is 4 ten-thousandths, which was carried from the product of 9 by the rejected term of the multiplicand. Adding these partial products, we have 15.0346± for the entire product, which is the same as that obtained by the ordinary method. Hence the

OPERATION.

$$\begin{array}{rl}
4.78567 & \\
95141.3 & \\
\hline
14.3570 & = 4.7856 \times 3 + .0002. \\
.4786 & = 4.785 \times .1 + .0001. \\
.1914 & = 4.78 \times .04 + .0002. \\
48 & = 4.7 \times .001 + .0001. \\
24 & = 4 \times .0005 + .0004. \\
4 & = 0 + .00009 + .0004. \\
\hline
15.0346 &
\end{array}$$

Rule.—I. *Write the terms of the multiplier in a reverse order, placing the units term under that term of the multiplicand which is of the lowest order in the required product.*

II. *Multiply each term of the multiplicand by the multiplier, rejecting those terms that are on the right of the term used as a multiplier, increasing each partial product by as many units as would have been carried to it from the product of the rejected part of the multiplicand, and one more when the second term towards the right in the product of the rejected terms is 5 or more than 5; and place the right hand terms of these partial products in the same column*

III. *Add the partial products, and point off in the sum the required number of decimal places.*

Notes.—1. If the number of decimal places in the multiplicand is less than the number required in the product, supply the deficiency by annexing ciphers.

2. In obtaining the number to be added to each partial product, it is generally necessary to multiply only one term at the right of the first term of the part of the multiplicand which is used; but if the terms are large, the multiplication should begin two places to the right.

3. We assume in the rule that the error caused by adding 1 to the partial product when the second term to the right in the product of the rejected terms is 5 or more than 5, will be balanced by the contrary error caused by neglecting the second term when it is less than 5. This may not always be the case, and hence the last term may not be exactly correct. The double sign $\pm$, read *plus or minus*, is placed after the last term to denote this uncertainty. If great accuracy is required, however, it may be attained by carrying the multiplication one place farther than required by the question.

4. If the decimal is a little less than 1, the rule given in Art. 114 may be used, the figures to the right of the decimal point representing the ciphers annexed, and the multiplication commencing with that term of the multiplicand which, multiplied by the lowest term of the multiplier, will give the last figure of the product of the required denomination. Thus, in the 6th example the product must be ten-thousandths, and the multiplicand is to be multiplied by .005, hence the multiplication must commence with the first decimal figure of the multiplicand. The 6th and 9th examples may be solved in this way more readily than by the rule given above.

Find the product of

2. $4379.765 \times .00476$ to 3 decimal places. *Ans.* $20.848\pm$.
3. 359.73485×1.00672 to 4 places. *Ans.* $362.1523\pm$.
4. $8\frac{2}{118} \times 6\frac{482}{741}$ to 5 decimal places. *Ans.* $52.78047\pm$.
5. $24.4379 \times 3\frac{5}{48}$ to 4 decimal places. *Ans.* $76.1553\pm$
6. $369.78347 \times .995$ to 4 places. *Ans.* $367.9345\pm$.
7. $561.745639 \times 54.7245$. *Ans.* $30741.24922\pm$.
8. $6534.65693145 \times 62.4376$. *Ans.* $408008.295623\pm$.
9. $7496.847679 \times .99997$. *Ans.* $7496.622773\pm$.

DIVISION OF DECIMALS.

277. Division of Decimals is the process of finding the quotient when one or both terms are decimals.

1. Divide 272.636 by 6.37.

OPERATION.

$$
\begin{array}{r}
6.37)272.636(42.8 \\
\underline{2548} \\
1783 \\
\underline{1274} \\
5096 \\
\underline{5096}
\end{array}
$$

SOLUTION 1st.—Dividing by 637, we would have for a quotient .428; but as the divisor 6.37 is $\frac{1}{100}$ of 637, the quotient must be 100 times .428 or 42.8.

Or, since the dividend is the product of the divisor and quotient, it must contain as many decimal places as both; hence the quotient must contain as many as the number in the dividend minus the number in the divisor; that is, 3 minus 2, or 1; hence the quotient is 42.8.

SOLUTION 2D.—$272.636 \div 6.37 = \frac{272636}{1000} \div \frac{637}{100} = \frac{272636}{1000} \times \frac{100}{637} = \frac{272636}{10 \times 637} = \frac{1}{10} \times \frac{272636}{637} = \frac{1}{10} \times 428 = 42.8$.

2. Divide .12 by .008.

SOLUTION.—We annex one cipher to the dividend in order to make the number of decimal places equal those in the divisor; then dividing, 8 thousandths is contained in 120 thousandths 15 times, or the number of decimals in the dividend and divisor being equal, the quotient is integral.

OPERATION.

$$.008\overline{).120}$$
$$\quad 15$$

3. Divide .072 by 2400.

SOLUTION.—Annexing two ciphers, so that the dividend may contain the divisor, we find 2400 is contained in 7200 hundred-thousandths, 3 hundred-thousandths times; or the divisor being integral, and the dividend containing five decimals, the quotient contains five decimals. From these solutions we derive the following rule:

OPERATION.

```
2400).07200(.00003
       7200
       ————
```

Rule.—I. *Annex ciphers to the dividend, if necessary to make the number of decimals equal to the number of decimal places in the divisor.*

II. *Divide as in whole numbers, annexing ciphers to the dividend when needed to continue the division.*

III. *Point off as many decimals in the quotient as the number of decimal places in the dividend exceeds the number in the divisor.*

NOTES.—1. We may divide, regarding the divisor as a whole number, and then change the position of the point in the quotient thus derived by comparing the actual divisor with itself used as a whole number.

2. When there are ciphers at the right of the divisor, cut them off, divide by the significant part, and then point off as many decimal places as before, plus the number of ciphers cut off.

3. Make complex decimals pure, or divide them like common mixed numbers, or multiply both by the L. C. M. of the denominators, and then divide.

4. Divide 563.717 by 3.85. *Ans.* 146.42.

5. Divide 101.6688 by 2.36. *Ans.* 43.08.

6. Divide 187.12264 by 123.107. *Ans.* 1.52.

7. Divide 381.9438688 by 7.072. *Ans.* 54.0079.

8. Divide .00020596611 by .005873. *Ans.* .03507.

9. Divide .0005094414 by 2.0709. *Ans.* .000246.

What is the value of

10. $.9 \div \frac{4}{5}$? *Ans.* 1.125.

11. $\frac{45}{112} \div \frac{15}{56}$? *Ans.* 1.5.

12. $.144 \div .02\frac{2}{3}$? *Ans.* 5.4.

13. $42\frac{3}{5} \div 12.25$? *Ans.* $3.47\frac{37}{49}$.

14. $.13 \div .026$? *Ans.* 5.

15. $.75 \div .025$? *Ans.* 30.

16. $7 \div .007$? *Ans.* 1000

17. $.4 \div .008$? Ans. 50

18. $.08 \div .008$? *Ans.* 10.

19. $.005 \div .0015$? *Ans.* $3\frac{1}{3}$.

20. $.16\frac{2}{3} \div 12\frac{1}{2}$? *Ans.* $.0133\frac{1}{3}$.

21. $\frac{3}{4} \div .00\frac{2}{3}$? *Ans.* 112.5

22. $.0003 \div 3.75$? *Ans.* .00008.

23. $.018 \div 3600$? *Ans.* .000005.

24. $1.56 \div 4800$? *Ans.* .000325.

25. $\frac{4}{5} \div 20\frac{3}{5}$? *Ans.* $.03883\frac{51}{103}$.

26. $.0\frac{7}{8} \div .20\frac{8}{9}$? *Ans.* $.418\frac{83}{94}$.

27. $.0004\frac{19}{20} \div .013\frac{1}{5}$? *Ans.* .0375.

28. $.0054 \div 144000$? *Ans.* .000000375.

29. $.003\frac{3}{4} \div 256000$? *Ans.* .0000000146484375.

30. $(16.12 - .04\frac{2}{5}) \div .00\frac{3}{8}$? *Ans.* $4286.93\frac{1}{3}$.

31. Divide four thousand three hundred and sixty-two and five hundredths by six hundred and ninety-five millionths.
Ans. $6276330.935\frac{35}{139}$.

CONTRACTIONS IN DIVISION OF DECIMALS.

278. Certain **Abbreviations** may be made in the division of decimals, which will facilitate the operation.

1. Divide 35.765342 by 8.76347, extending the quotient to four decimal places.

CONTRACTED OPERATION.

```
8.76347)35.765342(4.0811
        35 0539
        -------
           7114
           7010
           ----
            104
             88
            ---
             16
              9
             --
              7
```

SECOND METHOD.

```
8.7634|35.765342
1180.4|  7114
          104
           16
            7
```

SOLUTION.—In the first method of contraction, we compare 8 units, the first term of the divisor, with 35 tens, the first two terms of the dividend, and find that the first quotient place will be units, and since four decimal places are required, it will contain five terms. Taking, therefore, the five left-hand terms of the divisor, we find that 87634 is contained in 357653, 4 times; multiplying the contracted divisor by 4 and carrying 3 from the rejected part, and subtracting from the dividend, we have 7114 for a new dividend. Dropping the right-hand term of the divisor, and dividing by 8763, we find it is not contained in the dividend; we therefore place a zero in the quotient, and dropping another term from the divisor, we find it is contained in the dividend 8 times. Multiplying the divisor by 8, carrying 2 from the rejected part as in Contracted Multiplication, and subtracting, we have 104 left for a new dividend. Continuing this process till all the terms in the divisor are rejected, we have a quotient 4.0811, with a remainder of 7, and as this is more than

5, we may make the last quotient figure 2. By comparing the contracted with the common method, we shall see how much the work is abbreviated and how closely the intermediate results agree.

The second contracted method differs from the first in writing the quotient under the divisor in a reverse order, each term of the quotient being placed under that term of the divisor by which it is first multiplied, and the remainder only being set down, according to Case III. in Contracted Division, Art. 131.

Rule.—I. *Compare the divisor with the dividend to ascertain the number of terms in the quotient.*

II. *For the first contracted divisor, take as many terms of the divisor, beginning with the first significant term on the left, as there are terms in the quotient; and for each successive divisor, reject the right-hand term of the previous divisor, until all the terms of the divisor have been rejected.*

III. *In multiplying by the several terms of the quotient, carry from the rejected terms of the divisor as in contracted multiplication.*

NOTES.—1. Annex ciphers to either divisor or dividend, if necessary, before beginning the work. We take a divisor containing as many terms as the quotient, in order that all the terms of the divisor may be exhausted when we have obtained the required number of terms in the quotient.

2. It will be found convenient to write each term of the quotient as soon as found below the first term of the divisor into which it is first multiplied, as in the second contracted method, since greater accuracy is likely to be thus attained.

3. If a divisor is a little less than 1, the rule given in Art. 132 may be used, placing the dividing line to the right of that term of the dividend, which, multiplied by the difference between 1 and the divisor, gives a decimal of the required place. If the number of places in the quotient is not mentioned, the decimal point may be used as the dividing line. The last three examples may be most readily solved in this manner.

Find the quotient of

2. 36.7345 ÷ 4.7932, to 2 decimal places. *Ans.* 7.66+.

3. 487.355 ÷ 1.00567, to 3 places. *Ans.* 484.607+.

4. .847963 ÷ .92579, to 3 places *Ans.* .916—.

5. 57.643987 ÷ .63975, to 4 places. *Ans.* 90.1039+

6. 3 ÷ 1.0006785, to 5 places. *Ans.* 2.99797—.

7. 2347 ÷ 5675, to 6 places. *Ans.* .413568+.

8. 473 641 ÷ .999, to 4 places. *Ans.* 474.1151+.

9. 97.68397 ÷ .9994, to 5 decimal places.
Ans. 97.74261+.

10. 8574.3965 ÷ .99997, to 6 decimal places.
Ans. 8574.653740—.

MISCELLANEOUS EXAMPLES.

1. Value of $\frac{9}{10}\times\frac{18}{35}\times\frac{7}{24}\times\frac{16}{27}$? *Ans.* .08.

2. Of $\frac{229}{1728}\times 1.44$? *Ans.* $.19083\frac{1}{3}$.

3. Of $.25\times\frac{1-.5}{4}\times\frac{2-.5}{9}$? *Ans.* $.0052083\frac{1}{3}$.

4. Of $(.04\frac{7}{8}+.3\frac{3}{5}-.0075)\times 2\frac{3}{4}$? *Ans.* 1.1034375.

5. Of $\left(\frac{8-.4}{2}+\frac{16-.8}{4}-\frac{5}{2}\right)\times 7\frac{7}{10}$? *Ans.* 39.27.

6. Of $(6.05+3\frac{3}{4}-.004\frac{4}{5})\div .4$? *Ans.* 24.488.

7. Of $(.2\times .02\times .002)-(.01\times .001\times .0001\times 10\frac{2}{5})$? *Ans.* .0000079896.

8. Add $\frac{.4}{3.5}$, $\frac{4\frac{1}{2}}{.25}$, $\frac{7\frac{3}{4}}{33\frac{1}{3}}$, and $\frac{41.75}{21.3\frac{1}{7}}$. *Ans.* $20.3055\frac{1720}{2611}$.

9. Multiply $56\frac{7\frac{1}{2}}{.3\frac{1}{3}}$ by $72\frac{1-.5+2}{9-.99}$. *Ans.* $5676\frac{401}{801}$.

10. Divide $28\frac{\frac{1}{4}}{.7}$ by $1134\frac{\frac{2}{15}}{\frac{3\frac{1}{2}}{7\frac{1}{2}}}$. *Ans.* $.02\frac{1}{2}$.

11. Divide .006006 by $.024\frac{19\frac{4}{5}}{24\frac{1}{5}}$. *Ans.* .242.

12. Divide $.25\frac{\frac{1}{11}}{.9}$ by $4\frac{\frac{91}{99}}{\frac{55}{8}}$. *Ans.* $.06\frac{1627}{22508}$.

13. Divide $5.9001\frac{3}{7}$ by $.174\frac{1.125}{.18\frac{3}{4}}$. *Ans.* $32.778\frac{4}{7}$.

14. $\frac{7}{8}$ of 5.0356 is contained how many times in $\frac{8}{9}$ of 23.79321? *Ans.* 4.8.

15. Find the value of $\left(\frac{23\frac{1}{2}-4.6}{2\frac{1}{2}}+\frac{3.1515+3.08\frac{3}{5}}{12\frac{12}{25}-.005}+\frac{3.5}{1.1-\frac{3}{4}}\right)$ $\div .0025$. *Ans.* 7224.

16. Find the value of $\left(\frac{1.45\frac{3}{5}\times 4.65}{1.25\div .031\frac{1}{4}}\times\frac{16.74-4\frac{7}{25}}{4\frac{3}{4}-2.97}\right)\div .01\frac{2}{5}$. *Ans.* 84.63.

17. What number divided by $\frac{2}{3}$ of $3\frac{5}{8}$ of $\frac{6\frac{1}{15}}{8\frac{1}{10}}$ will give a quotient equal to the value of $\frac{5\frac{1}{2}}{8\frac{3}{4}}$ of $\frac{8}{7}$ of $7\frac{1}{8}$ of $\frac{7\frac{11}{18}}{4\frac{6}{7}}$? *Ans.* $14.96\frac{44}{81}$.

18. Divide $(.99+.7\frac{1}{2}-.4\frac{3}{16})\times\frac{.08\frac{1}{3}}{.2\frac{5}{6}}$ by $.7\frac{.7\frac{1}{4}}{.8}$.

Ans. $.49\frac{651}{4801}$.

19. What is the value of $4\frac{6\frac{4}{11}}{8\frac{3}{4}}+\frac{6\frac{1}{4}}{14\frac{2}{7}}-.33\frac{1}{3}-\frac{3\frac{3}{4}}{18\frac{3}{4}}$ of $\frac{\frac{3}{11}}{3\frac{1}{5}}+5\frac{1}{18}$?

Ans. $9.86\frac{197}{198}$.

20. What is the value of $\frac{50}{3.16\frac{2}{3}}\times\frac{6.3\frac{1}{3}}{33.33\frac{1}{3}}\times\frac{4.9}{18\frac{3}{4}}\times\frac{9.\frac{6}{1\frac{3}{5}}}{21.3\frac{1}{3}}+8.3\frac{1}{4}$ $-\frac{12\frac{6}{7}}{9.57\frac{1}{7}}$?

Ans. $7.32624766\frac{53}{67}$.

MISCELLANEOUS PROBLEMS.

1. If digging 26.54 rods of ditch cost \$176.25, what will 39.81 rods cost? *Ans.* \$264.37½.

2. How many solid feet in a pile of wood 7.3 feet long, 5.7 feet wide, and 6.5 feet high? *Ans.* 270.465.

3. From a cistern containing 2765 gallons, 56.25 barrels, of 31.5 gal. each, are drawn off; how many gallons remain?

Ans. 993.125 gallons.

4. A and B divide 897.25 bushels of corn between them, A taking .37½ and B .62½; how many bushels belong to each? *Ans.* A, 336.46⅞; B, 560.78⅛.

5. If I buy 4 loads of wood, the first containing 1.34 cords, the second 1.4 cords, the third .995 cords, and the fourth 1.16 cords; what would it cost at \$3.75 a cord?

Ans. \$18.35⅝.

6. Which will contain the most, a box 5.5 inches long, 4 inches wide, and 4.25 inches deep, or one 6.5 inches long, 4.5 inches wide, and 3.5 inches deep, the contents being equal to the product of the three dimensions?

Ans. 2d, 8.875 cu. in.

7. Mr. Jones gives .13 of his income in charity, spends .15 for books, .16 in traveling, .52 for his household expenses, and saves \$276.84; what is his income? *Ans.* \$6921.

8. How many barrels of flour, at \$9.66 a barrel, must a man give for 75.25 bushels of wheat at \$1.75 a bushel, 57.5 bushels of corn, at \$0.85 a bushel, and 65.75 bushels of *oats at \$0.56* a bushel? Ans. 22.5+ barrels.

9. A ship whose cargo was worth \$15,000, being disabled by a storm, .56$\frac{1}{4}$ of the whole cargo was thrown over board; how much would a merchant lose who owned .25 of the cargo? *Ans* \$2109.375.

10. The circumference of the fore wheel of a carriage is 13.25 ft., and of the hind wheel 15.75 ft.; how many revolutions will each make in going 25 miles, there being 5280 ft. in a mile? *Ans.* Fore, 9962$\frac{14}{53}$; hind, 8380$\frac{20}{21}$.

11. A grocer wished to buy an equal number of pounds of rice, hominy, and dried apples; the rice being 9 cents a pound, the hominy 13 cents, and the apples 15 cents; how many pounds of each can he buy for \$7.03? *Ans.* 19lb.

12. Mr. Bowman laid out \$779 in groceries, $\frac{1}{3}$ of the whole quantity being sugar at \$0.16 a pound, $\frac{1}{6}$ being tea at \$0.95 a pound, $\frac{3}{7}$ being coffee at \$0.35 a pound, and the remainder being starch at \$0.13 a pound to the amount of \$19.50; how many pounds of each did he buy?

Ans. 700 lb. sugar, 350 lb. tea, 900 lb. coffee, 150 lb. starch.

13. Mr. Thompson's will gave .5 of his property to his wife, $\frac{1}{3}$ of the remainder to each of his two sons, and the remainder to his daughter, who received \$1666.66$\frac{2}{3}$; what was the amount of his property and the share of each?

Ans. Amt., \$10,000; wife, \$5,000; each son, \$1666.66$\frac{2}{3}$.

14. James Williams left .2 of his property to his son John, .25 of the remainder to his son James, and $\frac{1}{6}$ of the remainder to his daughter, making his wife residuary legatee. The difference between the wife's and the daughter's share was \$1245.36; what was the whole amount of the property, and what did each receive?

Ans. Amount, \$3113.40; wife, \$1556.70; John, \$622.68; James, \$622 68; daughter, \$311.34.

15. A, B, C, and D, having built a stone wall, received a certain sum which was to be divided as follows: A received \$90.09 and $\frac{1}{13}$ of the remainder, B \$100.10 and $\frac{1}{13}$ of the remainder, C \$110.11 and $\frac{1}{13}$ of the remainder, and D what was left, when it was found that each received the same sum; what was the amount of their wages? *Ans.* \$480.48.

UNITED STATES MONEY.

279. United States Money, or the currency of the United States, is expressed in the decimal system.

280. The several **denominations** and their relation to each other are presented in the following table:

TABLE

10 mills	equal	1 cent.	10 dimes	equal	1 dollar.
10 cents	"	1 dime.	10 dollars	"	1 eagle.

NOTE.—$\frac{1}{4}$ of a dollar = 25 cents; $\frac{1}{2}$ of a dollar = 50 cents; $\frac{3}{4}$ of a dollar = 75 cents; $\frac{1}{2}$ of a cent = 5 mills.

281. The **dollar** is the unit and is indicated by the symbol **$**; the eagle and dollar are read as a number of dollars. Thus $385 is read 385 dollars.

282. The **dime** is one tenth of a dollar, and is expressed as tenths, the decimal point being placed between dimes and dollars. Thus $12.8 expresses 12 dollars and 8 dimes.

283. The **cent** is one tenth of a dime or one hundredth of a dollar, and is written in hundredths place. Thus $8.75 indicates 8 dollars 7 dimes and 5 cents. Dimes and cents, however, are usually read a number of cents. Thus $8.75 is read 8 dollars and 75 cents.

284. Since **dimes** and **cents** are regarded as a number of cents, when the number of cents is less than 10, a cipher must be written in tenths place. Thus 2 dollars and 8 cents are written $2.08.

285. The **mill** is one tenth of a cent or one thousandth of a dollar, and is written in thousandths place. Thus $12.375 is read 12 dollars 37 cents and 5 mills.

NOTES.—1. In checks, notes, drafts, etc., cents are usually written as hundredths of a dollar in the form of a common fraction, as $\$12\frac{75}{100}$.

2. When the final result of a business computation contains mills, if 5 or more they are reckoned one cent, and if less than 5 they are rejected. Thus $7.187 would be reckoned as $7.19 and $3.162 as $3.16.

REDUCTION OF UNITED STATES MONEY.

286. Reduction is the process of changing a number from *one denomination* to another without altering its value.

287. From the explanations given we have the following

PRINCIPLES.

1. *To reduce cents to mills, annex one cipher.*
2. *To reduce dollars to cents, annex two ciphers.*
3. *To reduce dollars to mills, annex three ciphers.*
4. *To reduce cents to dollars, place the point two places from the right.*
5. *To reduce mills to dollars, place the point three places from the right.*

NOTE.—In reducing a number of dollars and cents to cents, etc., remove the separatrix; thus, $5.25 = 525 cents, and $8.755 = 8755 mills.

EXAMPLES FOR PRACTICE.

1. Reduce 7 dollars 3 dimes 6 cents to mills; 19 dollars 5 dimes 6 cents to mills; 75 dollars 65 cents 7 mills to mills.

2. Reduce 4500 cents to dollars; 400 mills to cents; 460 dimes to dollars; 49000 mills to dollars.

3. Reduce 495 cents to dollars and cents; 567 cents to dollars, dimes, and cents; 5787 mills to dollars, cents, and mills; 97989 mills to dollars, cents, and mills.

FUNDAMENTAL OPERATIONS.

288. Since **United States Money** is expressed in the decimal system, all the operations may be performed as in decimals.

Rule.—*To add, subtract, multiply, or divide in United States money, proceed according to the corresponding operations in decimals.*

PRACTICAL PROBLEMS.

1. Subtract $12\frac{1}{2}$ cents from \$$12\frac{1}{2}$, and add the remainder to $12\frac{1}{2}$ dimes. *Ans.* $13.625.

2. From the sum of \$$18\frac{3}{4}$ and $18\frac{3}{4}$ cents, take the sum of $18\frac{3}{4}$ dimes $18\frac{1}{2}$ mills. *Ans.* $17.044.

3. From the sum of \$$62\frac{1}{2}$ and $62\frac{1}{2}$ cents, take 62 mills and add the result to $62\frac{1}{2}$ dimes. *Ans* $69.313.

4. A man bought a farm for $8750.45, and after keeping it for 5 years and making $525.67 above his expenses, he

sold it for \$650.50 more than he gave for it; what was his actual profit? *Ans.* \$1176.17.

5. Mr. Jones, on balancing his books, found the following debts: A, \$476$\frac{3}{4}$; B, \$768$\frac{5}{8}$; C, \$573$\frac{3}{4}$; D, \$969$\frac{3}{8}$; E, \$471$\frac{7}{8}$; and F, \$396.25; how much was due him? *Ans.* \$3656.62$\frac{1}{2}$.

6. Mr. Wilson bought a horse for \$250, a carriage for \$187$\frac{1}{2}$, and a set of harness for \$45$\frac{3}{4}$; he sold them for \$500; did he gain or lose, and how much? *Ans.* \$16.75 gain.

7. Mr. Stauffer bought 9 hogsheads of molasses, of 100 gallons each, at 45 cents a gallon, and sold them at 56$\frac{1}{4}$ cents a gallon; what was the gain? *Ans.* \$101.25.

8. A lady bought some calico for \$2.50, some delaine for \$5.20, some trimming for \$2.37$\frac{1}{2}$, some buttons for 50 cents, a paper of needles for 10 cents, and some elastic cord for 3 cents, and handed the clerk a ten-dollar bill and a five-dollar bill; what change would she receive? *Ans.* \$4.29.

9. Mr. Tomlinson bought of a farmer 5 cords of wood at \$7.50 a cord, and 7 tons of hay at \$9.75 a ton, and sold him in payment a barrel of sugar, containing 150 pounds at 8$\frac{1}{4}$ cents a pound, 21 pounds of tea at 65 cents a pound, 12 pounds of starch at 12$\frac{1}{2}$ cents a pound, and the remainder in cash; how much cash did he pay? *Ans.* \$78.22$\frac{1}{2}$.

10. A St. Louis merchant bought in New York 25 pieces of silk, 32$\frac{1}{4}$ yards each, at \$2.80 a yard; 200 pieces of calicoes, 35 yards each, at 9$\frac{3}{4}$ cents a yard; 8 pieces of broadcloth, 29$\frac{1}{2}$ yards each, at \$4.875 a yard; 6 pieces of merino, 21 yards each, at 87$\frac{1}{2}$ cents a yard. If he allowed \$4500 to purchase his stock, how much was left to be expended in gloves, etc.? *Ans.* \$299.25.

11. Mr. Page, on closing his year's accounts, found that his purchases amounted to \$3275 and his sales to \$7775, and that the cash on hand was $\frac{5}{6}$ more than at the beginning of the year; what was the amount of his capital at the close of the year, estimating his expenses at \$2500? *Ans.* \$4400.

12. A commission merchant received a consignment of petroleum; he sold 315 barrels at \$3.50, and the remainder at

\$4.20 a barrel; the whole amount of his sales was \$1459.50; how many barrels did he receive? *Ans.* 400 barrels.

13. A laborer's wages average \$1.62$\frac{1}{2}$ a day, and he works 26 days in a month; reckoning his expenses at \$22$\frac{1}{4}$ a month how long will it take him to save \$1250? *Ans.* 62$\frac{1}{2}$ mo.

14. A fish dealer bought Labrador herring at \$6.50 a barrel, mackerel at \$17 for No. 2, and \$8.50 for small No. 3, and pickled cod at \$5.25; he bought the same quantity of each, and the whole cost was \$447; how many barrels did he buy in all? *Ans.* 48 barrels.

15. John Everett sold a cart for \$65.75, and took in payment 25 bushels of wheat at \$1.14 a bushel, 10 bushels of rye at 95 cents a bushel, and the balance in corn at 75 cents a bushel; how much corn did he receive? *Ans.* 37 bu.

16. Bought 2500 lb. of dried peaches at 6 lb. for \$1.08, and exchanged them for canned peaches, which I sold at \$4.25 a dozen, and cleared \$60; how many cans of peaches did I handle? *Ans.* 120 dozen.

COMMERCIAL TRANSACTIONS.

289. In **Commercial Transactions** there are ordinarily three quantities considered, the *quantity*, the *price*, and the *cost*.

290. The **Quantity** is the amount bought or sold, estimated by the number of times it contains the *unit of measure.*

291. The **Price** is the value of one of the units of measure of any commodity. The *Cost* is the value of the whole quantity.

292. An **Aliquot Part** of a number is the whole or mixed number which will exactly divide that number.

ALIQUOT PARTS OF \$1.

5 cents = $\frac{1}{20}$.	16$\frac{2}{3}$ cents = $\frac{1}{6}$.
6$\frac{1}{4}$ cents = $\frac{1}{16}$.	20 cents = $\frac{1}{5}$.
8$\frac{1}{3}$ cents = $\frac{1}{12}$.	25 cents = $\frac{1}{4}$.
10 cents = $\frac{1}{10}$.	33$\frac{1}{3}$ cents = $\frac{1}{3}$.
12$\frac{1}{2}$ cents = $\frac{1}{8}$.	50 cents = $\frac{1}{2}$.

293. The simple operations of finding price, cost, and quantity have already been sufficiently indicated, and we shall here discuss only a few special cases.

CASE I.

294. ***To find the cost of a quantity, the price being an aliquot part of* $1.**

1. What cost 25 yards of muslin at $16\frac{2}{3}$ cents a yard?

SOLUTION.—At $1 a yard the cost would be $25; hence at $16\frac{2}{3}$ cents, which is $\frac{1}{6}$ of $1, the cost will be $\frac{1}{6}$ of $25, or $4.16\frac{2}{3}$. Hence the

OPERATION.

$$\begin{array}{r} 6)25.00 \\ \hline 4.16\frac{2}{3} \end{array}$$

Rule.—*Take such a fractional part of the given quantity as the price is of* $1.

2. What cost 120 pieces of Merrimac prints, each containing 35 yards, at $\$.06\frac{1}{4}$ a yard? *Ans.* $262.50.

3. Mr. Dawson bought 75 bags of Laguayra coffee, containing 43 lb. each, at $.25 a lb.; what was the whole cost? *Ans.* $806.25.

4. Bought 125 quarter boxes of new raisins at $\$.83\frac{1}{3}$ a quarter box, and sold them at $\$.87\frac{1}{2}$; what was the gain?
Ans. $\$5.20\frac{5}{6}$.

5. Bought 63 gallons of winter strained lard oil, at $\$1.12\frac{1}{2}$ a gallon, and 14 gallons crude sperm, at $\$1.37\frac{1}{2}$ a gallon; what was the cost? *Ans.* $\$90.12\frac{1}{2}$.

CASE II.

295. ***To find the cost, the quantity and the price of* 100 *or* 1000 *being given.***

1. What is the cost of 3508 feet of pine boards, at $\$3.37\frac{1}{2}$ a hundred?

SOLUTION.—If 100 feet cost $\$3.37\frac{1}{2}$, 1 foot will cost $\frac{1}{100}$ of $\$3.37\frac{1}{2}$, and 3508 feet will cost 3508 times $\frac{1}{100}$ of $\$3.37\frac{1}{2}$, which is the same as $\frac{1}{100}$ of 3508 times $\$3.37\frac{1}{2}$, which by multiplying and cutting off two places in the product, we find is $118.395, or $118.40.

OPERATION.

$$\begin{array}{r} 3.375 \\ 3508 \\ \hline 118.3950 \end{array}$$

Ans. $118.40

Rule.—*Multiply the price by the quantity, and point off in the product two places for price per hundred, or three places for price per thousand.*

NOTE.—The price per hundred or per thousand is expressed thus: ₩ C, or ₩ M.

2. What is the cost of 3145 Egg Harbor oysters, at $87\frac{1}{2}$ cents ₩ C? *Ans.* \$27.52.

3. How much will be paid for 525 feet of poplar boards at \$$17.12\frac{1}{2}$ ₩ M, 1867 feet of hemlock scantling at \$3.25 ₩ C, and 9850 feet of lath at \$$8.06\frac{1}{4}$ ₩ M? *Ans.* \$149.08.

4. A lumber merchant bought 12720 feet of white oak plank at \$53 ₩ M, 72450 feet of pine boards at \$33.25 ₩ M, and 4250 feet of siding at \$$1.87\frac{1}{2}$ ₩ C; what is his whole bill? *Ans.* \$3162.81.

5. Wishing to have some repairs made on my house I bought 1675 shingles at \$$6.37\frac{1}{2}$ ₩ M, 6725 bricks at \$$7.87\frac{1}{2}$ ₩ M, and 2487 feet of boards at \$14.75 ₩ M; what did my material cost? *Ans.* \$100.32.

CASE III.

296. *To find the cost, the quantity and the price of a ton of 2000 pounds being given.*

1. At \$76.45 a ton, what will be the cost of 6427 pounds of railroad iron?

OPERATION.

$$\begin{array}{r} 2)76.45 \\ \hline 38.22\frac{1}{2} \\ 6427 \\ \hline 245{,}672.07\frac{1}{2} \end{array}$$

Ans. \$245.67.

SOLUTION.—Dividing \$76.45, the price of a ton, by 2, we have \$$38.22\frac{1}{2}$, the price of 1000 pounds; and proceeding as in Case II., we have \$245.67 as the price of 6427 pounds.

Rule.—*Multiply half the price of a ton by the quantity, and remove the decimal point three places to the left.*

2. What is the cost of 7225 pounds of Lehigh red ash coal at \$8.25 a ton, and 5673 pounds of Trevorton coal at \$7.50 a ton? *Ans.* \$51.08.

3. Shipped on the Baltimore and Ohio Railroad 92793 lb. of pig iron at \$3.60 a ton, and 87437 lb. of English rails at \$2.40 a ton; what was the charge? *Ans.* \$271.95.

4. James Marter bought 750 pounds of potash at \$$6.87\frac{1}{2}$ ₩ C, and 8727 pounds of lime at \$$3.66\frac{2}{3}$ a ton, and sold the potash for 8 cents a pound and the lime at 22 cents ₩ C; what was his profit? *Ans.* \$11.64.

BILLS AND ACCOUNTS.

297. A **Bill** is a written statement of goods sold, services rendered, etc., giving the place, date, names of parties, and the price, quantity, and cost of each item.

298. An **Invoice** is a full statement, sent to a purchaser or agent, at the time the goods are forwarded, containing the marks, contents, and prices of each package, the charges paid, and the mode of forwarding. The terms *Invoice* and *Bill* are often used interchangeably.

299. The **Footing** of a bill is the amount of its items. To *extend* an item is to write its cost in the proper column. A bill is *receipted* when the person to whom it is due, or his agent, writes at the bottom of the bill "Received Payment," and signs his name.

300. An **Account Current** is a written statement of the business transactions between two parties for a given time. The party who owes is the *Debtor;* the party owed is the *Creditor.*

301. To **Balance an Account** we find the difference of the footings of the two sides and add it to the smaller side, so that the two amounts are equal.

NOTES.—1. The abbreviation *Dr.* signifies *debit* or *debtor;* *Cr.*, *credit* or *creditor* @ signifies *at*, and denotes the price of the *unit of measure;* ℅ stands for *account;* ¢ for *cents;* No. or ⋕ for *number;* pcs. for *pieces*, and 62^2, 18^3, 6^1, signify respectively $62\frac{1}{2}$, $18\frac{3}{4}$, $6\frac{1}{4}$; 3 pcs. sheeting 42, 42^1, 45^3 signifies that the pieces contain respectively 42, $42\frac{1}{4}$ and $45\frac{3}{4}$ yards.

2. Various marks, such as H enclosed in a diamond, called "diamond H," or K enclosed in a circle, called "circle K," and also different numbers, are placed upon goods and packages for convenience in distinguishing them in invoices.

3. *Accounts current* are frequently made out every month, and are then called *monthly statements*, and generally contain only the amounts bought at each date, bills of the items having been furnished at the time of buying.

4. Deductions are often made in bills, sometimes from the retail price of the items, and sometimes from the amount of the bill. Deductions from the retail price are generally made to customers buying in considerable quantities, and deductions from the amount are made for cash payments or payments within a short specified time, the prices in such bills being mostly wholesale. The symbol %, meaning *hundredth*, is frequently used; thus, less 6% means 6 hundredths deducted.

302. Required the *footings* and *balances* of the following bills and accounts:

PHILADELPHIA, *Dec. 9th*, 1874.

MR. JOHN WILSON,

Bought of ROEBUCK & CO.

Terms: 30 days.

Bbl.	45½ 2	43½	Galls. Bleached Whale Oil,	@		80		
Bbl.	867½ 60	41	" Linseed Oil,		1	00		
Bbl.	45 1	44	" Turpentine,			48		
Can		10	" Olive Oil Malaga,		1	45		
150			Cartage,				1	00
							$113	92

Received payment,

ROEBUCK & CO.

NOTE.—An allowance called "outs" is made for barrels not being filled; 7½ lb. of linseed oil make a gallon.

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411 MARKET ST., PHILADELPHIA, *Jan.* 12, 1875.

MR. THOMAS M. BROWN,

Mt. Airy, N. C.

Bought of ROGERS, CHAMBERS & CO.

Terms: 4 months.

11	5	Doz.	Child's Col'd Berlin Gloves,	3-5	1	30		
832	3	"	Misses' do do do	4-7	1	85		
621	10	"	Ladies' Cotton do	7½-8½		80		
468	2	"	do Lisle do	6½-7½	4	50		
3782	1	"	Boys' Sheep Driving do	7-8			6	
	2	"	Ladies' Colored Jouvin Kid Gloves,	6-7½	16	50		
4	10	"	Ladies' White Cotton Hose,			90		
312	4	"	do do do do rib'd		1	12½		
10B	3	"	do Iron Frame do		3	75		
316	1	"	do Balbriggan do				6	50
3360	2	"	Misses' White Cotton do	5 8	1	50		
3173	1	"	do do do do (full regular)					
			$\frac{230}{5}$ $\frac{260}{5\frac{1}{2}}$ $\frac{300}{6}$ $\frac{340}{6\frac{1}{2}}$ $\frac{380}{7}$ $\frac{425}{7\frac{1}{2}}$ $\frac{450}{8}$					
3420	2	"	Child's White Cotton Hose,	3-5		80		
			Box, Strapping and Cartage,				1	25
							129	00

NOTE.—3-5 refers to the numbers from 3 to 5; in the next to the last item, 1 dozen $\frac{230}{5}$ $\frac{260}{5\frac{1}{2}}$ etc., the meaning is that there are 1 dozen No. 5 @ $2.30, 1 dozen 5½ @ $2.60 and so on. It will be noticed that here, where the dozen is repeated, the price is the upper term of the fraction and the size the lower, while in subsequent examples the number of dozens is the upper term, together with the size if given, and the price the lower. Thus when the price is the upper term the quantity is to be repeated as many *times as there are different prices.*

PHILADELPHIA, *Oct.* 2, 1875

Messrs. J. G. MARTER & Co.,
Athens, Ga.
Bought of HOOD, DAVIS & CO.

Terms: { *7½ pr. ct. off 10 days.* / *6 pr. ct. off 30 days.* }

60	2	Beaver Black Alpaca,	54 each	108	87^2		
7	2	Philada. Ribbed Poplin,	45 43^1	88^1	52^2		
	2	Silk & Wool "	54 42 metres	104	137^2		
	2	Brown Barege,	14 16 auns	37^2	87^2		
846	1	Blue "	12 auns	15	45		
783	3	French Merinoes,	54 41 47 metres	$153\frac{1}{1}\frac{9}{2}$	105		
715	2	Japanese Silk,	45 47 metres	99^2	62^2		
7214	1	Black Gros Grain Silk,	94 auns	117^2	237^2		
611	1	" Taffeta "	72 auns	90	185		
		Less 6% for 30 days,					
						915	48

Received payment, Nov. 1.,
HOOD, DAVIS & Co.,
Per John Albright.

NOTE.—This bill shows a different manner of making out accounts. The first column gives a mark of the goods; the first item written out would read: 2 *pieces* beaver black alpaca, 54 *yards* each, 108 *yards* @ 87½¢, the last column containing the cost of the item, as usual. *Metres* and *auns* are French measures, the former being equal to $1\frac{1}{12}$ yd. and the latter to 1¼ yd. They are reduced to yards in the yards column.

NEW YORK, *August 17th*, 1875.

MR. JOHN WALKER,
Newburg, N. Y.,
Bought of BROUGHER & TILDEN.

	2	Set W. G. Tea Ware 48 pcs.	$4.50			
	8	" " Teas Handled,	95			
	6	Dos. " Plates $\frac{2\text{-}6\text{ in.}}{125}$ $\frac{2\text{-}7\text{ in.}}{140}$ $\frac{2\text{-}8\text{ in.}}{160}$,				
	2	" " Pitchers $\frac{\frac{1}{2}\mid 6}{900}$ $\frac{\frac{1}{2}\mid 12}{650}$ $\frac{\frac{1}{2}\mid 24}{425}$ $\frac{\frac{1}{2}\mid 30}{250}$,				
	1	" Soup Tureen,		3	50	
	2	Glass Bowls & Cover,	1.10			
	½	Doz. Glass Syrup Cans,	4.00			
1 Bx	6	" Sun Chimneys #1,	65 Bx. 40			
	2	" Stand Lamps #12,	8.50			
	3	" Glass Dishes oval $\frac{1\text{-}3\text{ in.}}{100}$ $\frac{1\text{-}6\text{ in.}}{150}$ $\frac{1\text{-}7\text{ in.}}{200}$,				
		1 Crate and Porterage,		2	50	74 23

NOTE.— $\frac{2\text{-}6\text{ in.}}{125}$ $\frac{2\text{-}7\text{ in.}}{140}$ $\frac{2\text{-}8\text{ in.}}{160}$ means that there are 2 dozen 6 inch plates @ $1.25, 2 doz. 7 inch @ $1.40, and 2 dozen 8 inch @ $1.60, making the 6 dozen.

Invoice of 142 Bales of Cotton, shipped per Steamer Ocean Wave, Jones Master, consigned to Wells & Davis, New York, for sale on account of Lafourche & Meurice.

50 Bales Good Ordinary, 20150 lb. @ $12\frac{1}{8}$¢,			
45 " " " 19248 lb. @ $11\frac{1}{4}$¢,			
47 " Middling, 18941 lb. @ $14\frac{3}{8}$¢,			
CHARGES.			
Drayage,	30 25		
Labor,	13 75		
Insurance,	50 00	94	00
		7593	01

NEW ORLEANS, *Oct.* 25, 1875.

LAFOURCHE & MEURICE.

ST. LOUIS, MO., *May* 16, 1875.

MESSRS. THOMSON, JONES & CO.,
Independence, Mo.,
Bought of TRUITT, WATSON & ROGERS.

1 dozen Corn Hoes, #140,		4 75		
$\frac{1}{2}$ " Disston's Hand Saws, #7,	2000			
$\frac{1}{4}$ " Hdld Socket Chisels Ea., $\frac{770}{\frac{3}{4}}$ & $\frac{880}{1 \text{ in.}}$,				
56 lbs. Hoop Iron, 1 inch,	$7\frac{1}{2}$			
$60\frac{1}{2}$ " Sad Iron,	$5\frac{1}{4}$			
1 M Tin Iron Rivets Ea., $\frac{45}{1}$ $\frac{55}{1\frac{1}{2}}$ & $\frac{90}{3 \text{ lbs.}}$,				
$\frac{3}{4}$ " Truitt's Needles Sharps,	200			
1 C Phila. Carriage Bolts, $\frac{233}{\frac{1}{4}\times 2}$ $\frac{252}{\times 2\frac{1}{2}}$,				
$\frac{1}{2}$ " " " " $\frac{472}{\frac{3}{8}\times 4\frac{1}{2}}$ $\frac{577}{\times 6\frac{1}{2}}$ $\frac{600}{\times 7}$,				
$\frac{1}{4}$ Great Gross Steel Shoe Tacks,	300			
2 Gross Screws, 1 in. $\frac{56}{\#8}$ $\frac{60}{9}$ & $\frac{64}{10}$,				
$\frac{1}{2}$ doz. Pocket Knives, $\frac{350}{\#1031}$ & $\frac{450}{562}$,				
$\frac{1}{2}$ " Scissors, $\frac{350}{\#1131}$ $\frac{525}{981}$,				
$\frac{1}{4}$ " Razors, $\frac{650}{9872}$ $\frac{850}{701}$,			53	15

NOTE.—In this example $\frac{472}{\frac{3}{8}\times 4\frac{1}{2}}$ $\frac{577}{\times 6\frac{1}{2}}$ $\frac{600}{\times 7}$ signifies that we have $\frac{1}{2}$ C bolts $\frac{3}{8}$ inches in diameter by $4\frac{1}{2}$ long @ \$4.72, $\frac{1}{2}$ C $\frac{3}{8}$ inches in diameter by $6\frac{1}{2}$ long @ \$5.77, $\frac{1}{2}$ C $\frac{3}{8}$ inches by 7 long @ \$6.00, and 2 gross screws 1 in. #8 *@ 56, 2 gross #9 @ 60¢.*

BOSTON, *Dec. 31st*, 1874.

JAMES MASTERS,

To JORDAN, MARSH & CO., DR.

1874.							
Jan.	25	To 1 piece Irish Linen,	16 yds.@57½				
"	"	" " " Swiss Mull,	20 " " 32½				
Feb.	19	2 pieces Russia Crash,	31 " " 15½				
April	30	1 piece Marseilles Vesting,	12 " " 110				
May	20	" " ¾ Buff Dress Linen,	35 " " 37½				
July	31	" " Fisher's Tweed,	17 " " 64				
Sept.	30	" " ½ Red Wool Flannel,	27 " " 25				
Nov.	28	" " ¾ White Flannel,	31 " " 32½				
		CR.					
Feb.	1	By 1 Box Peerless Bright Navy Tobacco,			62½		
May	19	" $1\frac{1}{10}$ M Key West Cigars,		65	00		
July	11	" $\frac{1}{10}$ " Partaga "		35	00		
Aug.	31	" 1 Jar Garrett's Snuff,			65		
"	"	" 1 Case Violet Cut and Dry Smok'g Tobacco		40	25		
Dec.	12	" ½ Gross Brierwood Pipes, No. 143,		65	50		
		Balance due James Masters,				182	49

MERCER & WAY,

In % *with* FOSTER & MARTIN,

DR. CR.

1875									
Oct.	1	To 24 pcs Troy Prints, 52 yds@6¼¢			Nov	1	By 2 pcs French Tricot, 25 yds @\$5.25		
"	"	" 30 " Eng. Prints, 25 yds@11¼¢			"	28	" 3 pcs Eng'h Broadcloth, 15 yds @\$6		
"	"	" 20 " Solid Black, 47½ yds@11¢			Dec.	15	" 2pcs French Broadcloth, 20 yds @\$4.75		
					1876				
Nov.	15	" 25 " Delaines, 35 yds@20¢			Jan.	2	Note for balance @30 da.	290	72
"	30	" 13 " White Paper Cambric, 41 yds@10½¢							
Dec.	12	To 20 pcs Peq. Cottonade, 47 yds@25¢							
"	"	To 10 pcs Bl'k Tabby Velvet, 12 yds@57½¢							
"	"	To 5 pcs Doeskin Jean 52 yds@57½¢							

FOSTER & MARTIN.

PHILADELPHIA, *Jan.* 1, 1876.

303. Make out bills, etc., from the following data, using the names of the pupil and teacher as seller and buyer, and the present date and place:

1. 25 lb. Tea @ $\frac{10}{75¢}$, $\frac{15}{65¢}$; 16 lb. Coffee @ 35¢; 15 lb. Granulated Sugar @ 14¢; 10 lb. Brown Sugar @ 10¢; 1 barrel No. 1 Mackerel, \$28; 3 gal. N. O. Molasses @ 50¢; 12 lb. Dried Apples @ 12½¢; 3 lb. Butter @ 45¢, and 2 doz. Eggs @ 35¢; paid \$20 on account. Required balance due. *Ans.* \$39.

2. Bo't, Aug. 2, 12 bbl. Patent Minnesota Extra Flour @ \$8.15; Aug. 3, 20 bbl. Good Extra Western @ \$6; Aug. 11, 15 bbl. Good White Western Extra @ \$7.50; Aug. 15, paid \$120 on account; Aug. 17, bo't 16 bbl. Very Choice Minnesota Extra @ \$10; Aug. 20, paid \$200 on account; Aug. 22, bo't 50 bbl. Western Corn Meal @ \$3.45; Aug. 25, 300 lb. Buckwheat Flour @ \$3.25 ℔ C.; Aug. 30, sold 20 hhd. Porto Rico Molasses, 2235 gal., @ 45¢; required monthly statement of account and balance Sept. 1st. *Ans.* \$653.20.

3. Bo't, Sept. 19, 10 lb. Passaic Patent Thread @ $\frac{90}{\#30}$, $\frac{1.10}{40}$, less 25 %; 2 Packs Brass Pins @ $\frac{70}{\#4}$, $\frac{80}{3}$, $\frac{90}{2}$, less 10 %; 1 Gross Rubber Coat and Vest Buttons each \$1 and 75¢; 3 pcs. Silk Velvet Ribbon @ $\frac{47^2}{\#1\frac{1}{4}}$, $\frac{65}{1\frac{3}{4}}$, $\frac{72^2}{2}$, $\frac{87^2}{2\frac{1}{2}}$, $\frac{95}{3}$, $\frac{135}{4}$, $\frac{200}{6}$, less 10%; 4 pcs. Velveteen Ribbon @ $\frac{30}{\#1\frac{1}{2}}$, $\frac{35}{2}$, $\frac{40}{2\frac{1}{2}}$, $\frac{45}{3}$, $\frac{55}{3\frac{1}{2}}$, $\frac{60}{4}$, $\frac{70}{5}$, $\frac{85}{6}$, $\frac{105}{7}$, $\frac{115}{8}$, less 25%; $3\frac{4}{10}$ M Dickens Collars @ \$7; $2\frac{6}{10}$ Jerome Collars @ \$17.50; 31 doz. Coats's Spool Cotton @ 76¢; 1¼ M Needles @ \$2; 2 doz. Rubber Long Combs @ \$4.50; 1 doz. Mahogany Back Hair Brushes, \$8.50; ¼ Gross Honey and Glycerine Soap, each @ \$5.50. Payments on account Sept. 19, \$50; Nov. 25, \$50; what is due at the final settlement? *Ans.* \$74.85.

CIRCULATING DECIMALS.

304. A **Circulating Decimal**, or **Circulate**, is a decimal in which one or more figures repeat in the same order.

305. A **Repetend** is the figure or set of figures which repeat; thus, in .3636 etc., the repetend is 36.

306. A **Repetend** of one figure is expressed by placing a dot over the figure; thus, $.\dot{3}$ expresses .333 etc.

307. A **Repetend** of more than one figure is expressed by placing a dot over the first and last figure; thus $6.\dot{3}4\dot{5}$ expresses 6.345345 etc.

308. A **Pure Circulate** is one which contains no figures but those which repeat; as, $.\dot{3}4\dot{5}$.

309. A **Mixed Circulate** is one which contains one or more figures before the repeating part; as, $.37\dot{4}3\dot{5}$.

310. In a **Mixed Circulate** the two parts are distinguished as the *repeating* and *non-repeating* parts, or as the *repetend* and the *finite* part.

311. A **Simple Repetend** contains but one figure, as, $.\dot{3}$. A *Compound Repetend* contains more than one figure; as, $.\dot{3}4\dot{2}$.

312. **Similar Repetends** are those which begin and end at the same decimal place; as, $.4\dot{2}\dot{7}$ and $.5\dot{3}\dot{6}$.

313. **Dissimilar Repetends** are those which either begin or end at different decimal places; as, $.5\dot{3}6$, $.\dot{7}4\dot{2}$, and $.3\dot{7}6\dot{5}$.

314. A **Perfect Repetend** is one which contains as many decimal places, less 1, as there are units in the denominator of the equivalent common fraction; thus, $\frac{1}{7} = .\dot{1}4285\dot{7}$

315. Repetends are said to be *conterminous* when they end at the same decimal place, and *coöriginous* when they begin at the same place.

ORIGIN OF CIRCULATES.—Circulates had their origin in reducing common fractions to decimals.

NOTES.—1. The subject of circulating decimals was first developed by Dr. Wallis, Professor of Geometry at Oxford, born in 1616.

2. Circulates which begin at the same place are usually called *similar*, and those which end at the same place, *conterminous*. It is more precise to include both of these in *similar* and give another name to those which begin at the same place. Surely circulates are not entirely *similar* unless they *begin* and *end* alike.

3. There being no word employed to denote a similarity of origin, the term *coöriginous*, expressing a *co-origin*, is suggested. Its appropriateness may be seen by comparing it with *conterminous*, a *co-termination*.

4. In reading a mixed circulate, read the decimal and then name the repeating part; thus $.2\dot{0}\dot{6}$ is read, "the mixed circulate 206 thousandths, in which 06 repeats."

REDUCTION OF CIRCULATES.

316. The **Reduction of Circulates** is conveniently treated under four cases.

CASE I.

317. ***To reduce a common fraction to a circulate.***

1. Reduce $\frac{7}{22}$ to a circulate.

OPERATION.

$$22)\underline{7.000}$$
$$.31818+$$
$$=.3\dot{1}\dot{8}$$

SOLUTION.—Annexing ciphers to the 7 and dividing by 22, as in Art. 271, we find $\frac{7}{22}$ equals the circulate $.3\dot{1}\dot{8}$. Hence the following

Rule.—*Annex ciphers to the numerator and divide by the denominator, until the terms begin to repeat, and then place the period over the first and the last terms of the repeating part.*

Reduce the following common fractions to circulates:

2. $\frac{10}{11}$. *Ans.* $.\dot{9}\dot{0}$.

3. $\frac{13}{33}$. *Ans.* $.\dot{3}\dot{9}$.

4. $\frac{43}{44}$. *Ans.* $.97\dot{7}\dot{2}$.

5. $\frac{11}{13}$. *Ans.* $.\dot{8}4615\dot{3}$.

6. $\frac{13}{14}$. *Ans.* $.9\dot{2}8571\dot{4}$.

7. $\frac{34}{35}$. *Ans.* $.9\dot{7}1428\dot{5}$.

8. $\frac{32}{33}$. *Ans.* $.\dot{9}\dot{6}$.

9. $\frac{21}{22}$. *Ans.* $.9\dot{5}\dot{4}$.

10. $\frac{20}{21}$. *Ans.* $.\dot{9}5238\dot{0}$.

11. $\frac{27}{28}$. *Ans.* $.96\dot{4}2857\dot{1}$.

12. $\frac{327}{328}$. *Ans.* $.996\dot{9}512\dot{1}$.

13. $\frac{415}{416}$. *Ans.* $.99759\dot{6}1538\dot{4}$.

Prove, by actual division, the following principles:

1. That $\frac{1}{9}=.\dot{1}$.

2. That $\frac{1}{99}=.\dot{0}\dot{1}$.

3. *That* $\frac{1}{999}=.\dot{0}0\dot{1}$.

4. That $\frac{1}{9999}=.\dot{0}00\dot{1}$.

5. That $\frac{1}{99999}=.\dot{0}000\dot{1}$.

6. That $\frac{1}{999999}=.\dot{0}0000\dot{1}$

ABBREVIATED METHOD OF REDUCTION.

318. An Abbreviated Method may be employed when the circulate consists of many figures.

1. Reduce $\frac{1}{7}$ to a circulate.

SOLUTION.—$\frac{1}{7}=.14\frac{2}{7}$; now $\frac{2}{7}$ is 2 times $\frac{1}{7}$, hence $\frac{2}{7}$ equals $2\times.14\frac{2}{7}$ or $.28\frac{4}{7}$; substituting the value of $\frac{2}{7}$, we have $.1428\frac{4}{7}$; now $\frac{4}{7}=4$ times $\frac{1}{7}$, hence, $\frac{4}{7}=4\times.1428\frac{4}{7}$, or $.5714\frac{2}{7}$; substituting this value, we have $\frac{1}{7}=.14285714\frac{2}{7}$, which we see begins to repeat; hence, $\frac{1}{7}=.\dot{1}4285\dot{7}$.

NOTE.—The solution so clearly indicates the method, that no rule need be given for it.

Reduce the following fractions to circulates:

2. $\frac{1}{41}$. *Ans.* $.\dot{0}243\dot{9}$.
3. $\frac{1}{21}$. *Ans* $.\dot{0}4761\dot{9}$.
4. $\frac{1}{13}$. *Ans.* $.\dot{0}7692\dot{3}$.
5. $\frac{1}{39}$. *Ans.* $.\dot{0}2564\dot{1}$.
6. $\frac{1}{73}$. *Ans.* $.\dot{0}136986\dot{3}$.
7. $\frac{1}{53}$. *Ans.* $.\dot{0}18867924528\dot{3}$.
8. $\frac{1}{79}$. *Ans.* $.\dot{0}12658227848\dot{1}$.
9. $\frac{1}{17}$. *Ans.* $.\dot{0}58823529411764\dot{7}$.
10. $\frac{1}{19}$. *Ans.* $.\dot{0}5263157894736842\dot{1}$.

CASE II.

319. ***To reduce a pure circulate to a common fraction.***

NOTE.—There are three distinct methods of explaining this case, two of which are given here and the other under Geometrical Progression.

1. Reduce $.\dot{6}4\dot{8}$ to a common fraction.

SOLUTION 1ST.—Since $.\dot{0}0\dot{1}$ equals $\frac{1}{999}$, as shown in Art. 317, $.\dot{6}4\dot{8}$, which is 648 times $.\dot{0}0\dot{1}$, equals 648 times $\frac{1}{999}$, which is $\frac{648}{999}$, and this, reduced to its lowest terms, is $\frac{24}{37}$.

OPERATION.

$.\dot{0}0\dot{1}=\frac{1}{999}$

$.\dot{6}4\dot{8}=\frac{648}{999}=\frac{24}{37}$, *Ans.*

SOLUTION 2D.—Let F represent the common fraction, then we will have F=.648648 etc; multiplying by 1000 to make a whole number of the repeating part, we have 1000 times the fraction equals 648.648 etc.; subtracting *once* the fraction from 1000 times the fraction, we have 999 times the fraction equals 648; hence the fraction equals $\frac{648}{999}=\frac{24}{37}$.

OPERATION.

$$\begin{aligned} F &= .648648 \text{ etc.} \\ 1000\,F &= 648.648648 \text{ etc.} \\ 999\,F &= 648 \\ F &= \tfrac{648}{999}=\tfrac{24}{37}, \textit{ Ans.} \end{aligned}$$

Rule.—*Take the repetend for the numerator of a fraction, and as many 9's as there are places in the repetend for the denominator, and reduce the fraction to its lowest terms.*

Reduce the following circulates to common fractions:

2. $.\dot{0}5\dot{4}$.	*Ans.* $\frac{2}{37}$.	7. $.\dot{8}8\dot{8}$.	*Ans.* $\frac{8}{9}$.
3. $.\dot{3}2\dot{4}$.	*Ans.* $\frac{12}{37}$.	8. $.\dot{9}80\dot{1}$.	*Ans.* $\frac{99}{101}$.
4. $.\dot{3}7\dot{0}$.	*Ans.* $\frac{10}{27}$.	9. $.\dot{8}6013\dot{9}$.	*Ans.* $\frac{123}{143}$.
5. $.\dot{2}9\dot{6}$.	*Ans.* $\frac{8}{27}$.	10. $.\dot{9}8601\dot{3}$.	*Ans.* $\frac{141}{143}$.
6. $.\dot{9}6\dot{2}$.	*Ans.* $\frac{26}{27}$.	11. $.\dot{9}2307\dot{6}$.	*Ans.* $\frac{12}{13}$.

CASE III.

320. ***To reduce a mixed circulate to a common fraction.***

1. Reduce $.3\dot{1}\dot{8}$ to a common fraction.

OPERATION.

$$.3\dot{1}\dot{8}=\frac{3\frac{18}{99}}{10}=\frac{3\frac{2}{11}}{10}=\frac{35}{110}=\frac{7}{22}.$$

SOLUTION 1ST.—$.3\dot{1}\dot{8}=\frac{1}{10}$ of $3.\dot{1}\dot{8}$, which by the preceding case equals $\frac{1}{10}$ of $3\frac{18}{99}$, or $\frac{1}{10}$ of $3\frac{2}{11}$, which equals $\frac{35}{110}$, and this reduced to its lowest terms equals $\frac{7}{22}$.

OPERATION

$$\begin{aligned} F &= .3181818 \text{ etc.}\\ 10\ F &= 3.181818 \text{ etc.}\\ 1000\ F &= 318.1818 \text{ etc.}\\ 990\ F &= 315\\ F &= \tfrac{315}{990} = \tfrac{7}{22}, \textit{ Ans.} \end{aligned}$$

SOLUTION 2D.—Let F represent the common fraction, then we shall have $F=.31818$ etc.; multiplying by 10 to make a whole number of the non-repeating part, we have 10 times the fraction equals 3.1818 etc.; multiplying this by 100 to make a whole number of the repeating part, we have $1000\ F=318.1818$ etc.; subtracting 10 F from 1000 F, we have $990\ F=315$; hence $F=\frac{315}{990}=\frac{7}{22}$.

Rule I.—*Write beneath the repetend as many 9's as there are places in the repetend, annex this to the finite part, and divide the result by 1 with as many ciphers annexed as there are places in the finite part.*

Rule II.—*Subtract the finite part from the whole circulate, and write under the remainder as many 9's as there are figures in the repetend, with as many ciphers annexed as there are places in the finite part, and reduce the resulting fraction to its lowest terms.*

2. Reduce $.7\dot{7}\dot{2}$ to a common fraction.

OPERATION.

$$\begin{array}{r} .772\\ 7\\ \hline \frac{765}{990}=\frac{17}{22}. \end{array}$$

SOLUTION.—Subtract 7, the finite part, from .772, and we have .765; dividing by two 9's with one cipher annexed, we have $\frac{765}{990}$, which, reduced to its lowest terms, equals $\frac{17}{22}$.

Reduce the following circulates to common fractions:

3. $.9\dot{5}\dot{4}$. *Ans.* $\frac{21}{22}$.	12. $4.00\dot{6}$. *Ans.* $4\frac{1}{150}$
4. $.52\dot{7}$. *Ans.* $\frac{19}{36}$.	13. $5.0\dot{0}\frac{3}{5}$. *Ans.* $5\frac{1}{150}$.
5. $.\dot{4}0\dot{5}$. *Ans.* $\frac{15}{37}$.	14. $.045\dot{7}317\dot{0}$. *Ans.* $\frac{15}{328}$.
6. $.\dot{9}4\dot{5}$. *Ans.* $\frac{35}{37}$.	15. $.82\dot{1}4285\dot{7}$. *Ans.* $\frac{23}{28}$.
7. $\dot{4}.8\dot{1}$. *Ans.* $4\frac{22}{27}$.	16. $.910\dot{7}1428\dot{5}$. *Ans.* $\frac{51}{56}$.
8. $.79\dot{5}\dot{4}$. *Ans.* $\frac{35}{44}$.	17. $.96\dot{6}04938 27\dot{1}$. *Ans.* $\frac{313}{324}$.
9. $.65\dot{9}\dot{0}$. *Ans.* $\frac{29}{44}$.	18. $3.\dot{4}\frac{1}{2}$; $2.\dot{0}\frac{1}{2}$. *Ans.* $3\frac{1}{2}$; $2\frac{1}{18}$
10. $\dot{2}8.571\dot{4}$. *Ans.* $28\frac{4}{7}$.	19. $.\dot{0}\frac{1}{4}\dot{0}\frac{1}{2}$; $2.\dot{0}\frac{1}{2}\dot{0}\frac{1}{4}$. *Ans.* $\frac{1}{88}$; $2\frac{7}{132}$
11. $4\dot{8}5714\dot{2}$. *Ans.* $\frac{17}{35}$.	20. $.\dot{0}\frac{1}{4}\dot{1}\frac{1}{2}$; $.\dot{0}\frac{1}{3}\dot{0}\frac{4}{5}$. *Ans.* $\frac{4}{99}$; $\frac{62}{1485}$

CASE IV.

321. ***To reduce dissimilar repetends to similar ones.***

322. To solve this case we need to remember the following principles:

PRINCIPLES.

1. *Any terminate decimal may be considered interminate, its repetend being ciphers.* Thus, $.45 = .45\dot{0}$ or $.45\dot{0}0\dot{0}$, etc.

2. *A simple repetend may be made compound by repeating the repeating figure.* Thus, $.\dot{3} = .\dot{3}\dot{3} = .\dot{3}33\dot{3}$, etc.

3. *A compound repetend may be enlarged by moving the right hand dot towards the right over an exact number of periods.* Thus, $.2\dot{4}\dot{5} = .2\dot{4}54\dot{5}$, etc.

4. *Both dots of a repetend may be moved the same number of places to the right without changing its value.* Thus, $.5\dot{3}7\dot{8} = .53\dot{7}8\dot{3}$, or $.537\dot{8}3\dot{7}$, etc., for each expression developed will give the same result.

5. *Dissimilar repetends may be made coöriginous by moving both dots of the repetend to the right until they all begin at the same place.*

6. *Dissimilar repetends may be made conterminous by moving the right hand dots of each repetend over an exact number of periods of each repetend until they end at the same place.*

1. Make $.\dot{4}\dot{5}$, $.4\dot{3}6\dot{2}$ and $.81\dot{3}69\dot{4}$ similar.

OPERATION.

$$.\dot{4}\dot{5} = .45\dot{4}5454545454\dot{5}$$
$$.4\dot{3}6\dot{2} = .43\dot{6}2362362362\dot{3}$$
$$.81\dot{3}69\dot{4} = .81\dot{3}6943694369\dot{4}$$

SOLUTION.—To make these repetends similar they must be made to begin and end at the same place. To do this we first move the left hand dots so that the repetends begin at the same place (Prin. 5), and then move the right hand dots over an exact number of periods so that they will end at the same place. Now the number of places in the periods are respectively 2, 3, and 4; hence the number of places in the new periods must be a common multiple of 2, 3, and 4, which is 12; we therefore move the right hand dot so that each repetend shall contain 12 places.

Rule.—I. *Expand the repetends, and move the left hand dots toward the right so that they all begin at the same place.*

II. *Move the right hand dots so that the number of terms in each period shall be the least common multiple of the number of terms in the given periods.*

2. Make $25.\dot{3}$, $.3\dot{7}\dot{5}$, and $.\dot{4}7\dot{3}$ similar.

3. Make $4.6\dot{3}\dot{2}$, $.\dot{3}2\dot{5}$, and $43.3\dot{2}$ similar.

4. Make $6.\dot{3}2\dot{4}$, $\dot{3}.3\dot{4}$, $.6\dot{5}3\dot{2}$, and $\dot{1}1.0\dot{1}$ similar.

5. Make $.3\dot{2}\dot{7}$, $.435$, $3.\dot{7}64\dot{2}$, and $\dot{6}.78\dot{9}$ similar.

6. Make $46.3\dot{2}\dot{6}$, $4\dot{6}.32\dot{6}$, and $\dot{4}6.32\dot{6}$ similar.

7. Make $\dot{1}0.\dot{1}$, $20.1\dot{2}$, $3401.0\dot{1}$, and $.0\dot{7}$ similar.

8. Make $.64$, $4.3\dot{2}$, $44.\dot{5}\dot{3}$, and $\dot{3}.2\dot{5}$ similar.

9. Make $.\dot{0}\frac{1}{2}\dot{0}\frac{1}{4}$, $\dot{2}.0\frac{1}{2}\dot{0}\frac{7}{8}$, $3\dot{4}5.\dot{3}$, and $.0\dot{0}04\dot{3}$ similar.

ADDITION OF CIRCULATES.

323. Addition of Circulates is the process of finding the sum of two or more circulates.

1. Find the sum of $3.\dot{2}\dot{4}$, $.6\dot{8}\dot{5}$, and $\dot{4}.3\dot{2}$.

OPERATION.

$$\begin{array}{rcl} 3.\dot{2}\dot{4} &=& 3.2\dot{4}2424\dot{2} \\ .6\dot{8}\dot{5} &=& .6\dot{8}5858\dot{5} \\ \dot{4}.3\dot{2} &=& \underline{4.3\dot{2}4324\dot{3}} \\ && 8.2\dot{5}2607\dot{1} \end{array}$$

SOLUTION.—Since only similar fractional units can be added, the repetends must first be made similar. Having done this, we add as in finite decimals, observing to add 1 to the right hand column, since this would be necessary if the repetends were expanded, and we have for the sum $8.2\dot{5}2607\dot{1}$.

Rule.—I. *Make the repetends similar if they are not so.*

II. *Add as in finite decimals, increasing the right hand term by the amount which would be added to it if the circulates were expanded, and make a repetend in the sum similar to those above.*

2. Add $\dot{2}.3\dot{4}$, $3.\dot{0}\dot{1}$, $.7\dot{5}\dot{6}$, $6.\dot{0}\dot{5}$. *Ans.* $12.159514\dot{0}$.
3. Add 72.43, $2.01\dot{2}$, $65.\dot{1}\dot{3}$, $18.5\dot{7}\dot{6}$. *Ans.* $158.15\dot{3}\dot{6}$.
4. Add 18.96, $5.7\dot{3}$, $17.67\dot{1}$, $4.\dot{1}\dot{9}$, *Ans.* $46.55\dot{6}9241\dot{9}$.
5. Add $8.29\dot{3}0\dot{4}$, $.\dot{4}\dot{7}$, $7.00\dot{5}$, $3.9\dot{2}3\dot{6}$. *Ans.* $1\dot{9}.\dot{6}$.
6. Add $5.0\dot{5}4936\dot{7}$, $1.\dot{5}\dot{3}$, $8.0\dot{7}6\dot{3}$, $4.\dot{5}$. *Ans.* $19.\dot{2}$.
7. Add $.346\dot{7}$, $.054\dot{3}$, $.0\dot{8}$, $.06\dot{5}$, $.\dot{4}$. *Ans.* 1.
8. Add $\dot{6}3.4\dot{5}$, $14.5\dot{7}\dot{2}$, $8.124\dot{3}71\dot{5}$, $\dot{2}.735\dot{4}$. *Ans.* $88.\dot{8}$.

SUBTRACTION OF CIRCULATES.

324. Subtraction of Circulates is the process of finding the difference between two circulates.

1. From $\dot{6}.0\dot{4}$ take $2.0\dot{5}\dot{7}$.

SOLUTION.—Having made the repetends similar, we subtract as in finite decimals, observing to diminish the right hand term by unity, since this would be necessary if the circulate were expanded, and we have $3.9\dot{8}8470\dot{2}$.

OPERATION.

$$\dot{6}.0\dot{4} = 6.0\dot{4}6046\dot{0}$$
$$2.0\dot{5}\dot{7} = \underline{2.0\dot{5}7575\dot{7}}$$
$$3.9\dot{8}8470\dot{2}$$

Rule.—I. *Make the repetends similar if they are not so.*

II. *Subtract as in finite decimals, diminishing the right hand term of the remainder by* 1, *when it would be necessary if the circulates were expanded, and make a repetend in the result similar to those above.*

2. Subtract $\dot{5}.6\dot{2}$ from $20.5\dot{4}7\dot{8}$. *Ans.* $14.9\dot{2}$.
3. Subtract $4.2\dot{2}9\dot{6}$ from $12.3\dot{7}$. *Ans.* $\dot{8}.1\dot{4}$.
4. Subtract $7\dot{1}.\dot{3}$ from $7\dot{4}.32\dot{5}$. *Ans.* $\dot{3}.01\dot{2}$
5. Subtract $.\dot{2}9\dot{6}$ from $\frac{8}{9}$. *Ans.* $.\dot{5}9\dot{2}$.
6. Subtract $.\dot{4}3746\dot{5}$ from $\frac{141}{143}$. *Ans.* $.\dot{5}4\dot{8}$.
7. Subtract $1.7\dot{8}3629\dot{0}$ from $10.0\dot{5}6\dot{3}$. *Ans.* $8.\dot{2}\dot{7}$.
8. Subtract $\dot{7}9.365\dot{0}$ from $88.5\dot{3}1746\dot{0}$. Ans. $9.1\dot{6}$.

MULTIPLICATION OF CIRCULATES.

325. Multiplication of Circulates is the process of finding a product when one or both terms are circulates.

1. Multiply $.25\dot{4}\dot{6}$ by $4.6\dot{3}$.

OPERATION.

$$\begin{array}{r} .25\dot{4}\dot{6} \\ 4.6\frac{1}{3} \\ \hline .152\dot{7}8787\dot{8} \\ 1.01\dot{8}58585\dot{8} \\ .008\dot{4}8821\dot{5} \\ \hline 1.17\dot{9}86195\dot{2} \end{array}$$

SOLUTION.—$4.6\dot{3}$ equals $4.6\frac{1}{3}$. Multiplying by .6 and carrying to the right hand term as much as would be necessary if the repetend were continued, we have $.152\dot{7}\dot{8}$; multiplying by 4 in the same manner, we have $1.01\dot{8}\dot{5}$; multiplying by $.0\frac{1}{3}$ we have $.008\dot{4}8821\dot{5}$; making these partial products similar, and adding, we have $1.17\dot{9}86195\dot{2}$.

Rule.—I. *If the multiplier contains a repetend, reduce it to a common fraction.*

II. *Multiply as in finite decimals, adding to the right hand term of each partial product the amount necessary if the repetend were expanded.*

III. *Make the partial products similar and find their sum.*

Find the value of

2. $\dot{8}.2\dot{5} \times 4.8\dot{3}\dot{9}$. *Ans.* $39.\dot{9}6\dot{4}$.
3. $.\dot{9}5238\dot{0} \times .7\dot{6}\dot{3}$. *Ans.* $0.\dot{7}\dot{2}$.
4. $16.\dot{2}0\dot{4} \times 32.\dot{7}\dot{5}$. *Ans.* $530.\dot{8}1044\dot{6}$.
5. $6.\dot{2}1\dot{7} \times 1.5\dot{3}$. *Ans.* $9.5\dot{3}3066399\dot{7}$.
6. $4.\dot{9}2307\dot{6} \times .\dot{4}8\dot{1}$. *Ans.* $2.\dot{3}7\dot{0}$.
7. $8.\dot{5}9\dot{4} \times 6.2\dot{9}\dot{0}$. *Ans.* $54.0\dot{6}7813\dot{2}$.
8. $.\dot{9}62566844919786\dot{0} \times .7\dot{5}$. *Ans.* $.\dot{7}\dot{2}$.

DIVISION OF CIRCULATES.

326. Division of Circulates is the process of finding a quotient when one or both terms are circulates.

1. Divide $.95\dot{6}9\dot{8}$ by $.3\dot{7}\dot{6}$.

OPERATION.

$$\begin{array}{rl} .37\dot{6}7\dot{6} & .95\dot{6}9\dot{8} \\ 376 & 956 \\ \hline 37300) & 94742(2.54 \\ & 746 \\ \hline & 2014 \text{ etc.} \end{array}$$

SOLUTION.—If we make the repetends similar and subtract the finite part of each repetend from the whole repetend, the remainders will be numerators of fractions having a common denominator, Art. 320. Dividing the one by the other, we have 2.54.

Rule.—*Make the repetends similar, subtract the finite part from the entire repetend, omit the dots, and use the results for the dividend and divisor.*

NOTE.—When the divisor is not a circulate divide as in finite decimals, bringing down the figures of the repetend instead of ciphers.

Find the value of

2. $.09\dot{2}\dot{9} \div .\dot{3}\dot{6}$. *Ans.* $.2\dot{5}$.
3. $39.\dot{9}6\dot{4} \div 4.8\dot{3}\dot{9}$. *Ans.* $\dot{8}.2\dot{5}$.
4. $4.9\dot{5}\dot{6} \div .75$. *Ans.* $6.6\dot{0}8754\dot{2}$.
5. $3.973\dot{4}\dot{8} \div .208\dot{3}$. *Ans.* $19.0\dot{7}\dot{2}$.
6. $7.\dot{7}1428\dot{5} \div .\dot{9}5238\dot{0}$. *Ans.* 8.1.
7. $54.0\dot{6}7813\dot{2} \div 8.\dot{5}9\dot{4}$. *Ans.* $6.2\dot{9}0$.
8. Divide $.\dot{7}\dot{2}$ by $.7\dot{5}$. *Ans.* $.\dot{9}62566844919786\dot{0}$.

GREATEST COMMON DIVISOR OF DECIMALS.

327. The **Greatest Common Divisor** of two or more decimals, either finite or infinite, is the greatest decimal that will exactly divide them.

1. Find the greatest common divisor of $.3\dot{7}\dot{5}$ and $.\dot{4}2\dot{3}$.

SOLUTION.—We make the two circulates similar, and subtract the finite part, which reduces them to fractions having a common denominator. (Art. 320.) We then find the greatest common divisor of their numerators, 1638, which is the numerator of the G. C. D., the denominator being of the same denomination as the original dividend and divisor; hence the G. C. D. is $.0\dot{0}0163\dot{8}$.

OPERATION.

$.3\dot{7}\dot{5}7575$	$.4\dot{2}3423\dot{4}$	
3	4	
3757572	4234230	1
	3757572	
3813264	476658	8
55692	501228	9
49140	24570	2
6552	26208	4
6552	1638	4

$\frac{1638}{9999990} = .0\dot{0}0163\dot{8}$, G. C. D.

Rule.—*Reduce the decimals to a common denominator, find the G. C. D. of their numerators, write the result over the common denominator, and reduce the resulting fraction to a decimal.*

NOTE.—The G. C. D. can be found by reducing the decimals to common fractions, and applying the rule given in Art. 255, but the process here given is generally less tedious and more direct.

Find the G. C. D.

2. Of 3.85 and 2.365. *Ans.* .055.

3. Of $.3\dot{1}$ and $.0\dot{2}1\dot{6}$. *Ans.* $.0\dot{0}1\dot{2}$.

4. Of $.\dot{0}6349\dot{2}$ and $.4\dot{4}7619\dot{0}$. *Ans.* $.0\dot{0}3174\dot{6}$.

5. Of .41, $.41\dot{6}$, and $.01\dot{6}\dot{9}$. *Ans.* $.00\dot{0}\dot{3}$.

6. Of .326, $.32\dot{6}$, and $.32\dot{6}$. *Ans.* $.000\dot{0}0\dot{2}$.

LEAST COMMON MULTIPLE OF DECIMALS.

328. The **Least Common Multiple** of two or more decimals is the least number that will exactly contain each of them.

1. Find the L. C. M. of $.3\dot{2}\dot{7}$, $\dot{1}.01\dot{1}$ and $.0\dot{7}\dot{5}$.

SOLUTION.—We reduce the circulates to fractions having a common denominator, as in the previous case. The least common multiple of these numerators is 275699700, which is the numerator of the L. C. M., the denominator being the common denominator of the fractions. Reducing $\frac{275699700}{99990}$, the L. C. M., to whole numbers and decimals, we have $275\dot{7}.\dot{2}$, the L. C. M. Hence the

OPERATION.

	$.3\dot{2}72\dot{7}$	$1.0\dot{1}11\dot{0}$	$.0\dot{7}57\dot{5}$
	3	10	0
3	32724	101100	07575
4	10908	33700	2525
25	2727	8425	2525
101	2727	337	101
	27	337	1

$3\times4\times25\times101\times27\times337=275699700$

$\frac{275699700}{99990}=2757.\dot{2}72\dot{7}$, L. C. M.

$=275\dot{7}.\dot{2}$

Rule.—*Reduce the decimals to a common denominator, find the L. C. M. of their numerators, write the result over the common denominator, and reduce the resulting fraction to a decimal.*

NOTE.—The L. C. M. may be found by reducing the decimals to common fractions and applying the rule given in Art. 256; but the process here given is often more direct.

Find the L. C. M.

2. Of 42.123 and 45.6. *Ans.* 33698.4.

3. $.\dot{6}$, $.5\dot{4}\dot{5}$, and $.7\dot{8}\dot{7}$. *Ans.* 78.

4. Of $8.\dot{4}$, $5.2\dot{7}$ and $16.\dot{1}8\dot{5}$. *Ans.* $971.\dot{1}$.

5. Of $.6\dot{8}5714\dot{2}$, 1.44, .3, and .35. *Ans.* 100.8.

6. Of 6.6, $7.4\dot{6}$, 9.35, and $10.0\dot{5}\dot{4}$. *Ans.* 992745.6.

PRINCIPLES OF CIRCULATES.

329. These **Principles of Circulates** will be found to embrace some interesting and practical properties.

1. *A common fraction whose denominator contains no other prime factors than* 2 *and* 5, *can be reduced to a simple decimal.*

Since 2 and 5 are factors of 10, if we annex as many ciphers to the numerator as there are 2's or 5's in the denominator, the numerator will then be exactly divisible by the denominator. Therefore, etc.

2. *The number of places in the simple decimal to which a common fraction may be reduced is equal to the greatest number of* 2*'s or* 5*'s in the denominator.*

For, to make the numerator contain the denominator we must annex a cipher for every 2 or 5 in the denominator, and the number of places in the quotient, which is the decimal, will equal the number of ciphers annexed. Therefore, etc.

3. *Every common fraction, in its lowest terms, whose denominator contains other prime factors than* 2 *or* 5, *will give an interminate decimal.*

For, since 2 and 5 are the only factors of 10, if the denominator contains other prime factors, the numerator with ciphers annexed will not exactly contain the denominator, hence the division will not terminate and the result will be an interminate decimal. Therefore, etc.

4. *Every common fraction which does not give a simple decimal gives a circulate.*

In reducing there cannot be more different remainders than there are units in the denominator; hence if the division be continued, a remainder must occur which has already been used, and hence we shall have a series of remainders and dividends like those already used, therefore the terms of the quotient will be repeated.

5. *A common fraction whose denominator contains* 2*'s or* 5*'s with other prime factors will give a mixed circulate, and the number of places in the non-repeating part will equal the greatest number of* 2*'s or* 5*'s in the denominator.*

This principle is evident from Prins. 2 and 4, and may be illustrated as follows: $\frac{1}{140}=\frac{1}{2^2\times5\times7}=\frac{100}{2^2\times5\times7\times100}=\frac{5}{7\times100}=.\frac{05}{7}$ which will evidently give a mixed repetend, the repeating part beginning at the third decimal place.

6. *The number of figures in a repetend cannot exceed the number of units in the denominator of the common fraction which produces it, less one.*

In reducing a common fraction to a decimal, when the number of decimal places equals the number of units in the denominator less one, all the possible different remainders will have been used, and hence the dividends, and therefore the quotients which constitute the circulate, will begin to repeat at this point, if not before.

7. *When the reciprocal of any prime number is reduced to a repetend, the remainder which occurs at the close of the period is* 1.

For, since the reduction of the fraction to a circulate commenced with a dividend of 1 with ciphers annexed, that the quotients may repeat we must begin at the close of the period with the same dividend, and therefore the remainder at the close of the period must be 1.

8. *The number of places in a repetend is always equal to the prime denominator of the common fraction producing it, less one, or to some factor of this number.*

For, the repetend must end when it reaches the point where it has as many places less 1 as there are units in the denominator of the producing fraction; hence if it ends before this, the number of places must be an exact part of the denominator less 1, that it may end when it has as many places as the denominator less 1.

9. *When the reciprocal of any prime number is reduced to a repetend, the remainder which occurs when the number of decimal places is one less than the prime is* 1.

For, since the number of decimal places in the period equals the denominator less 1, or is a factor of the denominator less 1, at the close of a period consisting of as many places as the denominator less 1, there will be an exact number of repeating periods, and therefore the remainder will be 1.

10. *A number consisting of as many* 9's *as there are units in any prime except* 2 *and* 5, *less* 1, *is divisible by that prime.*

For, if we divide 1 with ciphers annexed by a prime, after a number of places 1 less than the prime the remainder is 1; hence 1 with the same number of ciphers annexed minus 1, would be exactly divisible by the prime, but this number will be a series of 9's; therefore, etc. Thus, 999999 is divisible by 7.

11. *A number consisting of as many* 1's *as there are units in any prime (except* 3), *less* 1, *is divisible by that prime.*

For, since the prime is a divisor of a series of 9's, Prin. 10, which is equal to 9 times a series of 1's, and 9 and the prime are relatively prime, it must be a divisor of a series of 1's. Thus, 111111 is divisible by 7; 1111111111 is divisible by 11.

12. *A number consisting of any digit used as many times as there are units in a prime (except* 3), *less* 1, *is divisible by that prime.*

For, since such a series of 1's is divisible by the prime, any number of times such a series will be divisible by the prime. Hence 222222, 333333, 444444, etc., are divisible by 7.

13. *The same perfect repetend will express the values of all proper fractions having the same prime denominator, by starting at different places.*

Thus, $\frac{1}{7} = .14285714285$ etc. But $\frac{1}{7} = .1\frac{3}{7}$, hence the part that follows 1 in the repetend of $\frac{1}{7}$ is the repetend of $\frac{3}{7}$, that is, $\frac{3}{7} = .\dot{4}2857\dot{1}$. Again, $\frac{1}{7} = .14\frac{2}{7}$, hence the part that follows .14 in the repetend of $\frac{1}{7}$ is the repetend of $\frac{2}{7}$, that is, $\frac{2}{7} = .\dot{2}8571\dot{4}$. In a similar manner we find $\frac{6}{7} = .\dot{8}5714\dot{2}$, $\frac{4}{7} = .\dot{5}7142\dot{8}$; and the same thing we see is generally true.

14. *In reducing the reciprocal of a prime to a decimal, if we obtain a remainder* 1 *less than the prime, we have one-half of the period, and the remaining half can be found by subtracting the terms of the first half respectively from* 9.

Take $\frac{1}{7}$, and let us suppose, in decimating, we have reached a remainder of 6; now what follows will be the repetend of $\frac{6}{7}$, and the repetend of $\frac{6}{7}$ added to the repetend of $\frac{1}{7}$ must equal 1, since $\frac{6}{7} + \frac{1}{7} = 1$; hence the sum of these two repetends must equal .999999 etc. (since .999 etc. = 1). Now in adding the terms of these two repetends together, that the sum may be a series of 9's, there must be just as many places before the point where 6 occurred as a remainder as after, hence 6 occurred as a remainder when we were half through the series.

Again, since the sum of the terms of the latter and the former half of the repetend equals a series of 9's, each term of the first half of the repetend subtracted from 9 will give the corresponding term of the latter half of the series.

NOTES.—1. All perfect repetends possess this property, and a large number of those which are not perfect. Repetends possessing this property are called *complementary repetends.*

2. The last two properties are of great practical value in reducing common fractions to repetends.

COMPLEMENTARY REPETENDS.

330. Complementary Repetends are those in which the terms of the first half of the period are respectively equal to 9 minus the corresponding terms of the second half of the period.

331. Complementary Repetends include all perfect repetends, and many that are not perfect. The following *curious properties* illustrate the principles presented:

1. *If the last half of the terms of a perfect repetend are written in order under the first half, and added to the terms in the first half, the sum will be a succession of* 9's.

Thus, the fraction $\frac{1}{19} = .\dot{0}5263157894736842\dot{1}$, and this repetend written and added as suggested, will give

$$\begin{array}{r} 052631578 \\ 947368421 \\ \hline 999999999 \end{array}$$

2. *If the remainders obtained in reducing the common fraction to a repetend are written in the same way, and added, each sum will be the denominator of the common fraction.*

Thus, the remainders in reducing $\frac{1}{19}$ are

$$\begin{array}{rrrrrrrrr} 10, & 5, & 12, & 6, & 3, & 11, & 15, & 17, & 18 \\ 9, & 14, & 7, & 13, & 16, & 8, & 4, & 2, & 1 \\ \hline 19, & 19, & 19, & 19, & 19, & 19, & 19, & 19, & 19 \end{array}$$

which, added, give 19, 19, 19, 19, 19, 19, 19, 19, 19

3. *If we subtract the unit term of the denominator of the common fraction from* 10 *and multiply any term of the repetend by the remainder, the unit term of the product will be the unit term of the corresponding remainder.*

Thus, in $\frac{1}{19}$ =.0, 5, 2, 6, 3, 1, 5, etc., terms of repetend.
$10 - 9 = 1$

0, 5, 2, 6, 3, 1, 5, etc., unit terms of product and remainders.

4. *A complementary repetend, by beginning at different points, will be the repetend of all proper fractions having the same denominator as the fraction which produced it.*

Thus, $\frac{10}{19}$ = .52631 etc, which begins with the 2d figure of the circulate equal to $\frac{1}{19}$. Again, $\frac{5}{19}$ = .263157 etc., which begins with the 3d figure of the circulate equal to $\frac{1}{19}$, etc.

5. *The numerator of the fraction equal to any one of the several repetends beginning with the successive terms of a complementary repetend, is the remainder left when the preceding term of the repetend was obtained.*

Thus, in reducing $\frac{10}{19}$ to a circulate, when the first 5 of the circulates was obtained, 5 was the remainder, and 5 is the numerator of the fraction equal to the circulate .26315 etc.

Which of the following give complementary repetends?

$\frac{1}{11}$; $\frac{1}{13}$; $\frac{1}{21}$; $\frac{1}{73}$; $\frac{1}{101}$; $\frac{1}{53}$.

NOTE.--For a fuller discussion of circulates see the author's *Philosophy of Arithmetic.*

CONTINUED FRACTIONS.

332. A **Continued Fraction** is a fraction whose numerator is 1, and denominator an integer plus a fraction whose numerator is also 1 and denominator an integer plus a similar fraction, and so on.

Thus, $\frac{49}{155} = \cfrac{1}{3+\cfrac{1}{6+\cfrac{1}{8}}}$.

333. Several recent authors, for convenience, write a continued fraction with the sign of addition between the denominators; thus, $\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{4+}\,\frac{1}{5}$.

334. There are **Two Cases**: 1st. To reduce a common fraction to a continued fraction; 2d. To reduce a continued fraction to a common fraction.

NOTE.—Continued fractions were proposed about the year 1670, by Lord Brouncker, President of the Royal Society.

CASE I.

335. ***To reduce a common fraction to a continued fraction.***

1. Reduce $\frac{68}{157}$ to a continued fraction.

OPERATION.

$$\frac{68}{157} = \cfrac{1}{2+\frac{21}{68}}$$

$$\frac{68}{157} = \cfrac{1}{2+\cfrac{1}{3+\frac{5}{21}}}$$

$$\frac{68}{157} = \cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{5}}}}$$

SOLUTION.—Dividing both numerator and denominator by 68, we have 1 divided by $2+\frac{21}{68}$. Dividing the terms of $\frac{21}{68}$ by 21, we have 1 divided by $3+\frac{5}{21}$. Dividing again by 5, we have 1 divided by $4+\frac{1}{5}$, which completes the reduction, as the numerator of the last fraction is unity. The terms $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, etc., are called the first, second, third, etc., partial fractions. It will be seen that the same result may be obtained by dividing as in finding the greatest common divisor, and taking the reciprocals of the several quotients for the partial fractions. Hence we derive the following

Rule.—*Find the greatest common divisor of the terms of the given fraction; the reciprocals of the successive quotients will be the partial fractions which constitute the continued fraction required.*

2. Reduce $\frac{28}{121}$ to a continued fraction. *Ans.* $\cfrac{1}{4+\cfrac{1}{3+\cfrac{1}{9}}}$

3. Reduce $\frac{288}{1795}$ to a continued fraction.

Ans. $\cfrac{1}{6+\cfrac{1}{4+\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{6}}}}}}$.

4. Reduce $\frac{24375}{51691}$ to a continued fraction.

Ans. $\cfrac{1}{2+\cfrac{1}{8+\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{8+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{23}}}}}}}}$.

CASE II.

336. ***To reduce a continued fraction to a common fraction.***

1. Reduce $\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{4}}}}$ to a common fraction.

SOLUTION 1ST.—Reducing the complex fraction formed by the last two partial fractions to a simple fraction, we have $\frac{4}{5}$. Taking this result, and the preceding partial fraction together, and reducing, we have $\frac{5}{19}$. Writing this with the preceding partial fraction, and reducing, we have $\frac{19}{43}$, the value of the fraction.

OPERATION.

$\cfrac{1}{1+\frac{1}{4}}=\frac{4}{5}$

$\cfrac{1}{3+\frac{4}{5}}=\frac{5}{19}$

$\cfrac{1}{2+\frac{5}{19}}=\frac{19}{43}$, *Ans.*

SOLUTION 2D.—Approximate values may be obtained by beginning at the first partial fraction and reducing respectively two, three, or more, of the partial fractions to simple fractions. Thus, the first approximate value is $\frac{1}{2}$; the second is $\cfrac{1}{2+\frac{1}{3}}$, or $\frac{3}{7}$; the third is $\cfrac{1}{2+\cfrac{1}{3+\frac{1}{1}}}$, or $\frac{4}{9}$; the true value is $\frac{19}{43}$.

By exhibiting the second method in an analytic form, a law may be discovered which gives us a simple and practical rule for the reduction.

2. Find the approximate values of the continued fraction $\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{5+\cfrac{1}{4}}}}$.

SOLUTION.—The work may be written as follows:

$\frac{1}{2} \qquad = \frac{1}{2}$, 1st approx. val.

$\cfrac{1}{2+\frac{1}{3}}=\frac{3}{3\times2+1} \qquad =\frac{3}{7}$, 2d " "

$\cfrac{1}{2+\cfrac{1}{3+\frac{1}{5}}}=\cfrac{1}{2+\cfrac{5}{3\times5+1}}=\frac{3\times5+1}{(3\times2+1)\times5+2}=\frac{3\times5+1}{7\times5+2}=\frac{16}{37}$, 3d " "

$\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{5+\frac{1}{4}}}}=\cfrac{1}{2+\cfrac{5+\frac{1}{4}}{3\times(5+\frac{1}{4})+1}}=\frac{16\times4+3}{37\times4+7}=\frac{67}{155}$, true value.

We take $\frac{1}{2}$, the first term of the continued fraction, for the 1st approximate value. Reducing the first two partial fractions, we have $\frac{3}{7}$ for the 2d approximate value. Continuing the reduction, we obtain $\frac{16}{37}$ and $\frac{67}{155}$ for the remaining values. Examining the last two reductions, we find that the 3d approximate fraction is obtained by multiplying the terms of the 2d approximate fraction by the denominator of the 3d partial fraction, and adding to these products the corresponding terms of the first approximate fraction. We see also that the 4th approximate value is equal to the product of the terms of the third approximate value by the denominator of the 4th partial fraction, plus the corresponding terms of the 2d approximate fraction. The value obtained by using the last partial fraction is the exact value of the fraction. Hence we derive the following

Rule.—I. *For the first approximate value, take the first partial fraction.*

II. *For the second value, reduce the complex fraction formed by the first two partial fractions.*

III. *For each succeeding approximate value, multiply both terms of the approximation last obtained by the denominator of the following partial fraction, and add to the products the corresponding terms of the preceding approximation.*

IV. *The last value thus obtained will be the common fraction required.*

3. Reduce the continued fraction $\cfrac{1}{5+\cfrac{1}{6+\cfrac{1}{7+\cfrac{1}{8}}}}$ to a common fraction.

Ans. $\frac{350}{1807}$.

4. Reduce the continued fraction $\cfrac{1}{3+\cfrac{1}{7+\cfrac{1}{9+\cfrac{1}{5}}}}$ to a common fraction.

Ans. $\frac{327}{1027}$.

5. Find the approximate values of $\cfrac{1}{11+\cfrac{1}{9+\cfrac{1}{8+\cfrac{1}{2}}}}$.

Ans. $\frac{1}{11}$, $\frac{9}{100}$, $\frac{28}{811}$, $\frac{65}{722}$.

6. Find the approximate values of the fraction $\frac{29}{121}$.

Ans. $\frac{1}{4}$, $\frac{5}{21}$, $\frac{6}{25}$, $\frac{29}{121}$.

Note.—For a fuller discussion of the subject see *Brooks's Philosophy of Arithmetic.*

SECTION VI.

DENOMINATE NUMBERS.

337. A **Denominate Number** is a concrete number in which the unit is a *measure;* as, 3 *feet*, 4 *pounds*, etc.

338. A **Measure** is a unit by which quantity of *magnitude* or *continuous* quantity is estimated numerically; as, a *yard*, a *pound*, etc.

339. A **Compound Number** is a number which expresses several different units of the same kind of quantity; as, 4 yd. 8 ft. 6 in.

340. The **Terms** of a compound number are the *numbers* of its *different units.* Thus, in £12 10 s. 8 d, the terms are £12, and 16 s., and 8 d.

341. **Similar Compound Numbers** are compound numbers which express the same kind of quantity.

342. **Denominate Numbers** may be embraced under four distinct classes: *Value, Weight, Extension,* and *Time.*

343. Some of these classes contain several subdivisions of so much importance that the following is regarded as the most convenient classification:

1. Value.	5. Volume.
2. Weight.	6. Capacity.
3. Length.	7. Time.
4. Surface.	8. Angles.

ORIGIN.—There are two kinds of quantity; quantity of *multitude* and quantity of *magnitude.* Quantity of multitude exists in individual things, and is immediately expressed in *numbers.* Quantity of magnitude is a mass, and can only be expressed *numerically* by fixing upon a *unit* of *measure* and finding *how many times* the quantity considered contains the unit of measure. Those who prefer may use the terms *discrete* and *continuous*, for *multitude* and *magnitude.*

344. A **Denominate Number** may also be defined to be a numerical expression of quantity of magnitude, or of continuous quantity.

MEASURES OF VALUE.

345. The **Value** of anything is its worth, or that property which makes it useful or estimable.

346. **Value** depends principally upon utility and difficulty of attainment. Value is usually estimated in something called *Money*.

347. **Money** is the measure or representative of the value of things. It is of two kinds, *coin* and *paper money*.

348. **Coin,** or **Specie,** is metal prepared and authorized by government to be used as money. The metals generally used are *gold*, *silver*, *copper*, and *nickel*.

349. **Paper Money** consists of printed promises to pay the bearer a certain amount, duly authorized to circulate as money.

350. **Currency** (from *curro*, I run,) is that which circulates as money. It is of two kinds, *specie currency* and *paper currency*.

351. **Legal Tender** is a term applied to money which is required by law to be accepted in payment of debts.

352. An **Alloy** is a baser metal compounded with either gold or silver for the purpose of rendering it harder and more durable. In coinage the alloy is considered as having no value.

UNITED STATES MONEY.

353. **United States Money** is the legal currency of the United States.

TABLE.

10 mills (m.)	equal	1 cent	ct.
10 cents	"	1 dime	d.
10 dimes	"	1 dollar	$.
10 dollars	"	1 eagle	E.

SCALE.—Ascending and descending uniform by 10.

1. NAME.—United States money is so called because it is the money of the United States. It is called *Federal Money* because it was the money of the Federal Union. It was adopted by Act of Congress, Aug. 8, 1786.

II. Terms.—The term *dollar* is from *Dale* or *Daleburg*, a town where it was first coined ; or *thal*, a dale or valley ; or from the Anglo-Saxon *dael*, *a portion*, it being a portion of a ducat. *Dime* is from the French *disme*, meaning *a tenth ; cent* is from the Latin *centum*, *a hundred ; mill* is from the Latin *mille*, *a thousand ; eagle* is from the name of the national bird. The cent was proposed by Robert Morris, and named by Thomas Jefferson.

III. Symbols.—There are several theories for the origin of the dollar mark :

1st. That it is a combination of U. S., the initials of United States.

2d. That it is a modification of the figure 8, the dollar being formerly called a *piece of eight*, and designated by the symbol $\frac{8}{}$.

3d. That it is derived from a representation of the "Pillars of Hercules," consisting of two pillars connected with a scroll. The old Spanish coins containing this were called "*pillar dollars*."

4th. That it is a combination of HS. the mark of the Roman money unit.

5th. That it is a combination of P. and S. from the Spanish *peso duro*, signifying *hard dollar*. In Spanish accounts *peso* is contracted by writing the S over the P, and placing it after the sum.

IV. Unit.—The *unit* is the *gold dollar*. The currency is founded upon the decimal system, dimes, cents, and mills being written as decimals. This gives great simplicity to the operations.

V. Coins.—The coins are of *gold*, *silver*, *nickel*, and *bronze*. The *gold coins* are the *double eagle*, *eagle*, *half-eagle*, *quarter-eagle*, and *one dollar*. The *silver coins* are the *dollar*, *half dollar*, *quarter-dollar*, and *dime*. The *nickel coin* is the *five-cent piece*. The *bronze coin* is the *cent*. The gold three-dollar piece, the silver twenty-cent piece, half-dime, and three-cent piece, the nickel three-cent piece and cent, the bronze two-cent piece, and the old copper cent and half-cent, although occasionally seen in circulation, are no longer coined. The *mill* has never been a coin ; it is merely a convenient name for the tenth part of a cent.

VI. Composition.—The gold and silver coins consist of 9 parts of pure metal and 1 part alloy. The alloy of the silver coin consists of pure copper ; the alloy of the gold consists of silver and copper, the silver not to exceed $\frac{1}{10}$ of the alloy. The nickel coins contain $\frac{1}{4}$ nickel and $\frac{3}{4}$ copper. The bronze coins consist of 95 parts copper and 5 parts tin and zinc.

VII. Weight.—The gold dollar weighs 25.8 gr., and the other gold coins proportionally ; the silver dollar weighs $412\frac{1}{2}$ gr. ; the half-dollar weighs 192.9 gr. ; the quarter-dollar, 96.45 gr. ; the dime, 38.58 gr. ; the nickel five-cent piece, 77.16 gr. ; the bronze cent, 48 gr.

VIII. Legal Tender.—Gold coins are a legal tender for any amount ; silver coins, of the present coinage, for any amount not exceeding $5 in any one payment ; bronze and nickel coins for any amount not exceeding 25 cents in any one payment.

STATE CURRENCIES.

354. Previous to the establishment of the decimal currency, we employed the currency of England, that is, pounds, shillings, and pence. Some of the States still use shillings and pence, though not with the same values.

355. This difference of value was caused by the difference of depreciation of colonial currency in different States when the decimal system was adopted (1786).

NOTE.—In New York currency, used in New York, Michigan, Ohio, and North Carolina, \$1=8 s.; hence 1 s.=12½¢. In Pennsylvania currency, used in Pennsylvania, New Jersey, Delaware, and Maryland, \$1=7 s. 6 d., and 1 s. =13⅓¢. In New England, Virginia, Kentucky, Tennessee, Texas, Mississippi, Indiana, Illinois, \$1=6 s., and 1s.=16⅔¢. In Georgia and South Carolina, \$1=4 s. 8 d., and 1 s.=21³⁄₇¢.

ENGLISH, OR STERLING MONEY.

356. English, or Sterling Money, is the legal currency of England.

TABLE.

4 farthings (far. or qr.)	=	1 penny	. .	d.
12 pence	. . =	1 shilling	. .	s.
20 shillings	. . =	1 pound or sovereign,		£
21 shillings	. . =	1 guinea	. .	G.

£		s.		d.		far.
1	=	20	=	240	=	960
		1	=	12	=	48

SCALE.—Ascending, 4, 12, 20; descending, 20, 12, 4.

I. NAME.—The term *Sterling* is supposed to be derived from *Easterling*, the name given to early German traders, who came from the east to England. Their money was called Easterling Money, which was contracted into *Sterling Money*.

II. TERMS.—The term *farthing* is a modification of "four things," the old English penny being marked with a cross so deeply impressed that it could be broken into two or four pieces, called respectively *half-penny* and *four things*. The *pound*, as a measure of value, was derived from the pound as a measure of weight, 240 pence formerly weighing a pound. The *guinea* is so called because it was first made of gold brought from Guinea.

III. SYMBOLS.—The symbols £ s. d. qr. are the initials of the Latin words, *libra*, *solidus*, *denarius*, and *quadrans*, signifying respectively, pound, shilling, penny, and quarter. The old *ſ*, the original abbreviation for shillings, was formerly written between shillings and pence. The *ſ* has since been changed into /; thus, 7s. 6d. are sometimes written 7/6.

IV. UNIT.—The unit is the pound, represented by the sovereign and £1 bank note. Its value by late act of Congress is fixed at \$4.8665.

V. COINS.—The coins are of three classes; *gold*, *silver*, and *copper*. The *gold coins* are the *sovereign* (=£1), and *half sovereign* (=10s.), *guinea* (21 s.), and *half guinea* (10 s. 6 d.). The *silver coins* are the *crown* (=5 s.), the *half-crown* (=2 s. 6 d.), the *florin* (=2 s.), the *shilling*, and the *six-penny*, *four-penny*, and *three-penny* pieces. The *copper coins* are the *penny*, *half-penny*, and *farthing*.

The pound is not a coin; it is represented by the sovereign and £1 bank-note. The guinea (=21 s.) and half-guinea (10 s. 6d.) are old gold coins no longer coined, though some of them are still in circulation. The crown and half-crown, also, although still in circulation, are no longer coined.

VI. COMPOSITION.—The standard for gold coins is 22 carats fine, that is, 11 parts pure gold and 1 part alloy. The standard for silver is 37 parts pure silver and 3 parts alloy; hence the silver coins are $\frac{37}{40}$ pure, and $\frac{3}{40}$ copper. Pence and half-pence are made of pure copper.

VII. WEIGHT.—The sovereign weighs 123.274 grains; the shilling weighs 87.27 grains; the penny weighs 240 gr., or ½ oz. Troy.

CANADA MONEY.

357. The **Currency of Canada** is the same as that of the United States, the table and denominations being the same.

358. The decimal currency was adopted in 1858, the Act taking effect in 1859, previous to which their currency was the same as the English.

I. COINS.—The coins consist of silver and copper. The *silver coins* are the 50-*cent piece*, the 25-*cent piece*, the *shilling* or 20-*cent piece*, the *dime*, the *half-dime*. The *copper coin* is the *cent*.

II. VALUE.—The coins are nominally equal to the corresponding coins of United States money, but the intrinsic value is a little less. The *eagle* of the United States is the legal tender for sums of $10 and upwards.

III. COMPOSITION.—The silver coins consist of 925 parts silver and 75 parts copper; or 37 parts silver to 3 parts copper, the same as the English silver coins.

FRENCH MONEY.

359. French Money is the legal currency of France. The *unit* is the *franc*, whose value is 19.3 cents.

360. The **Franc** is divided into tenths and hundredths, called respectively *decimes* and *centimes*. The *decime*, like our dime, is not used in business calculations, but is expressed by *centimes;* thus, instead of 5 decimes we say 50 centimes.

I. COINS.—The principal French gold coins are the 20-franc and 10-franc pieces; in silver the 5-franc and 2-franc pieces and franc; in copper, 10-centime and 5-centime pieces.

II. COMPOSITION.—The gold and silver coins, like those of the United States, are $\frac{9}{10}$ pure metal.

GERMAN MONEY.

361. The **German Empire** has adopted a new and uniform system of coinage.

362. The **Unit** is the *mark* (*Reichsmark*) worth 23.85 cents, and this is divided into 100 *pfennige*, or *pennies*.

COINS.—The principal German coins are gold coins of 20 marks, 10 marks, and 5 marks; silver coins of 5 marks, 2 marks, and 1 mark; nickel ten-penny and five-penny pieces; and copper two-penny and penny pieces.

A pound of gold, .9 pure, is divided into 139½ coins, and the tenth part of this coin is called a *mark*.

EXAMPLES FOR PRACTICE.

1. *How many* pence in £26 15 s. 10 d.?

SOLUTION.—In one pound there are 20 shillings, and in £26 there are 26 times 20 shillings, which increased by 15 shillings, are 535 shillings: in one shilling there are 12 pence, and in 535 shillings there are 535 times 12 pence, which, increased by 10 pence, equal 6430 pence.

OPERATION.

£	s.	d.
26	15	10
20		
535		
12		
6430, *Ans.*		

2. How many pounds, shillings, and pence in 6430 pence?

SOLUTION.—There are 12 pence in one shilling, hence in 6430 pence there are as many shillings as 12 is contained times in 6430, which are 535 shillings, and 10 pence remaining: there are 20 shillings in one pound, hence in 535 shillings there are as many pounds as 20 is contained times in 535, which are £26, and 15 shillings remaining; hence in 6430 d. there are £26 15 s. 10 d.

OPERATION.

12)6430

2|0)53|5 – 10 d.

£26 – 15 s.

Ans. £26 15 s. 10 d.

3. How many dollars in £16? *Ans.* $77.864

4. How many pounds in 40 guineas? *Ans.* £42.

5. How many guineas in £8 8 s.? *Ans.* 8 G.

6. How many dollars in £20 5 s.? *Ans.* $98.546.

7. How many dollars in 12 sovereigns? *Ans.* $58.398.

8. How many francs in $22.45? *Ans.* 116 fr. 32 cent

9. How many marks in $84.75? *Ans.* 355 marks, 34.6 pf.

10. How many dollars in 565.40 francs? *Ans.* $109.12.

11. Dollars in 256 marks, 25 pfennige? *Ans.* $61.115.

12. How many dollars in 6 sovereigns 8 crowns and 3 florins? *Ans.* $40.39.

13. A lady bought in Boston, in 1856, 56 yd. of merino at 7 s. 6 d. a yard; what did it cost? *Ans.* $70.

14. A lady bought in New York, in 1846, 26 yd. of alpaca at 5 s. 3 d. a yard; what did it cost? *Ans.* 17.06\frac{1}{4}$.

15. What cost, in 1836, in Philadelphia, 25 lb. of candles, at 2 s. 6 d. a pound? *Ans.* 8.33\frac{1}{3}$.

16. What would a man's wages amount to for 17 weeks at 15 s. 9 d. a week, in Georgia currency? *Ans.* $57.375.

17. In 1858 a young man living in New York visited Boston and bought a pair of kid gloves, the price being 11 s. 6 d.; he handed the clerk the exact change in New York currency; how great was his mistake? *Ans.* 47$\frac{11}{12}$ cents

18. A gentleman returning from Europe, had 10 sover-

eigns, 4 crowns, 50 francs, and 6 marks; what was their value in U. S. money? *Ans.* $64.61.

MEASURES OF WEIGHT.

363. Weight is the measure of the force with which a body is drawn towards the centre of the earth by the attraction of gravitation.

364. There are three kinds of weight in common use: *Troy Weight, Apothecaries' Weight,* and *Avoirdupois Weight.*

TROY WEIGHT.

365. Troy Weight is used in weighing gold, silver, jewels, in philosophical experiments, etc.

TABLE.

24 grains (gr.)	=	1 pennyweight	pwt.
20 pennyweights	=	1 ounce	oz.
12 ounces	=	1 pound	lb.

lb.		oz.		pwt.		gr.
1	=	12	=	240	=	5760
		1	=	20	=	480

SCALE.—Ascending, 24, 20, 12; descending, 12, 20, 24.

I. NAME.—The name *Troy* is derived from *Troyes*, the name of a town in France, where this weight was first used in Europe. It was brought from Cairo in Egypt during the Crusades of the 12th century. The name is derived by some, however, from Troy Novant, a name given to London in monkish chronicles.

II. TERMS.—The term *pound* is from the Latin *pendo*, to *bend* or *weigh*. The term *ounce* is from the Latin *uncia*, a *twelfth part*, the ounce being one-twelfth part of a pound. The *pennyweight* was the weight of the old English silver penny. The term *grain* is from *grains of wheat* which were formerly used for weighing. These were taken from the middle of the ear, and well dried; at first the pennyweight contained 32 of these grains, but afterwards it was divided into 24 parts, which, though still called grains, were much heavier than a grain of wheat.

III. SYMBOLS.—The symbol *oz.* is from the Spanish word *onza*, signifying *ounce*, though Webster derives it from the use of the termination 3, to express abbreviations, which was afterwards changed to *z*; *lb.* is from *libra*, the Latin for pound. *Pwt.* is a combination of *p.* for *penny* and *wt.* for *weight*; *dwt.*, from *denarius* and *weight*, is nearly obsolete.

IV. UNIT.—The *standard unit* of weight is the *Troy pound.* It is equal to the weight of 22.794377 cubic inches of distilled water, at the temperature of 39.83° Fahrenheit, barometer at 30 inches, and is identical with the Imperial Troy pound of Great Britain.

V. At the United States Mint, the Troy ounce is adopted as the standard, and all weights are expressed in multiples and decimal subdivisions of the *ounce.*

APOTHECARIES' WEIGHT.

366. Apothecaries' Weight is used in prescribing and mixing dry medicines. Medicines are bought and sold by Avoirdupois Weight.

TABLE.

20 grains (gr.xx) .	=	1 scruple . .	℈.
3 scruples (℈iij)	=	1 dram	ʒ
8 drams (ʒviij)	=	1 ounce . .	℥.
12 ounces (℥xij) .	=	1 pound . .	℔

℔.		℥		ʒ		℈		gr.
1	=	12	=	96	=	288	=	5760.
		1	=	8	=	24	=	480.

SCALE.—Ascending, 20, 3, 8, 12; descending, 12, 8, 3, 20.

I. NAME.—The name arises from the weight being used by *apothecaries*.

II. TERMS.—The term *scruple* is from the Latin *scrupulus*, a *little stone*. The term *dram* is from *drachma*, a Greek weight.

III. SYMBOLS.—The symbols have been supposed to be modifications of the figure 3, suggested by there being 3 scruples in a dram. Champollion, however, has traced them back to the hieroglyphics of Egypt.

IV. UNIT.—The *Unit* is the *pound*, and is identical with the Troy pound, as are also the ounce and grain, the ounce being differently divided.

V. NOTATION.—Physicians use the Roman notation in writing prescriptions, using the small letters, preceded by the symbols, and writing *j* for *i* when it terminates a number. Thus, 12 gr. is written gr.xij.; 2 scruples. ℈ij. ℞ is an abbreviation for *recipe, take;* $\bar{a}$ or $\overline{aa}$ (from the Greek ἀνά) means *of each*, referring to two or more preceding ingredients; *ss.* for *semis* or *half*, as ℈ivss., means 4½ scruples; *P.* for *particula*, or *little part*, *P. aeq.* for *equal parts; q. p., quantum placet, as much as you please.*

AVOIRDUPOIS WEIGHT.

367. Avoirdupois Weight is used for weighing everything except jewels, precious metals, etc.

TABLE.

16 ounces (oz.) .	=	1 pound . .	lb.
100 pounds . .	=	1 hundredweight	cwt.
20 hundredweight .	=	1 Ton . .	T.

T.		cwt.		lb.		oz.
1	=	20	=	2000	=	32000
		1	=	100	=	1600.

SCALE.—Ascending, 16, 100, 20; descending, 20, 100, 16.

I. NAME.—The term *Avoirdupois* is probably from the French *avoir du poids, to have weight.* It has also been derived from an old French verb

averer, to verify, from the old French *aver de pes, goods of weight*, and from the old Norman French *avoir du poids, goods or chattels of weight.*

II. TERMS.—The term *ton* is from the Saxon *tunne*, a *cask*. The origin of the other terms has already been given. The symbol *cwt.* is from *centum, hundred*, and *weight*. The term *dram* has been used for $\frac{1}{16}$ of an ounce, but is obsolete, fractions of an ounce being now used.

III. UNIT.—The unit is the *pound*. It is derived from the Troy pound, and contains 7000 grains Troy. It is equal to the weight of 27.7015 cubic inches of water at 39.83° Fah., the barometer being at 30 inches, or to 27.7274 cubic inches at 62° Fah., barometer 30 inches. The Imperial pound avoirdupois of Great Britain is derived by the latter method.

IV. In Great Britain 28 lb. equal 1 qr., 112 lb. equal 1 cwt., and 2240 lb. equal 1 ton. These are called the *long hundred* and *long ton*; they were formerly used in this country, but are now used only at the custom-houses in invoices of English goods, in the wholesale iron and plate trade, and in wholesaling and freighting coal from the coal mines of Pennsylvania.

V. OLD WEIGHTS.—A *stone* of iron or lead=14 lb.; 21½ stone=1 *pig*, and 8 pigs = 1 *fother*; a *stone* of fish or butcher's meat = 8 lb.; a *stone* of glass = 5 lb. A *seam* of glass = 24 stone; a *truss* of hay = 56 lb; a *truss* of new hay, until the 1st of Sept.= 60 lb.; a *truss* of straw = 36 lb. In weighing wool, 7 lb. = 1 *clove*; 2 cloves = 1 *stone*; 2 stones = 1 *tod*; 6½ tods = 1 *wey*; 2 weys = 1 *sack*; 12 sacks = 1 *last*. A *pack* of wool = 240 lb. In weighing cheese and butter, 8 lb. equal 1 *clove*. A *bale* of cotton in Egypt weighs 90 lb.; in America a commercial bale is 400 lb., though it varies in different localities from 280 to 720 lb. A bale of Sea Island cotton is 300 lb.

VI. The following denominations are frequently used:

25 lb.	of powder	make	1 barrel.	100 lb.	of raisins	make	1 cask.
56 "	" butter	"	1 firkin.	196 "	" flour	"	1 barrel.
84 "	" "	"	1 tub.	200 "	" pork, beef or fish		1 barrel.
100 "	" grain or flour	"	1 cental.	240 "	" lime,	"	1 cask.
100 "	" dry fish	"	1 quintal.	280 "	" salt at N.Y.S.w'ks		1 barrel.
100 "	" nails	"	1 keg.	600 "	" rice	"	1 barrel.

COMPARISON OF WEIGHTS.

368. The **Troy Pound** and the **Apothecaries' Pound** each contains 5760 Troy grains; the *Avoirdupois* pound contains 7000 Troy grains. From this we readily derive the following table:

AVOIRDUPOIS.		TROY GR.	TROY OR APOTH.		TROY GR
1 lb.	=	7000	1 lb.	=	5760
1 oz.	=	437½	1 oz.	=	480
144 lb. Av.=175 lb. Troy.			192 oz. Av.=175 oz. Troy		

DIAMOND WEIGHT.

369. Diamond Weight is used in weighing diamonds and other precious stones.

TABLE.

16 parts . . .	equal	1 carat grain	=.792	Troy grains.
4 carat grains	"	1 carat	=3.168	" "

NOTE.—The *carat of weight* must be carefully distinguished from the *assay carat*. The former is an absolute weight; the latter is used to denote the proportion of pure gold in a mass, and is a twenty-fourth part of the mass. Thus gold 18 carats fine has 18 parts gold and 6 parts alloy.

EXAMPLES FOR PRACTICE.

1. Change 7 lb. 3 oz. 6 pwt. 3 gr. to Apothecaries' Weight.
Ans. 7 lb. 3℥. 2ʒ. 1℈. 7 gr.

2. Change 14 lb. 6 oz. 12 pwt. 6 gr. to Avoirdupois Weight.
Ans. 11 lb. 15.574$\frac{6}{7}$ oz.

3. Change 4 cwt. 72 lb. 8 oz. to Troy Weight.
Ans. 574 lb. 2 oz. 12 pwt. 12 gr.

4. Change 6 cwt. 20 lb. 12 oz. U. S. to English weight.
Ans. 5 cwt. 2 qr. 4 lb. 12 oz.

5. Which is heavier, and how much, a pound of gold or a pound of iron? *Ans.* The latter, 1240 gr.

6. Which is heavier, and how much, an ounce of silver or an ounce of lead? *Ans.* The former, 42$\frac{1}{2}$ gr

7. If Commodore Nutt weighs 23 lb. Avoirdupois, how much would he weigh by Troy Weight? *Ans.* 27$\frac{137}{144}$ lb.

8. I shipped 125 tons of iron from England; how much did it weigh by United States weight? *Ans.* 140 tons.

9. What is the weight of 24 gold eagles and 72 silver dollars? *Ans.* 6 lb. 2 oz. 15 pwt. 12 gr.

10. What is the weight of 10 sovereigns, 5 shillings, and 8 pence? *Ans.* 7 oz. 9 pwt. 13.09 gr.

11. What is the weight of $437.985 in English sovereigns; also in shillings?.

Ans. $\begin{cases} \text{1 lb. 11 oz. 2 pwt. 6.66 gr. in sovereigns.} \\ \text{27 lb. 3 oz. 5 pwt. 6 gr. in shillings.} \end{cases}$

12. What is the weight of the gold and also of the alloy in 720 sovereigns?

Ans. $\begin{cases} \text{Gold, 14 lb. 1 oz. 10 pwt. .84 gr.} \\ \text{Alloy, 1 lb. 3 oz. 8 pwt. 4.44 gr.} \end{cases}$

13. If I owe $1,000,000, and pay it in gold, what will be its weight Av.? *Ans.* 3685$\frac{5}{7}$ lb.

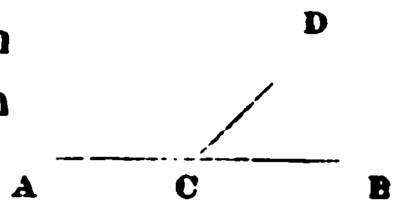
14. An apothecary bought 14 lb. 12 oz. of opium by Avoirdupois weight, at 62$\frac{1}{2}$ cts. an ounce, and retailed it at 5 cts. a scruple; how much did he gain? Ans $110.62½.

MEASURES OF LENGTH.

370. Measures of Length are used in measuring length, breadth, height, distance, etc.

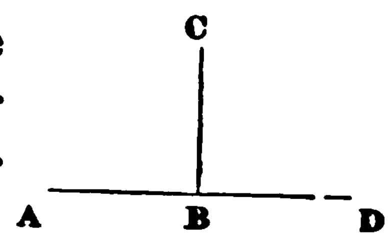
371. A **Line** is that which has length without breadth or thickness. It is estimated by ascertaining how many times it contains a *unit of measure.*

372. An **Angle** is the opening between two lines which diverge from a common point. Thus ACD and DCB are angles.



373. The **Vertex** of an angle is the point from which the two lines diverge; thus, C is the vertex of the angle BCD.

374. A **Right Angle** is formed by one line perpendicular to another; as, ABC or CBD One line is *perpendicular* to another when it makes the two adjacent angles equal.



LONG MEASURE.

375. Long Measure is used for the general purposes of measuring length and distances.

TABLE.

12 inches (in.)	. . =	1 foot . .	ft.
3 feet . . .	. =	1 yard . .	yd.
$5\frac{1}{2}$ yards, or $16\frac{1}{2}$ feet	. =	1 rod . .	rd.
320 rods . .	. =	1 mile . .	mi.
3 miles . .	. =	1 league . .	lea.
69.16 miles . .	. =	1 degree of latitude, deg. or ° or of longitude at the equator	

mi.		rd.		yd.		ft.		in.
1	=	320	=	1760	=	5280	=	63360
		1	=	$5\frac{1}{2}$	=	$16\frac{1}{2}$	=	198

SCALE.—Ascending, 12, 3, $5\frac{1}{2}$, 320, 69.16; descending, 69.16, 320, $5\frac{1}{2}$, 3, 12.

I. TERMS.—The units of length are nearly all derived from the different parts of the human body and from other objects. The ancient yard of England was the length of the arm of King Henry I. The term *inch* is from *uncia, a twelfth; foot* is from the human foot; *yard* was a *rod* or

shoot; *rod* is from a *measuring stick* or *rod*; *furlong*, now obsolete, is from *fur*, *furrow*, and *lang*, *long*, the *length of a furrow*; *mile* is from *mille passuum*, 1000 paces; *span* is the space measured from the end of the thumb to the end of the little finger extended; *cubit*, from the elbow to the end of the middle finger; *fathom*, the length of the two arms extended.

II. UNIT.—The *standard unit* of length is the *yard*, from which all other measures of length, and also those of capacity, weight, etc., are derived. It is identical with the Imperial yard of Great Britain, which, under William IV., was declared to be fixed by dividing a pendulum, which vibrates seconds in a vacuum, at the level of the sea, at 62° Fah., in the latitude of London, into 391393 equal parts, and taking 360000 of these parts for the yard. Subsequent scientific experiments have proved that such a standard is impracticable.—*See Brooks's Philosophy of Arithmetic.*

III. THE MILE.—The *geographic* or *nautical mile* is equal to 1 minute of one of the great circles of the earth; hence it equals $\frac{1}{60}$ of $\frac{1}{360}$ of the circumference of the earth, which equals about 1.15 statute miles. The English mile is the same as that of the United States. The German *short mile* equals 6857 yd., or about $3\frac{9}{10}$ statute miles; the German *long mile* equals 10125 yd., or about $5\frac{3}{4}$ statute miles; the Prussian mile equals 8237 yards, or about $4\frac{7}{10}$ statute miles. 3 statute miles make a *land league*; 3 nautical miles a *nautical league*.

IV. DEGREES.—A degree of longitude at any point is $\frac{1}{360}$ of the circle passing through the latitude of that point, and as these circles diminish as we pass from the equator, the degrees of longitude will diminish. Thus, at the equator, the length of a degree of longitude is about $69\frac{1}{6}$ statute miles; at 25° of latitude, $62\frac{7}{10}$ miles; at 40° of latitude, 53 miles; at 42°, $51\frac{1}{2}$ miles; at 49°, $45\frac{1}{2}$ miles; at 60°, $34\frac{7}{12}$ miles, etc. A degree of latitude also varies, being 68.72 miles at the equator; from 68.9 to 69.25 miles in middle latitude; and from 69.30 to 69.34 miles in the polar regions.

V. OTHER MEASURES.—The following denominations are frequently used: in clock-making, 6 *points* = 1 *line*, and 12 *lines* = 1 *inch*; in measuring the foot, 3 *barleycorns* or *sizes* = 1 *inch*; in measuring the height of horses, 4 *inches*=1 *hand*, the measure being taken directly over the shoulder; 1 *span* = 9 inches; 1 *common cubit* = 18 inches, and 1 *sacred cubit*=21.888 inches; 1 *pace*=3.3 feet; a *knot* is equal to a nautical mile. Formerly we had 40 *rods* equal 1 *furlong* and 8 *furlongs* one *mile*, but these are now seldom used.

SURVEYORS' LINEAR MEASURE.

376. Surveyors' Linear Measure is used by surveyors and engineers in measuring the dimensions of land, distances, etc.

TABLE.

7.92 inches (in.) . .	= 1 link . .	li.
100 links . . .	= 1 chain . .	ch.
80 chains . . .	= 1 mile . .	mi.

mi.		ch.		li.		in.
1	=	80	=	8000	=	63360
		1	=	100	=	792

SCALE.—Ascending, 7.92, 100, 80; descending, 80, 100, 7.92.

I. NAME.—*Gunter's chain* is named after the reputed inventor, Edmund Gunter, an English mathematician, born 1581.

II. UNIT.—The *unit* is a chain called *Gunter's Chain*, which consists of 100 links, and is 4 rods, 66 feet, or 792 inches long.

III. The denomination *rods* is seldom used by surveyors, distances being represented in chains and links. Since each link is $\frac{1}{100}$ of a chain, the number of links is generally expressed as a decimal; thus, 5 chains and 47 links are written 5.47 chains. Engineers generally use a chain 100 feet long.

MARINERS' AND CLOTH MEASURES.

377. Mariners' Measure is used by seamen in measuring distances, the depth of the sea, etc. *Cloth Measure* is used for measuring cloth, ribbons, etc.

MARINERS' MEASURE.			CLOTH MEASURE.		
6 feet	=	1 fathom.	1 yard	=	36 inches.
120 fathoms	=	1 cable length.	$\frac{1}{2}$ yard	=	18 inches.
880 fathoms	=	1 mile.	$\frac{1}{4}$ yard	=	9 inches.
			$\frac{1}{8}$ yard	=	$4\frac{1}{2}$ inches.

I. The *foot* and *yard* of these two measures are the linear foot and yard. The *nail* in Cloth Measure is obsolete. At the custom-houses, the yard is divided into tenths, hundredths, etc.

II. In the old table of Cloth Measure there were given 3 qr. = 1 Ell Flemish; 5 qr. = 1 Ell English; 6 qr. = 1 Ell French; 4 qr. $1\frac{1}{8}$ in. = 1 Ell Scotch.

EXAMPLES FOR PRACTICE.

1. Reduce 120 ch. 25 li., to miles. *Ans.* 1 mi. 161 rd.

2. Reduce 575 stat. miles to geog. miles. *Ans.* 500.

3. Reduce 12 cable lengths 60 fathoms, to chains. *Ans.* $136\frac{4}{11}$ ch.

4. Required the distance round the earth. *Ans.* 24897.6 mi.

5. If a horse is $16\frac{1}{2}$ hands high, what is its height in feet and inches? *Ans.* 5 ft. 6 in.

6. A ship was sailing in $12\frac{1}{2}$ fathoms of water; how deep was the water? *Ans.* 75 ft.

7. If a vessel sails 14 knots an hour, how many statute miles will it sail in 12 hours? *Ans.* 193 miles 64 rd.

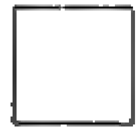
8. The soldier's common step is 28 inches, his double quick step 32 inches; how many of each must he take in marching a mile? *Ans.* $2262\frac{6}{7}$; 1980.

9. Two towns in Germany are 26 "long miles" from each other; what is the distance in "short miles" and in English miles? *Ans.* $38\frac{2684}{6857}$ short miles; $149\frac{101}{176}$ Eng. miles.

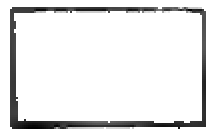
MEASURES OF SURFACE.

378. A Surface is that which has length and breadth without thickness.

379. A Square is a plane surface which has four equal sides and four right angles, as in the margin.

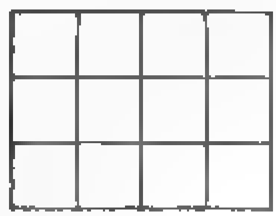


380. A Rectangle is a surface which has four sides and four right angles, as in the margin. A slate, a door, the sides of the room, etc., are examples of rectangles.



381. All **Surfaces** are *measured* by ascertaining the number of times they contain a *small square* regarded as the *unit of measure.*

Thus, in the surface in the margin there are three rows of squares, each row containing 4 squares; hence there are 3 times 4 or 12 squares in all; and since these make up the entire surface, the measure of the surface, called its *area*, is 12 square units.



SURFACE OR SQUARE MEASURE.

382. Surface or Square Measure is used in measuring surfaces, as land, boards, amount of painting, papering, plastering, paving, etc.

TABLE.

144 square inches (sq. in.)	=	1 square foot,	sq. ft.
9 square feet . .	=	1 square yard,	sq. yd.
$30\frac{1}{4}$ square yards, or $272\frac{1}{4}$ square feet .	=	1 perch or sq. rod,	P.
160 perches . . .	=	1 acre, . .	A
640 acres . . .	=	1 square mile,	sq. mi

A.		P.		sq. yd.		sq. ft.		sq. in.
1	=	160	=	4840	=	43560	=	6272640
		1	=	$30\frac{1}{4}$	=	$272\frac{1}{4}$	=	39204
				1	=	9	=	1296

SCALE.—Ascending, 144, 9, $30\frac{1}{4}$, 160, 640; descending, 640, 160, $30\frac{1}{4}$, 9, 144.

I. TERMS.—*Perch* is from the French *perche*, a *pole*; *acre* was primarily an open plowed or sowed field.

II. UNIT.—The unit for land is the *acre*; for other surfaces it is usually the *square yard*.

III. The *perch* is a surface equal to a *square rod*. The *rood* is found now only in old title-deeds and surveys; it is equal to 40 perches.

IV. A square piece of land, measuring 209 feet, or about 70 paces on each side, equals very nearly one acre.

SURVEYORS' SQUARE MEASURE.

383. Surveyors' Square Measure is used by surveyors in computing the area or contents of land

TABLE.

10,000 square links (sq. li.)	=	1 square chain,	sq. ch.
10 square chains	=	1 acre,	A.
640 acres	=	1 square mile,	sq. mi.
36 sq. mi. (6 miles square)	=	1 township,	Tp

Tp.		sq. mi.		A.		sq. ch.		sq. li.
1	=	36	=	23040	=	230400	=	2304000000
		1	=	640	=	6400	=	64000000
				1	=	10	=	100000

SCALE.—Ascending, 10,000, 10, 640, 36; descending, 36, 640, 10, 10,000.

I. Also 625 sq. li. = 1 perch; 16 perches = 1 sq. chain; 10 sq. ch. = 1 acre; or, 40 perches = 1 rood; 4 roods = 1 acre. The *perch* and *rood* are not so much used as formerly, the contents of land being commonly estimated in square miles, acres, and hundredths.

EXAMPLES FOR PRACTICE.

1. How many square chains in 10 A. 150 P.?
Ans. 109.375 sq. ch.

2. Reduce 5 A. 120 P. to sq. in. *Ans.* 36067680 sq. in.

3. Reduce 89794172 sq. in. to acres.
Ans. 14 A. 50 P. 13 sq. yd. 1 sq. ft. 20 sq. in.

4. Reduce 78985432184 sq. li. to townships.
Ans. 34 Tp. 10 sq. mi. 94 A. 3 sq. ch. 2184 sq. li.

5. Required the value of a field containing 45 sq. chains at $120 an acre. *Ans.* $540.

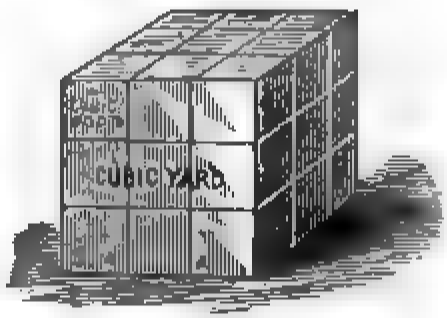
6. Bought 12 A. 100 P. of land at $160 an acre, and sold it for $16½ a square chain; what did I gain? *Ans.* $63.12½.

7. What is the difference in area between a garden bed *5 feet square* and one containing 5 square feet?
Ans. 20 sq. ft.

MEASURES OF VOLUME.

384. A **Volume** is that which has length, breadth, and thickness or height. A volume is also called a *solid.*

385. A **Cube** is a volume bounded by six equal squares.



386. A **Rectangular Volume** or **Solid** is a volume bounded by rectangles. Cellars, boxes, rooms, etc., are examples of rectangular volumes.

387. All **Volumes** are *measured* by ascertaining the number of times they contain a *small cube* regarded as a *unit of measure.*

Thus, in the cube in the margin, it will be seen that there are 3 times 3, or 9 cubes upon one surface, and since there are three such layers, there are 3 times 9, or 27 little cubes in all; and since these make up the entire volume, the measure of the cube, called its *contents*, is 27 cubic units.

CUBIC OR SOLID MEASURE.

388. **Cubic or Solid Measure** is used in measuring things which have length, breadth, and thickness.

TABLE.

1728 cubic inches (cu. in.)	=	1 cubic foot,	cu. ft.
27 cubic feet . .	=	1 cubic yard,	cu. yd.
16 cubic feet . .	=	1 cord foot,	cd. ft.
8 cord feet, or 128 cubic feet .	=	1 cord of wood,	cd.

cd.		cu. yd.		cu. ft.		cu. in.
1	=	$4\frac{20}{27}$	=	128	=	221184
		1	=	27	=	46656

SCALE.—Ascending, 1728, 27; descending, 27, 1728.

I. A *cord of wood*, so named from being originally measured by a *cord*, or *string*, is a pile 8 ft. long, 4 ft. wide, and 4 ft. high. A *cord foot* is a part of this pile 1 ft. long, it equals 16 cubic feet. See Art. 492.

II. The *ton* of 40 ft. for *round*, or 50 ft. for *hewn* timber is seldom used

EXAMPLES FOR PRACTICE.

1. Reduce 8 cd. 6 cd. ft. to cu. ft. *Ans.* 1120 cu. ft.

2. Reduce 78976 cd. ft. to cords. *Ans.* 9872 cd.

3. Reduce 8797 cu. ft. to cords. *Ans.* 68 cd. 93 cu. ft.

4. In 798765432 cu. in. how many cubic yards?
Ans. 17120 cu. yd. 8 cu. ft. 888 cu. in.

5. What is the difference between a 4 inch cube and 4 cubic inches? *Ans.* 60 cu. in.

MEASURES OF CAPACITY

389. Measures of Capacity are volumes used to determine the quantity of fluids and many dry substances.

390. Measures of Capacity are, therefore, of two kinds, *Measures of Liquids* and *Measures of Dry Substances.*

391. Liquid Measures are of two kinds, *Liquid* or *Wine Measure* and *Apothecaries' Fluid Measure.*

LIQUID OR WINE MEASURE.

392. Liquid or **Wine Measure** is used for measuring all kinds of liquids.

TABLE.

4 gills (gi.)	=	1 pint	pt.
2 pints	=	1 quart	qt.
4 quarts	=	1 gallon	gal.
$31\frac{1}{2}$ gallons	=	1 barrel	bar.
63 gallons or 2 bar.	=	1 hogshead	hhd.

hhd.	bar.	gal.	qt.	pt.	gi.
1 =	2 =	63 =	252 =	504 =	2016
	1 =	$31\frac{1}{2}$ =	126 =	252 =	1008
		1 =	4 =	8 =	32

SCALE.—Ascending, 4, 2, 4, $31\frac{1}{2}$, 2; descending, 2, $31\frac{1}{2}$, 4, 2, 4.

I. NAME.—It is called *Wine Measure* because wine was measured by it, while beer was measured by another measure.

II. TERMS.—*Gill* is from Low Latin *gilla*, a drinking glass; *pint* is from the Anglo-Saxon *pyndan*, *to shut in*, *to pen*, or from the Greek *pinto*, *to drink*; *quart* is from the Latin *quartus*, a *fourth*. The derivation of *gallon* is not clear; in the French, a *galon* is a *grocer's box.*

II. UNIT.—The *standard unit* of wine measure is the *gallon*, which contains 231 cubic inches, and will hold a little more than $8\frac{1}{3}$ lb. Av. of distilled water. This is called the *Winchester gallon*, from the standard having been formerly kept at Winchester, England. The *Imperial gallon*, now adopted by Great Britain, contains 277.274 cu. in., or 10 lb. Av. of distilled water, temperature 62° Fah., the barometer standing at 30 inches.

IV. Barrels and *hogsheads* are of variable capacity. The above values

are used in estimating the capacity of wells, cisterns, vats, etc. In Massachusetts, the barrel is estimated at 32 gallons. A pint of water weighs nearly one pound, hence the old adage, "A pint's a pound the world around."

V. Besides the above the following denominations are frequently given: 42 gal. = 1 tierce; 84 gal. = 1 puncheon; 2 hhd., or 126 gal. = 1 pipe or butt; 2 pipes = 1 tun. These are not measures, however, but vessels of no uniform capacity; they are usually gauged and have their capacities marked upon them.

VI. Ale, beer, and milk were formerly sold by a *gallon* of 282 cu. in., the subdivisions being *quarts* and *pints*. The measure was greater than wine measure, as beer was less costly than wine.

APOTHECARIES' FLUID MEASURE.

393. Apothecaries' Fluid Measure is used for measuring liquids in preparing medical prescriptions.

TABLE.

60 minims (♏)	=	1 fluidrachm	f℈.
8 fluidrachms	=	1 fluidounce	f℥.
16 fluidounces	=	1 pint	O.
8 pints	=	1 gallon	Cong.

SCALE.—Ascending, 60, 8, 16, 8; descending, 8, 16, 8, 60.

I. TERMS.—*Minim* is from the Latin *minimus*, *the least*, the minim being the smallest fluid measure used. Several of the other terms are formed by prefixing *fluid* to the terms of Apothecaries' Weight.

II. SYMBOLS.—Cong. is the abbreviation of *congius*, the Latin for gallon, O. is the initial of *octarius*, the Latin for *one-eighth*, the pint being one-eighth of a gallon. Drops are indicated in a physician's prescription by *gtt.*, for the Latin *guttæ*.

III. In estimating the quantity of fluids, 45 drops equal about a fluidrachm; a common teaspoon holds about one fluidrachm; a common tablespoon, about ½ a fluidounce; a wineglass, about 1½ fluidounces; a common teacup, about 4 fluidounces. The minim is equivalent to a drop of water; but the drops of different liquids vary in size according to the tenacity of the liquid.

DRY MEASURE.

394. Dry Measure is used in measuring dry substances, such as grain, fruit, salt, coal, etc.

TABLE.

2 pints (pt.)	=	1 quart	qt.
8 quarts	=	1 peck	pk.
4 pecks	=	1 bushel	bu.

bu.		pk.		qt.		pt.
1	=	4	=	32	=	64
		1	=	8	=	16

SCALE.— Ascending, 2, 8, 4; descending, 4, 8, 2.

I. TERMS.—*Peck* is supposed to be a corruption of *pack*, or to be derived from the French *picotin*, a *peck*.

II. UNIT.—The unit is the Winchester bushel, formerly used in England, and named from the place where the standard was preserved. In form it is a cylinder, 18½ in. in diameter, and 8 inches deep. Its volume is 2150.42 cu in., and contains 77.627413 lb Av of distilled water, at its maximum density. The New York bushel is nearly identical with the Imperial bushel of Great Britain, which contains 2218 192 cu. in.

III. The *Cental* of 100 lb. is a standard recently recommended by the Boards of Trade of New York, Cincinnati, Chicago, and other large cities for estimating grain, seeds, etc. Bushels are changed to centals, by multiplying by the number of pounds in 1 bushel, and dividing the product by 100. The remainder will be hundredths of a cental.

IV. The *Chaldron*, consisting in some places of 36 bu., and in others of 32 bu., is used in some parts of the United States for measuring coal and coke, but is being discontinued here, as it has been in England. The coal bushel contained 1 quart more than the Winchester bushel. Twenty-one chaldrons made a *score*. Foreign coal is imported by the chaldron, but American coal is generally bought and sold in large quantities by the ton, in small quantities by the bushel.

V. Where fruit and vegetables are sold by the basket or barrel, a peach basket should hold 2 pk., a potato basket 3 pk., and a barrel 3 potato baskets. Barrels made for measuring articles for market usually hold 100 quarts.

895. The **Weight of a Bushel** of the principal kinds of grain, seeds, and dried fruit has been fixed by statute in many of the States, as shown by the following

ARTICLES.	Cal.	Conn.	Del.	Ill.	Ind.	Iowa.	Ky.	La.	Me.	Mass.	Mich.	Minn.	Mo.	N. H.	N. J.	N. Y.	Ohio.	Or.	Penn.	R. I.	Vt.	W. T.	Wis.
Barley,	50			48	48	48	48	32		46	48	48	48		48	48	48	46	47		46	45	48
Beans,				60	60	60	60						60			62							
Blue Grass S'd,				14	14	14	14						14			[illegible]							
Buckwheat,	40	45		40	50	52	52			46	42	42	52		50	48		42	48		46	42	42
Clover Seed,				60	60	60	60				60	60	60		64	60	60	60				60	60
Dried Apples,				24	25	24					28	28	24					28				28	28
Dried Peaches,				33	33	33					28	28	33					28				28	28
Flax Seed,				56	56	56	56						56		55	55	56						56
Indian Corn,	52	56	56	52	56	56	56	56		56	56	56	52		56	58	56	56	56		56	56	56
Oats,	32	28		32	32	35	33½	32	30	30	32	32	35	30	30	32	32	34	32		32	36	32
Onions,				57	48	57	57			52			57							50		50	
Potatoes,		60		60	60	60	60		60				60	60	60	60		60		60	60	60	60
Rye,	54	56		54	56	56	56	32		56	56	56	56		56	56	56	56	56		56	56	56
Salt,					50	50	50						50			56							
Timothy Seed,				45	45	45	45						45			44							46
Wheat,	60	56	60	60	60	60	60	60		60	60	60	60		60	60	60	60	60		60	60	60

NOTE.—In Pennsylvania 80 lb. coarse, 70 lb. ground, or 62 lb. fine salt make 1 bushel, and in Illinois 50 lb. common, or 55 lb. fine salt make 1 bushel.

EXAMPLES FOR PRACTICE.

1. How many minims in 4 Cong. 2 O. 15 f℥. 7 fʒ.?

Ans. 268740 ♏.

2. How many Cong. in 8472347 ♏.?

Ans. 137 Cong. 7 O. 2 f℥. 5 fʒ. 47 ♏.

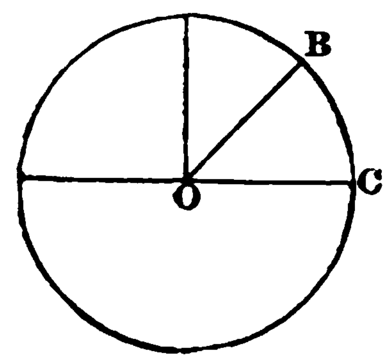
3. How many bushels in 78954 pints of timothy seed? *Ans.* 1233 bu. 2 pk. 5 qt.

4. What cost 8 gal. 3 qt. 1 pt. of kerosene at 3 cts. a pint? *Ans.* $2.13.

5. What cost 7 bu. 3 qt. 1 pt. of strawberries at 12¢ a pint? *Ans.* $54.60.

CIRCULAR MEASURE.

396. Circular Measure is used to measure angles and directions, latitude and longitude, etc.



397. A **Circle** is a plane figure bounded by a curved line, every point of which is equally distant from a point within called the *centre.*

398. The **Circumference** of a circle is the bounding line; any part of the circumference, as BC, is an *arc.* An arc of one-fourth of the circumference is called a *quadrant.*

399. For the purpose of measuring angles, the circumference is divided into 360 equal parts, called *degrees;* each degree into 60 equal parts, called *minutes;* each minute into 60 equal parts called *seconds.*

400. Any angle having its vertex at the centre, is measured by the arc included between its sides; thus, COB is measured by the arc BC. A right angle is measured by 90 degrees, or a quadrant; half a right angle, by 45 degrees, etc.

TABLE.

60 seconds (″) . .	=	1 minute . .	′
60 minutes . . .	=	1 degree . .	°
30 degrees . .	=	1 sign . . .	S.
12 signs, or 360° .	=	1 circumference.	C.

C		S.		°		′		″
1	=	12	=	360	=	21600	=	1296000
		1	=	30	=	1800	=	108000
				1	=	60	=	3600

Scale.—Ascending, 60, 60, 30, 12; descending, 12, 30, 60, 60.

I. TERMS.—The division of the circumference of the circle into 360 equal parts, took its origin from the length of the year, which (in round numbers) was supposed to contain 360 days, or 12 months of 30 days each. The 12 *signs* correspond to the 12 months. The term *minute* is from the Latin *minutum*, which signifies a *small part*. The term *second* is an abbreviated expression for *second* minutes, or minutes of the *second order*. *Signs* are used in astronomy as a measure of the zodiac.

II. UNIT.—The *unit* is the *degree*, which is $\frac{1}{360}$ of the circumference of a circle. A *quadrant* is one-fourth of a circumference, or 90°. A *minute* of the earth's circumference is called a *geographic mile*.

III. DIVISIONS.—The divisions of the circumference are not of absolute length, but are merely *equal parts*, indicating the size of angles. Thus, a *quadrant*, whether the circle is large or small, measures a *right angle*.

MEASURES OF TIME.

401. Time is a portion of duration. The *measures* of time are fixed by the revolution of the earth on its axis and around the sun.

402. A **Day** is the time of the revolution of the earth upon its axis; a *Year* is the time of the revolution of the earth around the sun

TABLE.

60 seconds (sec.)	=	1 minute	min
60 minutes	=	1 hour	h
24 hours	=	1 day	da.
365 days	=	1 common year	yr.
366 days	=	1 leap year	yr.
100 years	=	1 century	cen.

ALSO.

7 days	=	1 week	wk.
4 weeks	=	1 lunar month	mo.
12 calendar months, or 13 lunar months, 1 da., 6 h.,	=	1 year	yr.

yr.		mo.		wk.		da.		h.		min.		sec.
1	=	12	=			365	=	8760	=	525600	=	31536000
						366	=	8784	=	527040	=	31622400
				1	=	7	=	168	=	10080	=	604800
						1	=	24	=	1440	=	86400
								1	=	60	=	3600

Scale.—Ascending, 60, 60, 24, 7; descending, 7, 24, 60, 60.

I. TERMS.—*Second* and *minute* are parts of an hour, corresponding to the parts of a degree in Circular Measure. *Hour* is derived from the Latin *hora*, originally a definite space of time fixed by natural laws; a *day*, derived from the Saxon *daeg*, is the time of the revolution of the earth upon its axis; a *week* is a period of uncertain origin, but which has been used from time immemorial in Eastern countries; a *month*, from Saxon *monadh*, from *mona*, the moon, is the time of one revolution of the moon around the earth; a *year*, from Saxon *gear*, is the time of the earth's revolution around the sun; a *century* comes from the Latin *centuria*, a collection of a hundred things.

II. UNIT.—The *unit of time* is the *day;* it is determined by the revolution of the earth on its axis. The *Sidereal Day* is the exact time of the revolution of the earth on its axis. The *Solar Day* is the time of the apparent revolution of the sun around the earth. The *Astronomical Day* is the solar day, beginning and ending at noon. The *Civil Day* is the average length of all the solar days of the year; it begins at 12 o'clock midnight, and consists of two periods of 12 hours each.

THE CALENDAR.

403. **The Calendar** is a division of time into periods adapted to the purposes of civil life.

404. The **Year** is divided into 12 calendar months, three of which constitute a period called a *Season.*

405. The seasons, months, and number of days in each, are given in the following table:

No. of Mo.	Month.	Season.	No. of Days.
1	January,	Winter,	31
2	February,		28 or 29
3	March,	Spring,	31
4	April,		30
5	May,		31
6	June,	Summer,	30
7	July,		31
8	August,		31
9	September,	Autumn,	30
10	October,		31
11	November,		30
12	December,	Winter,	31

1. NAMES.—January is derived from *Janus*, the god of the year, to whom this month was sacred. February is from *februa*, the Roman festival of expiation, celebrated on the 15th of this month. January and February were added to the Roman calendar by Numa, Romulus having previously divided the year into 10 months. March is from *Mars*, the god of war and reputed father of Romulus. It was the first month of the Roman calendar. April is probably from the Latin *aperire*, to open, from the opening of the buds or the bosom of the earth in producing vegetation. May is from *Maia*, the mother of Mercury, to whom the Romans offered sacrifices on the first day of this month. June is from *Juno*, the sister and wife of Jupiter, to whom it was sacred. July was named by Mark Antony after *Julius* Cæsar, who was born in this month. It was previously called Quintilis. August was named after *Augustus* Cæsar, who entered upon his first *consulate* in this month. It was formerly called Sextilis, the

sixth month. September, October, November, December, are respectively named from the Latin numerals, *Septem*, *Octo*, *Novem*, and *Decem*, as, when the year began in March, they were the seventh, eighth, ninth, and tenth months, as their names indicate. It will be noticed that we have derived our names of the months directly from the Romans, as have most of the nations of modern Europe, while the days of the week in English are derived from the Saxons.

II. The number of days in each month is easily remembered by the following stanza:

Thirty days hath September,
April, June, and November;
All the rest have thirty-one,
Excepting February alone;
To which we twenty-eight assign,
Till leap year gives it twenty-nine.

406. The time from any day of one month to any day of another month in the same year is readily found by the following table:

TABLE

SHOWING THE NUMBER OF DAYS FROM ANY DAY OF ONE MONTH TO THE SAME DAY OF ANY OTHER MONTH IN THE SAME YEAR.

FROM ANY DAY OF	TO THE SAME DAY OF											
	Jan.	Feb.	Mar	Apr	May	J'ne	July	Aug	Sep.	Oct.	Nov	Dec
January	365	31	59	90	120	151	181	212	243	273	304	334
February	334	365	28	59	89	120	150	181	212	242	273	303
March	306	337	365	31	61	92	122	153	184	214	245	275
April	275	306	334	365	30	61	91	122	153	183	214	244
May	245	276	304	335	365	31	61	92	123	153	184	214
June	214	245	273	304	334	365	30	61	92	122	153	183
July	184	215	243	274	304	335	365	31	62	92	123	153
August	153	184	212	243	273	304	334	365	31	61	92	122
September	122	153	181	212	242	273	303	334	305	30	61	91
October	92	123	151	182	212	243	273	304	335	365	31	61
November	61	92	120	151	181	213	242	273	304	334	365	30
December	31	62	90	121	151	182	212	243	274	304	335	365

METHOD OF USING THE TABLE.—Suppose we wish to find the number of days from March 10th to November 16th. We find March in the vertical column, and November at the top, and at the intersection we find 245, to which adding 6 days we have 251, the number of days required.

The table being constructed for February 28 days, the proper allowance must be made for leap year.

ADJUSTMENT OF THE CALENDAR.

407. **A True or Solar Year** is the exact time in which the earth revolves around the sun. It varies a little as given by different authorities, but Laplace, Herschel, and

some others, reckon it at 365 da. 5 h. 48 min. 49.7 sec. Now since it is inconvenient to reckon the fractional part of a day each year, it is necessary to arrange a correct calendar in which each year may have a whole number of days This is done by causing some years to consist of 365 days and others of 366 days. The former are called *common years*, the latter *Bissextile* or *Leap* years.

408. The **Calendar** is reckoned according to the following rule:

Rule.—*Every year that is divisible by 4, except the centennial years, and every centennial year divisible by 400, is a leap year; all the other years are common years.*

NOTE.—The *centennial years* are the hundredth years, or those whose expressions in figures end in two ciphers.

EXPLANATION.—I. If we reckon 365 days as one year, the time lost in the calendar in one year is 5 h. 48 min. 49.7 sec., and in four years is 23 h. 15 min. 18.8 sec., that is, *one day*, lacking only 44 min. 41.2 sec.; hence the first error can be corrected by adding *one* day every *four* years, making the year to consist of 366 days.

II. If every fourth year be reckoned as leap year, since we add 44 min., etc., too much, the time *gained* in the calendar in four years is 44 min. 41.2 sec., and in 100 years it will be 18 h. 37 min. 10 sec., that is, *one* day lacking 5 h. 22 min. 50 sec.; hence the second error may be corrected by deducting one day from each centennial leap year, thus calling each centennial year a common year of 365 days.

III. Again, if every centennial year be reckoned as a common year, since we do not add enough, the time lost in 100 years will be 5 h. 22 min. 50 sec., and in 400 years it will be 21 h. 31 min. 20 sec.; hence the time lost in 400 years will be 1 day lacking 2 h. 28 min. 40 sec., and this error may be rectified by making every 4th centennial year a leap year. In the same way we may make the calendar correct for any number of years.

NOTES.—1. The reckoning of time among the ancients, owing to their ignorance of astronomy, was very inaccurate. The calendar adopted by Romulus consisted of only ten months, but Numa added two more, and arranged a system of intercalations, which, had it been adhered to, would have made the year to average 365¼ days. But changes were frequently made for political reasons, and the calendar fell into such confusion that the civil equinox, in the time of Cæsar, differed from the astronomical by three months. The calendar was reformed by Julius Cæsar, 46 B. C., who decreed that the year should consist of 365¼ days, and since it was not convenient to count the ¼ of a day every year, every 4th year was made to consist of 366 days. This extra day, called the *inter-calary* day, was introduced by counting the 24th of February twice. This day, being the sixth before the kalends of March, the years containing it were called *bissextile* (*bis-sextile*), having two sixths. With us it is called Leap Year because it *leaps*, as it were, over a day.

2. The correction of Cæsar assumed the year to consist of 365 days, 6 hours, which is 11 min. 10.3 sec. too much; hence his correction introduced a slight error, which in 1582 had amounted to 10 days—the civil

year being 10 days behind the solar year. In 1582 Pope Gregory corrected the error by striking 10 days out of the calendar, calling the 5th of October the 15th, and ordering that henceforth only those centennial years should be leap years which are divisible by 400.

3. The Gregorian calendar was soon adopted by most Catholic countries. Great Britain adopted the change in 1752, calling the 3d of September the 14th, the error having amounted to 11 days. Russia and the other countries of the Greek Church still adhere to the Julian calendar, their dates being now about 12 days behind ours. The two calendars are distinguished as Old Style and New Style, marked O. S. and N. S. respectively. In the Old Style the *civil* or *legal* year commenced on the 25th of March, while the *historical* year commenced on the first of January, and dates between those days were marked with the number of both years; thus, January 30th, 1649, is frequently found written, Jan. 30th, $164\frac{8}{9}$. The New Style made the civil year commence also on the 1st of January.

EXAMPLES FOR PRACTICE

1. How many minutes in a leap year? *Ans.* 527040.
2. How many seconds in a solar year? *Ans.* 31556929.7.
3. How many leap years from 1800 to 1861? *Ans.* 15.
4. How many $''$ in 1 quadrant and $12°$? *Ans.* $367200''$.
5. How many days from June 15 to Dec. 9? *Ans.* 177.
6. How many days in the 16th century? *Ans.* 36525.
7. How many degrees in $17651''$? *Ans.* $4° 54' 11''$.
8. In 6 S. $25° 56'$, how many minutes? *Ans.* 12356.
9. How many minutes from 15 minutes past 9 A. M. to 20 minutes of 12 A. M.? *Ans.* 145.
10. How many hours and minutes from 7 h. 25 min. A. M. to 3 h. 45 min. P. M.? *Ans.* 8 h. 20 min.
11. In what time does a fixed point on the earth's surface pass through $60° 15' 30''$? *Ans.* 4 h. 1 min. 2 sec.
12. How many lunar months of 29 da. 12 h. 44 min. 2.7 sec. in a solar year? *Ans.* 12.36+.
13. The average daily motion of Mercury is $4° 5' 32.42''$; how long will it require to complete a revolution in its orbit? *Ans.* 87 da. 23 h. 15 min. 43.6— sec.
14. In the Julian calendar a year equals 365 da. 6 h.; in how many years was a day gained? *Ans.* 128.89+.
15. Venus revolves around the sun in 224 da. 16 h. 49 min. 7.98 sec.; what is its daily motion? *Ans.* $1° 36' 7.67''$.
16. A steamer sailing due east at the equator changes her longitude $3'$ every 15 minutes; how many knots an hour is she making? *Ans.* 12 knots.

MISCELLANEOUS TABLES.

409. The following tables are frequently used, the first in counting certain kinds of articles, and the second in the paper trade.

COUNTING.	PAPER.
12 units = 1 dozen.	24 sheets = 1 quire.
12 dozen = 1 gross.	20 quires = 1 ream.
12 gross = 1 great gross.	480 sheets = 1 ream.
20 units = 1 score.	

I. *Two* things of a kind are frequently called a *pair* and *six* a *set*.
II. Paper is sold at retail by sheets and quires, and at wholesale by reams.

BOOKS.

410. In printing books large sheets of paper are used, which are folded into leaves according to the size of the book. The terms *folio*, *quarto*, *octavo*, etc., as applied to printed books, are based on sheets about 18 × 24 in., about half the sizes now generally used, and indicate the number of leaves into which such a sheet is folded.

A sheet folded in 2	leaves is called	a folio, makes 4 pages.
A sheet folded in 4	"	a quarto or 4to, makes 8 pages.
A sheet folded in 8	"	an octavo or 8vo, makes 16 pages.
A sheet folded in 12	"	a 12mo, makes 24 pages.
A sheet folded in 16	"	a 16mo, makes 32 pages.
A sheet folded in 18	"	an 18mo, makes 36 pages.
A sheet folded in 24	"	a 24mo, makes 48 pages.
etc.		etc.

NOTE.—Printing paper is made of many sizes, according to the requirements of the printer. In book printing 24×38 inches, called *Double Medium*, is perhaps used most largely.

411. Clerks and copyists are often paid by the *folio* for making copies of legal papers, records, and documents.

72 words make 1 folio, or sheet of common law.
90 " " 1 " " " " chancery.

EXAMPLES FOR PRACTICE.

1. A cabinet-maker uses 48 screws a day; how many gross will he use in 4 weeks? *Ans.* 8 gross.

2. The agent of a Liverpool steamer ships for the voyage 4128 eggs packed in 8 boxes; how many dozen in a box?

Ans 43.

3. A weekly newspaper has 4750 subscribers; how much printing paper would it require in 1 year?

Ans. 514 reams, 11 quires, 16 sheets.

4. On taking an account of stock, a hardware merchant finds he has on hand 11 gross 3.75 dozen of white door knobs; what is the number of knobs? *Ans.* 1629.

5. How much paper is required to issue an edition of 2000 copies of a 16mo. book of 416 pp., allowing 1 quire in a ream for waste? *Ans.* 57 reams, 9 sheets.

6. A lady copies 11,700 words of common-law folios, at 10¢ per folio; what does she receive? *Ans.* $16.25.

7. A chancery case contains 561,420 words; what does the copying cost at $12\frac{1}{2}$¢ per folio? *Ans.* $779.75.

8. A stationer, on making an inventory at the close of the year, finds that he has on hand 11 packages of Gillott's steel pens of a dozen boxes each, a broken package containing 10 boxes, and an open box containing 7 doz. 5 pens; if each box contains 1 gross, how many pens has he on hand?

Ans. 20,537 pens.

THE METRIC SYSTEM.

412. In the **Metric System** we first establish the unit of each measure, and then derive the other denominations by taking decimal multiples and divisions of the unit. Any quantity consisting of several denominations is thus written and treated as an integer and decimal.

413. Names.—We first name the *unit* of any measure, and then derive the other denominations by prefixing words to the unit name.

414. The *higher denominations* are expressed by prefixing to the name of the unit,

DEKA	HECTO	KILO	MYRIA
10	100	1000	10000

415. The *lower denominations* are expressed by prefixing to the name of the unit,

DECI	CENTI	MILLI
$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$

416. Units.—The following are the different units, with the English pronunciation:

Measure.	Unit.	Pronunciation.	Measure.	Unit.	Pronunciation.
LENGTH,	**Meter,**	(meter.)	CAPACITY,	**Liter,**	(leeter.)
SURFACE,	**Are,**	(air.)	WEIGHT,	**Gram,**	(gram.)
VOLUME,	**Stere,**	(stair.)	VALUE,	**Dollar,**	

In 1795, France adopted a system of weights and measures based upon the decimal scale, called the *Metric System*. This has been adopted by Italy, Spain, Portugal, many parts of Spanish America, Belgium, Holland, Germany, Austria, Switzerland, Sweden, Denmark, Greece, Mexico, and Brazil. In 1864, the Parliament of Great Britain passed an act permitting its use throughout the United Kingdom wherever parties should agree to use it.

In 1866 its use was authorized in this country by Congress, and to furnish a convenient standard of comparison, and render the public familiar with the new measures, the five-cent piece issued at this time was ordered to be made 5 grams in weight, and $\frac{1}{50}$ of a meter in diameter. The system has not yet come into general use in this country, but is employed in the natural sciences, and to some extent in the U. S. Coast Survey and other branches of the public service.

The base of the system is the *meter*, which is $\frac{1}{10000000}$ of the distance from the equator to either pole, as determined with the greatest care by the measurement of an arc of the meridian.

MEASURES OF LENGTH.

417. The **Meter** is the *unit of length.* It is the ten-millionth part of the distance from the equator to the poles, and equals 39.37 inches, or 3.28 feet. The standard meter is a bar of platinum deposited in the archives of Paris.

TABLE.—**10 millimeters (mm.) = 1 centimeter (cm.); 10 centimeters = 1 decimeter (dm.); 10 decimeters = 1 meter, (M.); 10 meters = 1 decameter (DM.); 10 decameters = 1 hectometer (HM.); 10 hectometers = 1 kilometer (KM.); 10 kilometers = 1 myriameter (MM.).**

NOTES.—1. The *meter* is very nearly 3 feet, 3 inches, and 3-eighths of an inch in length, which may be easily remembered as the rule of *three threes.*

2. Cloth, etc., are measured by the *meter;* very small distances, by the *millimeter;* great distances, by the *kilometer.*

3. The 5-cent piece of 1866 is very nearly $\frac{1}{50}$ of a *meter* in diameter; hence its diameter is about $\frac{1}{5}$ of a *decimeter*, or 2 *centimeters.* It was ordered to be $\frac{1}{50}$ of a meter in diameter, but owing to the composition of the alloy it was necessary to make its diameter a little greater; 48.6 nickel 5-cent pieces laid side by side measure one meter.

4. A *decimeter* is about 4 inches; a *kilometer*, about 200 rods, or $\frac{5}{8}$ of a mile; a *millimeter*, about $\frac{1}{25}$ of an inch. The *inch* is about $2\frac{1}{2}$ centimeters; the *foot* 3 decimeters; the *rod*, 5 meters; the *mile*, 1600 meters, or 16 hectometers.

MEASURES OF SURFACE.

418. The **Are** is the *unit of surface* used to measure land. The *are* is a *square decameter.* It equals 119.6 sq. yd., or 0.0247 acre.

TABLE.—10 centiares (ca.) = 1 deciare (da.); 10 deciares = 1 are (A.); 10 ares = 1 decare (DA.); 10 decares = 1 hectare (HA.).

NOTES.—1. The *are*, *centiare*, and *hectare* are the denominations principally used, as these are exact squares. The centiare is a square whose side is 1 meter; the hectare is a square whose side is 100 meters. The *are* = 100 square meters. The *centiare* = 1 square meter. The *hectare* = 10,000 square meters.

2. The *deciare* is not a square, it is merely the tenth of an are; the *decare* is not a square, it is merely 10 ares.

3. A *hectare* equals very nearly $2\frac{1}{2}$ acres; a *centiare* equals nearly $1\frac{1}{5}$ sq yd. An *acre* is very nearly 40 ares.

MEASURES OF OTHER SURFACES.

419. All surfaces besides land are measured by the *square meter, square decimeter,* etc. The measures are shown by the following table:

TABLE.—100 sq. millimeters (mm^2.) = 1 sq. centimeter ($cm.^2$); 100 sq. centimeters = 1 sq. decimeter ($dm.^2$); 100 sq. decimeters = 1 sq. meter (M^2.).

NOTE.—The measures higher than these are not generally used. The usual method of notation is to write *sq.* before the denomination; but I suggest as an abbreviation that we indicate the square by an exponent.

MEASURES OF VOLUME.

420. The **Stere** is the *unit of volume.* It is a *cubic meter*, and equals 35.3166 cubic feet, or 1.308 cu. yd.

TABLE.—10 decisteres (ds.) = 1 stere (S.); 10 steres = 1 decastere (DS.).

NOTE.—Wood is measured by this measure. The *stere*, *decastere*, and *decistere* are principally used. 3.6 steres, or 36 decisteres very nearly equal the common cord.

MEASURES OF OTHER VOLUMES.

421. Other solid bodies are usually measured by the *cubic meter* and its divisions. The measures are shown by the following table:

TABLE.—1000 cubic millimeters ($mm.^3$) = 1 cubic centi

meter (cm.^{3}); 1000 cubic centimeters=1 cubic decimeter (dm.^{3}); 1000 cubic decimeters=1 cubic meter (M^3.).

NOTE.—The higher denominations are not generally used. I indicate the cubic measures with an exponent, instead of writing *cu.* before the denominations.

MEASURES OF CAPACITY.

422. The **Liter** is the *unit of capacity.* It equals a *cubic decimeter;* that is, a cubic vessel whose edge is one-tenth of a meter. It is used for measuring liquids and dry substances. The *liter* is a cylinder, and holds 2.1135 pints wine measure, or 1.816 pints dry measure.

TABLE.—10 milliliters (ml.)=1 centiliter (cl.); 10 centiliters=1 deciliter (dl.); 10 deciliters=1 liter (L.); 10 liters =1 decaliter (DL.); 10 decaliters=1 hectoliter (HL.); 10 hectoliters=1 kiloliter (KL.); 10 kiloliters=1 myrialiter (ML.).

NOTES.—1. The *liter* is principally used in measuring *liquids*, and the *hectoliter* in measuring grains, etc.

2. The *liter* equals nearly $1\frac{1}{18}$ liquid quarts, or $\frac{9}{10}$ of a dry quart, or nearly $\frac{1}{36}$ of a bushel measure.

3. The *hectoliter* is about $2\frac{5}{6}$ bushels or $\frac{5}{6}$ of a barrel. 4 liters are a little more than a *gallon;* 35 liters, very nearly a *bushel.*

MEASURES OF WEIGHT.

423. The **Gram** is the *unit of weight.* It is the weight of a cubic centimeter of distilled water at the temperature of melting ice. The gram equals 15.432 Troy grains.

TABLE.—10 milligrams (mg.)=1 centigram (cg.); 10 centigrams=1 decigram (dg.); 10 decigrams=1 gram (G.); 10 grams=1 decagram (DG.); 10 decagrams=1 hectogram (HG.); 10 hectograms=1kilogram (KG., or K.); 10 kilograms=1 myriagram (MG.).

NOTES.—1. The *gram* is used in weighing letters, in mixing and compounding medicines, and in weighing all very light articles. The five-cent coin adopted 1866 weighs 5 grams.

2. The *kilogram* is the ordinary unit of weight, and is generally abbreviated into *kilo.* It equals about $2\frac{1}{5}$ pounds avoirdupois. Meats, sugar, etc., are bought and sold by the *kilogram.*

3. In weighing heavy articles, two other weights, the *quintal* (100 kilograms) and the *tonneau* (1000 kilograms) are used. The *tonneau* is between our *short ton* and *long ton.*

4. The *avoirdupois ounce* is about 28 *grams;* the *pound* is a little less than $\frac{1}{2}$ a *kilo.* The U. S. post offices receive 15 grams, though a little overweight, as equivalent to an ounce avoirdupois.

5. Some of the old weights and measures are still used in France; 1 livre $=\frac{1}{2}$ a kilogram; 1 marc $=\frac{1}{2}$ a livre; 1 once $=\frac{1}{8}$ a marc; 1 gros $=\frac{1}{8}$ an once; 1 grain $=\frac{1}{72}$ gros; 1 toise $=2$ mètres; 1 pied or foot $=\frac{1}{3}$ mètre; 1 inch $=\frac{1}{12}$ pied or foot; 1 aune $=1\frac{1}{5}$ mètres; 1 boisseau or bushel $=12\frac{1}{2}$ litres; 1 litron $=1.074$ Paris pints. When these are employed, the word *usuel* is annexed to them, signifying *customary*.

424. Units of the common system may readily be changed to those of the Metric System by the following

TABLE.

1 Inch = 2.54 Centimeters.	1 Cu. Inch = 16.39 Cu. Centim.
1 Foot = 30.48 Centimeters.	1 Cu. Foot = 28320 Cu. Centim.
1 Yard = .9144 Meter.	1 Cu. Yard = .7646 Cu. Meters.
1 Rod = 5.029 Meters.	1 Cord = 3.625 Steres.
1 Mile = 1.6093 Kilometers.	1 Fl. Ounce = 2.958 Centiliters.
1 Sq. Inch = 6.4528 Sq. Centimeters.	1 Gallon = 3.786 Liters.
1 Sq. Foot = 929 Sq. Centimeters.	1 Bushel = .3524 Hectoliters.
1 Sq. Yard = .8361 Sq. Meters.	1 Troy Gr. = 64.8 Milligrams
1 Sq. Rod = 25.29 Centiares.	1 Troy lb. = .373 Kilo.
1 Acre = 40.47 Ares.	1 Av. lb. = .4536 Kilo.
1 Sq. Mile = 259 Hectares.	1 Ton = .907 Tonneau.

NUMERATION AND NOTATION.

425. In the **Metric System** the decimal point is placed between the unit and its divisions, the whole quantity being regarded as an integer and a decimal. Thus, 3 decagrams, 5 grams, 6 decigrams, 8 centigrams, are written 35.68 grams.

NOTE.—The *initials* of the denomination may be placed either before or after the quantity, though they are most frequently placed after it; thus, 27 grams may be written G27, or 27G.

EXERCISES IN NUMERATION AND NOTATION.

1. Read 48.64 M., 85.87 A., 48.89 M^2.
2. Read 854.17 S., 506.347 L., 4007.563 G.
3. Write 12 meters, 3 decimeters, 5 centimeters.
4. Write 8 hectares, 10 ares, 17 centiares.
5. 9 kilograms, 5 hectograms, 4 grams and 1 centigram

REDUCTION OF THE METRIC SYSTEM TO THE COMMON SYSTEM.

1. How many pounds Av. in 488.125 grams?
Ans. 1 lb. 1 oz. 95.245 gr.
2. Grams in 24 pounds Troy? *Ans.* 8958.009 G.
3. *Meters in* 4 mi. 240 rd? *Ans.* 7644.399 M.

4. Miles in 2000 meters? *Ans.* 1 mi. 77 rd. 11 ft. 2 in
5. Acres in 1011.2 ares? *Ans.* 24 A. 156.2624 P.
6. Ares in 11 A. 48 P.? *Ans.* 457.489 A.
7. Cu. ft. in 429.56 steres? *Ans.* 15170.5987 cu. ft.
8. Steres in 32 cu. yd. 16 cu. ft.? *Ans.* 24.917 S.
9. Gallons in 90.1 liters? *Ans.* 23 gal. 3 qt.
10. Liters in 73 gallons? *Ans.* 276.319 L.
11. Bushels in 130.5 liters? *Ans.* 3 bu. 2 pk. 6.49 qt

PRACTICAL PROBLEMS.

1. What cost 48.625 meters of cloth, if 9.725 meters cost $36.75? *Ans.* $183.75.

2. What must I pay for 75.25 steres of wood at the rate of $2.65 a stere? *Ans.* $199.41.

3. Bought 15.25 liters of wine in Bordeaux, at 75.5 francs a liter; what is the cost in U. S. money? *Ans.* $222.22

4. How much must be paid for 12.5 grams of jewels, at $6.50 a gram? *Ans.* $81.25.

5. What is the cost of 672.25 grams of opium at 62½¢ a gram? *Ans.* $420.16.

6. Mr. Brown imported for his house 35.429 meters of French carpet, at 19.75 francs a meter, including duty; required the whole cost. *Ans.* 699.72+fr.

7. Mr. Winslow bought a valuable gem in Paris which weighed 245.25 grams, @ 10.25 francs, duty $4.75; how must he sell it a gram to clear $100? *Ans.* $2.41.

8. An importer bought 428.5 grams of drugs in France, at 12.5 francs a gram, paid 31½ cents a gram duty and freight, and sold them for $2.25 a gram; how much was gained or lost? *Ans.* Lost $204.61.

9. I bought 175.25 liters of French brandy at 7.50 francs a liter, paid 15 cents a liter duty and freight, and sold it in New York at $1.65 a liter; how much did I gain? *Ans.* $9.20.

10. Jordan, Marsh, & Co. bought 200 meters of silk in Lyons, at 16.25 francs a meter; after paying $2 a yard duty and freight, they sold it in Boston at $6.12½ a yard; what was their *profit*? Ans. $274.98.

REDUCTION OF COMPOUND NUMBERS.

426. Reduction is the process of changing a number from one denomination to another, without altering its value.

427. There are **Two Cases**: *Reduction Descending* and *Reduction Ascending*.

These two cases have been considered in the examples under the tables, but we will present a few more problems under their proper heads.

REDUCTION DESCENDING.

428. Reduction Descending is the process of reducing a number to a lower denomination

1. Reduce £8 6 s. 4 d. to pence.

SOLUTION.—In 1 pound there are 20 shillings, and in £8 there are 8 times 20 shillings, plus 6 shillings are 166 shillings: in 1 shilling there are 12 pence, and in 166 shillings there are 166 times 12 pence, plus 4 pence equals 1996 pence. Therefore, etc.

OPERATION.

£	s.	d.
8	6	4
20		
166 s.		
12		
1996 d., *Ans.*		

Rule.—I. *Multiply the number of the highest denomination given, by the number of units of the next lower denomination which equals one of this higher, and to the product add the number given, if any, of this lower denomination.*

II. *Multiply this result as before, and proceed in the same manner until we arrive at the required denomination.*

2. Reduce 8 ℔. 4 oz. 6 pwt. 12 gr. to gr. *Ans.* 48156.

3. Reduce 9 ℔. 11 ℥ 3 ʒ 2 ℈ 4gr. to gr. *Ans.* 57344.

4. Reduce 124 A. 140 P. to sq. yd. *Ans.* 604395.

5. Reduce 120 cd. 6 cd. ft. to cubic feet. *Ans.* 15456.

6. Reduce 52 hhd. 24 gal. 3 qt. to pints. *Ans.* 26406.

7. 6 Circ. 10 S. 16° 20′ 20″ to seconds. *Ans.* 8914820.

8. Cong. vij. O. iv. f℥ vj. fʒ iij. to minims. *Ans.* 463860.

9. A farmer sold 16 A. 132 P. of land at $1.25 a square rod; how much did he receive? *Ans.* $3365.

10. A man bought 6 bu. 3 pk. 5 qt. of berries for $10.25, and sold them at 10 cents a quart; how much did he gain? *Ans.* $11.85.

REDUCTION ASCENDING.

429. Reduction Ascending is the process of reducing a number to a higher denomination.

1. In 246374 grains, how many pounds?

OPERATION.

gr.
24)246374
20)10265+14 gr.
12)513+5 pwt.
42 lb.+9 oz.

Ans. 42 lb. 9 oz. 5 pwt. 14 gr.

SOLUTION.—There are 24 gr. in 1 pwt., hence in 246374 gr. there are as many pwt. as 24 is contained times in 246374, which is 10265 pwt. and 14 gr. remaining: there are 20 pwt. in 1 oz., hence in 10265 pwt. there are as many ounces as 20 is contained times in 10265, which are 513 oz., and 5 pwt. remaining: there are 12 oz. in 1 pound, and in 513 oz. there are as many pounds as 12 is contained times in 513, which are 42 lb. and 9 oz. remaining. Therefore in 246374 grains there are 42 lb. 9 oz. 5 pwt. 14 gr.

Rule.—I. *Divide the given number by the number of units in that denomination which equals one of the next higher.*

II. *Divide the quotient in the same way, and thus proceed until we arrive at the required denomination.*

III. *The last quotient and the remainders, if any, will be the result required.*

2. 346256 gr. to ℔. *Ans.* 60 ℔. 1 ℥ 2 ʒ 2 ℈ 16 gr.

3. 4763254 li. to miles. *Ans.* 595 mi. 32 ch. 54 li.

4. 764325 cu. in. to cubic yards.
Ans. 16 cu. yd. 10 cu. ft. 549 cu. in.

5. 74625 ♏. to Cong.
Ans. 1 Cong. 1 O. 11 f℥ 3 fʒ 45 ♏.

6. 25627542 sq. li. to acres.
Ans. 256 A. 2 sq. ch. 7542 sq. li.

7. The side of a square field is 360 ft. long; how many rods of fence will enclose it? *Ans.* 87 rd. 1 yd. 1 ft. 6 in.

8. A dealer sold 1 ton of fish at $4.00 a quintal; what did it amount to? *Ans.* $80.00.

9. A miller sold 2560 lb. of flour at the rate of $9.00 a barrel; what did it amount to? *Ans.* $117.55.

10. Bought 7420 square rods of land at $172 an acre, and sold it for $7000; how much did I lose? *Ans.* $976.50.

MISCELLANEOUS PROBLEMS.

1. How many times will a wheel 11 ft. 6 in. in circumference revolve in going 50 miles? *Ans.* $22956\frac{12}{23}$ times.

2. I have a watch which is $18\frac{3}{4}$ carats fine; how much pure gold is there in it? *Ans.* $\frac{25}{32}$ of the whole.

3. My watch is $\frac{3}{4}$ and my chain $\frac{2}{3}$ pure gold; how many carats fine is each? *Ans.* 18 carats; 16 carats.

4. How long will it take to count 6 millions at the rate of 80 a minute, working 10 hours a day? *Ans.* 125 days.

5. A young lady weighs 125 lb. Troy weight; how much does she weigh Avoirdupois weight? *Ans.* $102\frac{6}{7}$ lb.

6. What is the weight of the silver in English silver coin worth $447.718? *Ans.* 25 lb. 9 oz. 8 pwt. 21.54 gr.

7. What is the difference in the weight of $960 in gold coin or in silver half-dollars? *Ans.* 60 lb.

8. How many demijohns can be filled from 4 hhd. of wine, each demijohn holding 2 gal. 3 qt. 1 pt.? *Ans.* $87\frac{15}{23}$.

9. The Gregorian calendar adds 97 days in 400 years; how long will it require to gain a day? *Ans.* $3874\frac{98}{223}$ years.

10. What day of the week and what day of the year is the 4th of July in a common year which begins on Monday?

11. A captain of a vessel, taking the soundings, found the water to be 760 fathoms in depth; what part of a mile was it? *Ans.* $\frac{19}{22}$.

12. How many copies of a duodecimo book can be printed on 66 reams 15 quires of paper, using 10 sheets per volume? *Ans.* 3204.

13. A grocer bought 100 bushels of coarse salt (50 lb. to a bushel) at 62¢, and sold it at $1\frac{7}{8}$ of a cent a lb.; what was his gain? *Ans.* $31.75.

14. Find the number of minims in the following prescription: Tincture of digitalis ♏xv, distilled vinegar fʒj, syrup fʒj, water f℥jss. *Ans.* 855♏.

15. The wheels of a locomotive are 15 ft. 3 in. in circumference and make 5 revolutions a second; in what time will the locomotive run 75 miles? *Ans.* 1 h. 26 min. $33\frac{27}{61}$ sec.

16. How many quart, pint, and half-pint bottles, of each

an equal number, may be filled from a vessel containing 54 gal. 1 qt.? *Ans.* 124 of each.

17. A boy brought from the bank a bag of gold weighing 1 lb. 9 oz. 10 pwt.; required its value. *Ans.* $400.

18. A man bought 50 cords of wood, $3\frac{1}{2}$ ft. long, and proposes to put it in a pile 12 ft. high; how long will the pile be? *Ans.* $152\frac{8}{21}$ ft.

19. William rises 40 minutes earlier and retires 30 minutes later than his companion; how much time will he gain in 4 school sessions of 26 weeks each?

Ans. 35 da. 9 h. 20 min.

20. An apothecary bought 50 lb. 8 oz. of opium at 45 cents an ounce Avoirdupois, and sold it at 3 cents a scruple; did he gain or lose, and how much? *Ans.* Gained $166.65.

21. A man bought 2 cwt. 87 lb. 10 oz. of sugar, at 6 cts. a pound, and retailed it at $6\frac{1}{2}$ cts. a pound, using by mistake Troy weights; how much did he make? *Ans.* $5.46\frac{1}{4}$.

22. In the following prescription find the number of grains of the solid and the number of minims of the liquid part:

℞. Mellis, confectionis rosæ caninæ āā ℥ij; aceti distillati f℥iij; acidi hydrochlorici ♏xxx; aquæ rosæ f℥j; aquæ puræ f℥vj. Misce. *Ans.* 240 gr.; 4830 ♏.

23. A man having a piece of ground which he wished accurately surveyed, employed two surveyors, the first of whom reported its contents to be 2 A. 54 P. 64 in.; the other reported 2 A. 53 P. 30 yd. 2 ft. 100 in. Wishing to know which was right, he employed a third, who gave the area as 2 A. 53 P. 272 ft. 100 in. How much did the three estimates differ from one another?

24. The distance from Boston to Chicago being about 1040 miles, if one man should start from Boston and one from Chicago on Monday, November 3, 1873, the first traveling 2 miles 299 rd. 12 ft. per hour, and the second 3 miles 20 rd. $4\frac{1}{2}$ ft. per hour, both starting at 9 A. M., traveling 7 hours a day, and resting on the Sabbath, at what time will they meet, and how far will each have traveled?

Ans. Dec. 1st, 2 h. 20 min. P. M.

ADDITION OF COMPOUND NUMBERS.

430. Addition of Compound Numbers is the process of finding the sum of two or more similar compound numbers.

1. Find the sum of £9 13 s. 11 d.; £17 15 s. 9 d.; £15 12 s. 5 d.; and £23 11 s. 10 d.

OPERATION.		
£	s.	d.
9	13	11
17	15	9
15	12	5
23	11	10
66	13	11

SOLUTION.—We write the numbers so that similar units shall stand in the same column, and begin at the right to add. 10 d. plus 5 d., plus 9 d., plus 11 d. are 35 d., which by reduction we find equals 2 s. and 11 d.; we write the 11 d. under the column of pence and reserve the 2 s. to add to the column of shillings: 2 s. plus 11 s., plus 12 s., plus 15 s., plus 13 s. are 53 s., which by reduction we find equals £2 and 13 s.: we write the 13 s. under the column of shillings and reserve the £2 to add to the column of pounds: £2 plus £23, plus £15, plus £17, plus £9, equals £66, which we write under the column of pounds.

Rule.—I. *Write the compound numbers so that similar units stand in the same column.*

II. *Begin with the lowest denomination and add each column separately, placing the sum, when less than a unit of the next higher denomination, under the column added.*

III. *When the sum equals one or more units of the next higher denomination, reduce it to this denomination, write the remainder under the column added, and add the quotient obtained by reduction to the next column.*

IV. *Proceed in the same manner with all the columns to the last, under which write the entire sum.*

Proof.—The same as in addition of simple numbers.

NOTE.—Addition of compound numbers is the same in principle as the addition of simple numbers. In each we carry for the number of units in the lower denomination which makes a unit of the next higher. The apparent difference is in their *scales*. In simple numbers the expression shows how much to carry; in denominate numbers we must reduce to see what to carry.

(2)

mi.	rd.	yd.
187	319	4
269	227	2
387	158	3
578	269	1
465	217	3

(3)

A.	P.	sq. yd.
789	109	27
891	143	19
134	79	17
234	108	27
678	157	18

(4)

deg.	mi.	rd.	ft.	in.
27	56	148	15	9
32	43	223	12	6
45	57	316	16	10
56	65	267	14	11
34	68	318	12	11

(5)

℔	℥	ʒ	℈	gr.
28	8	5	2	16
37	7	6	1	12
42	10	3	0	15
96	11	7	2	13
78	10	4	2	11

(6)

C.	cu. ft.	cu. in.
216	104	1316
135	117	1072
738	121	1527
217	108	1289
392	126	1132

(7)

T.	cwt.	lb.	oz.
25	16	68	11
43	12	40	14
67	15	23	12
85	17	92	15
61	19	14	13

8. Find the sum of 25 lb. 7 oz. 15 pwt. 20 gr.; 78 lb. 11 oz. 19 pwt. 23 gr.; 34 lb. 9 oz. 12 pwt. 15 gr.; 60 lb. 10 oz. 3 pwt. 4 gr.; 17 lb 6 oz. 18 pwt. 22 gr.

Ans. 217 lb. 10 oz. 10 pwt. 12 gr.

9. Find the sum of 21 mi. 67 ch. 3 rd. 21 li.; 28 mi. 78 ch. 2 rd. 23 li.; 47 mi. 6 ch. 2 rd. 18 li.; 56 mi. 59 ch. 2 rd. 16 li.; 25 mi. 38 ch. 3 rd. 23 li.; 46 mi. 75 ch. 2 rd. 21 li.

Ans. 227 mi. 7 ch. 2 rd. 22 li.

10. Find the sum of 145 sq. yd. 7 sq. ft. 116 sq. in.; 218 sq. yd. 3 sq. ft. 141 sq. in.; 317 sq. yd. 6 sq. ft. 108 sq. in.; 419 sq. yd. 5 sq. ft. 132 sq. in.; 381 sq. yd. 4 sq. ft. 136 sq. in.

Ans. 1483 sq. yd. 2 sq. ft. 57 sq. in.

11. Find the sum of 37 mi. 275 rd. 3 yd. 2 ft. 10 in.; 42 mi. 228 rd. 2 yd. 1 ft. 8 in.; 56 mi. 317 rd. 1 yd. 2 ft. 7 in.; 76 mi. 141 rd. 5 yd. 2 ft. 11 in.; 85 mi. 272 rd. 4 yd. 1 ft. 10 in.

Ans. 299 mi. 276 rd. 2 yd. 1 ft. 4 in.

SUBTRACTION OF COMPOUND NUMBERS.

431. Subtraction of Compound Numbers is the process of finding the difference between two similar compound numbers.

1. From 33 oz. 14 pwt. 23 gr., take 17 oz. 16 pwt. 11 gr.

OPERATION.

oz.	pwt.	gr.
33	14	23
17	16	11
15	18	12

SOLUTION.—We write the subtrahend under the minuend, placing similar units in the same column, and begin at the lowest denomination to subtract. 11 gr. from 23 gr. leaves 12 gr., which we write under the grains: 16 pwt. from 14 pwt. we cannot take, we will therefore take 1 oz. from the 33 oz., leaving 32 oz.; 1 oz. equals 20 pwt., which added to 14 pwt., equals 34 pwt.; 16 pwt. from 34 pwt. leaves 18 pwt., which we write under the pwt.: 17 oz. from 32 oz. (or, since it will give the same result, we may add 1 oz. to the 17 oz. and say 18 oz. from 33 oz.) leaves 15 oz. Hence the following

Rule.—I. *Write the subtrahend under the minuend so that similar units stand in the same column.*

II. *Begin with the lowest denomination and subtract each term of the subtrahend from the corresponding term of the minuend.*

III. *If any term of the subtrahend exceeds the corresponding term of the minuend, add to the latter as many units of that denomination as make one of the next higher, and then subtract; add* 1 *also to the next term of the subtrahend before subtracting.*

IV. *Proceed in the same manner with each term to the last.*

Proof.—The same as in subtraction of simple numbers.

NOTE.—The pupil will notice that the general principle of subtraction is the same as in simple numbers, the difference being in the irregularity of the scale, the units themselves being expressed in the decimal scale.

(2)

£	s.	d.	qr.
56	18	5	3
22	18	7	1

(3)

lb.	oz.	pwt.	gr.
48	10	18	13
27	11	12	18

(4)

mi.	rd.	yd.	ft.
72	45	2	1
48	272	4	2

(5)

mi.	rd.	yd.	ft.	in.
48	305	0	0	0
23	194	5	1	4

(6)

A.	P.	sq. yd.	sq. ft	sq. in.
48	147	00	00	00
25	155	30	3	71

7. From 28 deg. 160 rd. 1 ft., subtract 16 deg. 69 mi. 232 rd. 5 yd. 2 ft. 7 in.

Ans. 10 deg. 69 mi. 29 rd. 2 yd. $6\frac{1}{5}$ in.

8. From 1 circumference subtract 358 deg. 69 mi. 159 rd. 5 yd. 1 ft. 5 in. *Ans.* 68 mi. 262 rd. 2 yd. $8\frac{1}{5}$ in.

9. A has a field 15 rd. 5 yd. 2 ft. 11 in. long, B has one 16 rd. 1 ft. 4 in. long; which is the longer field and how much? *Ans.* A's, 1 inch.

10. From 56 A. 97 P. 8 sq. ft. 112 sq. in., take 49 A. 159 P. 30 sq. yd. 8 sq. ft. 120 sq. in.

Ans. 6 A. 97 P. 2 sq. ft. 28 sq. in

MULTIPLICATION OF COMPOUND NUMBERS.

432. Multiplication of Compound Numbers is the process of finding the product when the multiplicand is a compound number.

1. Multiply £15 12 s. 10 d. by 9.

£	s.	d.
15	12	10
		9
140	15	6

OPERATION.

SOLUTION.—We write the multiplier under the lowest denomination of the multiplicand, and begin at the right to multiply. 9 times 10 d. are 90 d., which by reduction we find equals 7 s. and 6 d.; we write the 6 d. under the pence, and reserve the 7 s. to add to the next product: 9 times 12 shillings are 108 shillings, plus the 7 s. equals 115 s., which by reduction we find equals £5 and 15 shillings; we write the 15 s. under the shillings and reserve the £5 to add to the next product: 9 times £15 are £135, plus the £5, equals £140, which we write under the pounds.

Rule.—I. *Write the multiplier under the lowest denomination of the multiplicand.*

II. *Begin with the lowest denomination, and multiply each term in succession as in simple numbers, reducing as in addition of compound numbers.*

Proof.—The same as in multiplication of simple numbers.

NOTE.—If the multiplier is a large composite number, it will be more convenient to multiply by its factors.

(2)

cwt.	lb.	oz.
25	94	12
		7

(3)

lb.	oz.	pwt.	gr.
25	10	16	21
			5

(4)

mi.	rd.	yd.
36	314	6
		8

(5)

℔	℥	ʒ	℈	gr.
24	11	7	2	19
				9

(6)

T.	cwt.	lb.	oz.
16	18	96	12
			12

(7)

A.	P.	sq. yd.	sq. ft.
96	150	16	7
			11

8. A lumberman has 15 piles of wood, each containing 8 cd. 76 cu. ft., how much wood has he?

Ans. 128 cd. 116 cu. ft.

9. A man traveled 25 mi. 224 rd. 5 yd. in one day, 6 times as far the next, 8 times as far the next, and the next as far as the second and third; how much did he lack of traveling 1000 miles? *Ans.* 254 mi. 197 rd. 3½ yd.

DIVISION OF COMPOUND NUMBERS.

433. Division of Compound Numbers is the process of finding the quotient when the dividend is a compound number.

434. There are two cases:

1st. To divide a compound number into equal parts.

2d. To divide one compound number by a similar one.

CASE I.

435. ***To divide a compound number into a number of equal parts.***

1. Divide £107 11 s. 6 d. into 6 equal parts.

SOLUTION.—We write the divisor at the left of the dividend, and begin at the highest denomination to divide. $\frac{1}{6}$ of £107 equals £17 and £5 remaining; £5 equals 100 s., which added to 11 s. equals 111 s.: $\frac{1}{6}$ of 111 s. equals 18 s. and 3 s. remaining; 3 s. equals 36 d. which added to 6 d. equals 42 d.: $\frac{1}{6}$ of 42 d. is 7 d. Hence the following

OPERATION.

	£	s.	d.
6)	107	11	6
	17	18	7

Rule.—I. *Begin with the highest denomination of the dividend and divide each term in succession, as in simple numbers.*

II. *If there is a remainder, reduce it to the next lower denomination, add it to the term of that denomination, and divide the result as before.*

III. *Proceed in the same manner until all the terms are divided.*

Proof.—The same as in division of simple numbers.

NOTE.—When the divisor is *large* and *composite*, and the factors not greater than 12, it is perhaps more convenient to divide by the factors.

(2)

	cwt.	lb.	oz.
7)	24	98	9

(3)

	lb.	oz.	pwt.	gr.
5)	124	6	16	11

(4)

	mi.	rd.	yd.
8)	138	65	2

(5)

	lb.	℥	ʒ	℈	gr
9)	64	6	4	2	17

(6)

	T.	cwt.	lb.	oz.
12)	133	0	88	12

(7)

	A.	P.	sq. yd.	sq. ft.
11)	33	80	30	3

8. If a car could run 640 mi. 298 rd. 15 ft. in a day, what distance will it average an hour? *Ans.* 26 mi. 225 rd. 13 ft.

9. The earth revolves around the sun in about 365 da. 5 h. 48 min. 49.7 sec.; in what time does it move 1 degree?

Ans. 1 da. 20 min. 58$\frac{427}{3600}$ sec.

10. The moon revolves around the earth in 29 da. 12 h. 44 min.; in what time does it move 6 degrees?

Ans. 11 h. 48 min. 44 sec.

11. Venus performs her revolution around the sun in about 224 da. 16 h. 49 min. 10 sec.; in what time does she move 45 degrees? *Ans.* 28 da. 2 h. 6 min. 8$\frac{3}{4}$ sec.

CASE II.

436. ***To divide one compound number by a similar one.***

1. Divide £78 18 s. 6 d. by £9 11 s. 4 d.

SOLUTION.—£78 18 s. 6 d. equals 18942 pence; £9 11 s. 4 d. equals 2296 pence; and dividing 18942 d. by 2296 d. we have a quotient of 8$\frac{1}{4}$. From this solution we have the following

OPERATION.

$$\frac{£78\ \ 18\text{s.}\ \ 6\text{d}}{£9\ \ 11\text{s.}\ \ 4\text{d.}} = \frac{18942\text{ d.}}{2296\text{ d.}}$$

2296)18942(8$\frac{1}{4}$
18942

Rule.—*Reduce both dividend and divisor to the lowest denomination mentioned in either, and then divide as in simple numbers.*

Proof.—The same as in division of simple numbers.

NOTE.—The division may also be made without reducing to the lowest denomination, and this will be shorter when the quotient is integral.

2. How long will it take a student to walk 376 mi. 220 rd. at the rate of 17 mi. 300 rd. a day? *Ans.* 21 days.

3. A farmer raises 60 bu. 3 pk. 6 qt. 1 pt. of grain on an acre; on how many can he raise 2925 bu. 3 pk.? *Ans.* 48 A.

4. In how many hours will a pipe discharge 163 tuns 7 gal. of water, at the rate of 2 tu. 3 hhd. 40 gal. 2 qt. 1 pt. an hour? *Ans.* 56 hours.

5. How long would it take a bird to fly across the Atlantic ocean, 3000 mi., at the rate of 25 mi. 185 rd. 4 yd. an hour?

Ans. 4 da. 21 h. 16 min. 38+ sec.

6 How long would it take a person to travel around the earth, at the average rate of 15 mi. 62 rd. 3 yd. 2 ft. in 4 h. 20 min. 30 sec.? *Ans.* 296 da. 9 h. 36 min.

DIFFERENCE BETWEEN DATES.

CASE I.

437. ***To find the difference of time between two dates.***

1. Shakespeare was born April 23d, 1564, and died April 25th, 1616; what was his age?

SOLUTION.—Dates are expressed in the *number* of the year, the month, and the day; hence the date of his birth is 1564 yr. 4 mo. 23 da., and the date of his death is 1616 yr. 4 mo. 25 da.; and the difference of these two dates will equal his age, which we find to be 52 yr. 2 da.

OPERATION.

Yr.	mo.	da.
1616	4	25
1564	4	23
52	0	2

Rule.—*Write the number of the year, month, and day of the earlier date under the year, month, and day of the later date, and take the difference of the numbers.*

NOTE.—In this method we reckon 30 days to the month; when greater accuracy is required, we reckon the actual number of days in each month. Thus from the 28th of Feb. (common year) to the 31st of July is exactly *five* months. The exact time between two dates of the same year is found by the table, Art. 406.

2. Milton was born Dec. 9th, 1608, and died Nov. 8th, 1675; what was his age? *Ans.* 66 yr. 10 mo. 29 da.

3. Andrew Jackson was born Mar. 15th, 1767, and died June 8th, 1845; what was his age?
Ans. 78 yr. 2 mo. 23 da.

4. Thomas was 16 yr. old May 25th, 1865; how old will he be Mar. 29th, 1873? *Ans.* 23 yr. 10 mo. 4 da.

5. How many days from Feb. 12th, 1861, to Sept. 17th of the same year? *Ans.* 217 days.

6. A note given Aug. 15th, 1860, was paid May 10th, 1865; how long was it on interest? *Ans.* 4 yr. 8 mo. 25 da.

7. A note dated Jan. 16th, 1860, is due Nov. 21st, 1860; what is the exact time it has to run? *Ans.* 310 days.

8. A was born Mar. 6th, 1820; B, July 9th, 1833; both died Sept. 19th, 1865; how much older was A than B?
Ans. 13 yr. 4 mo. 3 da.

9. A was born Jan. 1st, 1741, and B Jan. 1st, 1584; each died exactly 45 years after he was born; what was the difference of their ages? *Ans.* 1 day.

CASE II.

438. ***To find the day of the week upon which any given day of the month will fall, the day of the week of some other date being given.***

NOTE.—A common year begins one day later than the preceding year. A year following leap year begins two days later.

1. If the 12th of March be on Sunday, on what day of the week will the next 20th of October be?

SOLUTION.—By the table we find the difference of time to be 222 days: dividing by 7, the number of days in a week, we have 31 weeks and 5 days; the 20th of October must therefore be 5 days after Sunday, or on Friday.

OPERATION.
$222 \div 7 = 31, +5$

Rule.—*Find the number of days between the two dates, reduce this number to weeks; the number of days remaining will be the number of days from the given day of the week to the required day.*

2. If the 1st of May is on Tuesday, on what day is the 8th of August of the same year? *Ans.* Wednesday.

3. If a leap year begins on Friday, on what day will the 4th of July be? *Ans.* Monday.

4. In 1865 Christmas, the 25th of December, fell on Monday; on what day did the year commence? *Ans.* Sunday.

5. Christmas of 1863 came on Friday; on what day did 4th of July, 1864, come? *Ans.* Monday.

6. The battle of Bunker Hill was fought on Saturday, June 17, 1775; and Gen. Warren's statue was erected June 17, 1857; on what day was it erected? *Ans.* Wednesday.

7. Let the pupils now determine, from the above principles, the day of the week upon which they were born.

LATITUDE AND LONGITUDE.

439. The **Latitude** of a place is its distance from the equator, north or south. It is reckoned in degrees, minutes, and seconds, and cannot exceed 90°, or a quadrant.

440. The **Longitude** of a place is its distance, east or west, from a given meridian. It is reckoned in degrees, minutes, and seconds, and cannot exceed 180°, or a semi-circumference.

NOTE.—In adding two longitudes, if their sum exceed 180 degrees, it must be subtracted from 360 degrees for the correct difference of longitude.

441. From these principles, to find the difference of latitude or longitude, we have the following rule:

Rule.—*When the latitudes or longitudes are both of the same name, subtract the less from the greater; when they are of different names, take their sum.*

1. The latitude of Washington is 38° 53′ 39″ north, and that of Boston 42° 21′ 27″ north; what is the difference of latitude? *Ans.* 3° 27′ 48″.

2. The latitude of Philadelphia is 39° 56′ 39″ north, and that of Montreal 45° 35′ north; what is the difference of latitude? *Ans.* 5° 38′ 21″.

3. The latitude of New York is 40° 24′ 40″ north, and of Cape Horn 55° 58′ 30″ south; what is the difference of latitude? *Ans.* 96° 23′ 10″.

4. The long. of Phila. is 75° 9′ 5″ west, of San Francisco 122° 26′ 15″ west; what is the difference? *Ans.* 47° 17′ 10″.

5. The long. of San Francisco is 122° 26′ 15″ west, of Pekin 118° east; what is the difference? *Ans.* 119° 33′ 45″.

LONGITUDE AND TIME.

442. The earth revolves upon its axis from west to east once in 24 hours, which causes the sun to *appear* to revolve around the earth from east to west in the same time. Places east of a certain point have *later* time, those west of it *earlier* time, since the sun appears first to those on the east.

443. The circumference of a circle contains 360°, hence the sun appears to travel through 360° in 24 hours, and in 1 hour it travels $\frac{1}{24}$ of 360° = 15°; in 1 minute it travels $\frac{1}{60}$ of 15° = 15′; and in 1 second it travels $\frac{1}{60}$ of 15′ = 15″. Hence the following table:

TABLE OF LONGITUDE AND TIME.

15° of longitude	=	1 hour of time.
15′ of "	=	1 minute of time.
15″ of "	=	1 second of time.

CASE I.

444. ***To find the difference of time of two places when their difference of longitude is given.***

1. The difference of longitude between two places is 50° 45′; what is the difference of time?

SOLUTION.—Since 15° of longitude correspond to 1 h. of time, and 15′ of longitude to 1 min. of time, $\frac{1}{15}$ of the number of *degrees* and *minutes* will equal the number of *hours* and *minutes* difference in time. Dividing by 15 we have 3 h. 23 min. Hence the

OPERATION.

$$15)\underline{50° \ 45'}$$
$$3 \quad 23$$

Rule.—*Divide the difference of longitude expressed in* ° ′ ″ *by* 15; *the result will be the difference of time in* H. MIN. SEC.

2. The longitude of Philadelphia is 75° 9′ 5″ west, and that of New Orleans 90° west; what is the difference of time? *Ans.* 59 min. $23\frac{2}{3}$ sec.

3. The longitude of Boston is 71° 3′ 30″ west, and that of San Francisco 122° 26′ 15″ west; what is the time in Boston when it is 8 o'clock A. M. in San Francisco?

Ans. 11 h. 25 min. 31 sec. A. M.

4. The longitude of Edinburgh is 3° 11′ west, and that of Chicago 87° 44′ 30″ west; what change would it be necessary to make in our watches in coming from Edinburgh to Chicago? *Ans.* Set back 5 h. 38 min. 14 sec.

5. The longitude of Dubuque is 90° 38′ 30″ west; what change must we make in our watches in coming from Dubuque to Philadelphia? *Ans.* Set forward 1 h. 1 min. $57\frac{2}{3}$ sec.

6. The long. of Jerusalem is 35° 32′ east; what time is it when it is 7 A. M. in Boston? *Ans.* 2 h. 6 min. 22 sec. P. M.

7. St. Petersburg is in 30° 19′ east longitude; what is the time there when it is 23 min. past 10 P. M. in Philadelphia?

Ans. 5 h. 24 min. $52\frac{1}{3}$ sec. A. M. the day after.

CASE II.

445. ***To find the difference of longitude of two places when their difference of time is given.***

1. The difference of time between two places is 3 h. 23 min.; what is the difference of longitude?

SOLUTION.—Since 1 h. of time corresponds to 15° of longitude, and 1 min. of time to 15′ of longitude, 15 times the number of *hours* and *minutes* difference in time will equal the number of *degrees* and *minutes* difference in longitude. Multiplying by 15 we have 50° 45′. Hence the following

OPERATION.

h.	min.
3	23
	15
50°	45′

Rule.—*Multiply the difference of time expressed in* H. MIN · SEC. *by* 15*; the result will be the difference of longitude in* ° ′ ″.

2. The difference of time between New York (long. 74° 3′ W.) and Buffalo is 18 min. 48 sec.; required the longitude of Buffalo. *Ans.* 78° 45′ W.

3. The difference of time between Philadelphia (long. 75° 9′ 5″ W.) and St. Louis is 1 h. 24$\frac{11}{15}$ sec.; what is the longitude of St. Louis? *Ans.* 90° 15′ 16″ W.

4. When it is noon at London (long. 9′ 17″ W.) it is 7 h. 16 min. 23$\frac{2}{15}$ sec. A. M. at Boston; required the longitude of Boston. *Ans.* 71° 3′ 30″ W.

5. When it is 3½ o'clock P. M. at Cambridge, England (long. 5′ 21″ E.), it is 10 h. 45 min. 9$\frac{1}{5}$ sec. A. M. at Cambridge, Mass.; required the longitude of the latter place. *Ans.* 71° 7′ 21″ W.

6. When it is 7½ o'clock P. M. at Chicago (long. 87° 44′ 30″ W.), it is 3 h. 43 min. 6 sec. A. M. at Jerusalem; required the longitude of Jerusalem. *Ans.* 35° 32′ E.

7. In going from Detroit (long. 82° 58′ W.) to Baltimore, I found it necessary to set my watch forward 45 min. 32 sec.; what is the longitude of Baltimore? *Ans.* 71° 35′ W.

8. I left New Haven (long. 72° 55′ 24″) at 11½ o'clock A M. and when arriving in San Francisco I found it to be 9 P. M. by their time, while it was 12 h. 18 min. 3$\frac{2}{5}$ sec. A. M. by my watch; required the longitude of San Francisco. *Ans.* 122° 26′ 15″ W.

9. A captain of a vessel takes an observation and finds that by solar time it is 2 h. 25 min. 30 sec. past noon, but by his chronometer, set at Greenwich, it is 32 min. 42 sec. past 11 A. M.: what was his longitude? *Ans.* 43° 12′ E.

STANDARD TIME.

445a. Standard Time is time reckoned from certain fixed meridians instead of from the meridian of the place.

For the convenience of the business world the United States has been divided into four great time-belts, each 15° wide, and the "local time" of the central meridian of each belt is made the "standard time" for the entire belt. Places within 7° 30′ on each side of the central meridian have the time of the meridian.

445b. Standard time embraces four divisions—*Eastern Time, Central Time, Mountain Time,* and *Pacific Time.*

1. EASTERN TIME is the time of the 75th meridian.
2. CENTRAL TIME is the time of the 90th meridian.
3. MOUNTAIN TIME is the time of the 105th meridian.
4. PACIFIC TIME is the time of the 120th meridian.

To the east of the United States is also the time-belt of the 60th meridian, known as the *Intercolonial.*

445c. Standard time, as is evident, coincides with local time in very few places, and therefore it is frequently necessary, in accurate calculations, to obtain the local time.

By finding the difference of longitude between any place and its standard meridian, local time can be reduced to standard time or standard time to local time.

NOTES.—1. *Eastern Time* coincides very nearly with the local time of Philadelphia, Pa., and Cape May, N. J

2. *Central Time* coincides very nearly with the local time of St. Louis, Mo., Memphis, Tenn., and New Orleans, La.

3. *Mountain Time* is very nearly the same as the local time of Denver, Col., and Cheyenne, Wyoming.

4. *Pacific Time* is very nearly the same as the local time of Reno and Carson City, Nev., and Santa Barbara, Cal.

5. The limit of any belt is not a straight line, as it is convenient to select prominent cities, not on the same meridian, as the places for making a change of time.

WRITTEN EXERCISES.

1. The longitude of Boston is about 71° 3′ 30″ W.; what is the difference between standard time and local time?

SOLUTION.—Boston, whose longitude is within 7° 30′ of 75°, is in the Eastern time-belt; hence, its standard time is that of the meridian of 75°. The difference of longitude between Boston and the 75th meridian is 3° 56′ 30″; reducing this, we find the difference of time to be 15 min. 46 sec.

OPERATION.

$$\begin{array}{r} 75^\circ\ 00'\ 00'' \\ 71^\circ\ \ 3'\ 30'' \\ \hline 15)3^\circ\ 56'\ 30'' \\ \hline 15\text{ min. }46\text{ sec.} \end{array}$$

Rule.—*Find the difference between the longitude of a place and that of its standard meridian, and find the time corresponding to this difference.*

To determine the standard meridian of a place, see Art. 445b, noticing that each belt extends 7° 30′ on each side of the established meridian.

2. The longitude of New Haven is 72° 55′ 24″ W.; what is the difference between the standard and local time at New Haven? *Ans.* 8 min. $18\frac{2}{5}$ sec.

3. The longitude of Detroit is 82° 58′ W.; when it is noon by standard time, what is the local time?

Ans. 12 h. 28 min. 8 sec.

4. The longitude of San Francisco is 122° 26′ 15″ W.; when it is noon by local time, what o'clock is it by standard time? *Ans.* 12h. 9 min. 45 sec.

5. The local time at Galveston is 3 h. 40 min. 40 sec., when it is 4 P. M. by standard time; what is the longitude of Galveston? *Ans.* 94° 50′.

6. When it is 5 o'clock P. M. standard time in Chicago (87° 44′ 30″ W.), what time is it in London (9′ 17″ W.)?

Ans. 10 h. 59 min. $22\frac{13}{15}$ sec. P. M.

7. When it is noon in Jerusalem (35° 32′ E.) what will be the standard time in Boston? *Ans.* 4 h. 37 min. 52 sec.

8. When it is 6 A. M. standard time, in Philadelphia it is 4 h. 57 min. 26 sec. local time, in Dubuque; what is the longitude of Dubuque? *Ans.* 90° 38′ 30″ W.

9. If a telegram is sent from Portland, Ore. (122° 27′ 30″ W.), to Portland, Me. (70° 15′ 40″ W.), at 6.30 P. M. local time, at what hour local time will it be received? At what hour standard time?

Ans. 9 h. 58 min. $47\frac{1}{3}$ sec.; 9 h. 39 min. 50 sec.

10. Mr. Smith started from Boston Feb. 24, 1896, at 9 h. 30 min. P. M. local time, and arrived at San Francisco in 7 da. 2 h. 25 min. 35 sec. by his watch, keeping Boston local time: at what time did he arrive by San Francisco time? at what time by standard time?

Ans. Mar. 2, 8 h. 30 min. 4 sec. P. M.; 8 h. 39 min. 49 sec.

DENOMINATE FRACTIONS.

446. A **Denominate Fraction** is one in which the unit of the fraction is denominate; as, $\frac{2}{3}$ lb., .36bu.

447. Denominate Fractions may be expressed either as *common* fractions or as *decimals.*

448. The **Processes** are *Reduction, Addition, Subtraction, Multiplication, Division,* and *Relation.*

REDUCTION OF DENOMINATE FRACTIONS.

449. Reduction of Denominate Fractions is the process of changing them from one denomination to another without altering their value.

450. There are **two general cases,** *reduction descending* and *reduction ascending,* which for convenience of treatment, are subdivided into several other cases.

REDUCTION DESCENDING.

CASE I.

451. ***To reduce a common denominate fraction to a fraction of a lower denomination.***

1. Reduce $\frac{1}{64}$ of a shilling to farthings.

OPERATION.

$\frac{1}{64} \times \frac{12}{1} \times \frac{4}{1} = \frac{3}{4}$

SOLUTION.—Since there are 12 d. in one shilling, 12 times the number of shillings equals the number of pence; and since there are 4 farthings in one penny, 4 times the number of pence equals the number of farthings; hence $\frac{1}{64}$ of a shilling equals $\frac{1}{64} \times \frac{12}{1} \times \frac{4}{1}$ farthings, which, by cancelling and multiplying, becomes $\frac{3}{4}$ of a farthing.

Rule.—*Express the multiplication by the multipliers required, and reduce by cancellation.*

Reduce

2. $\frac{1}{640}$ of an oz. to the fraction of a gr. *Ans* $\frac{3}{4}$.

3. $\frac{1}{80}$ of a bu. to the fraction of a pint. *Ans.* $\frac{4}{5}$.

4. $\frac{5}{192}$ of a gal. to the fraction of a gill. *Ans.* $\frac{5}{6}$.

5. $\frac{3}{1792000}$ of a ton to the fraction of an ounce. *Ans.* $\frac{3}{56}$.

6. $\frac{7}{506880}$ of a mile to the fraction of an inch. *Ans.* $\frac{7}{8}$.

7. $\frac{1}{35478000}$ of a com. yr. to seconds. *Ans.* $\frac{8}{9}$

8. $\frac{1}{9408960}$ of an A. to the fraction of a sq. in. *Ans.* $\frac{2}{3}$

CASE II.

452. ***To reduce a common denominate fraction to integers of a lower denomination.***

1. What is the value of $\frac{7}{9}$ of a pound Troy?

SOLUTION.—There are 12 oz. in one pound, hence 12 times the number of pounds equals the number of ounces; 12 times $\frac{7}{9}$ equals $\frac{84}{9}$, or $9\frac{1}{3}$ oz.: there are 20 pwt. in one ounce, therefore 20 times the number of oz. equals the number of pwt.; 20 times $\frac{1}{3}$ equals $\frac{20}{3}$, or $6\frac{2}{3}$ pwt., etc.

OPERATION.

$\frac{7}{9}\times 12=\frac{84}{9}=9\frac{1}{3}$ oz.
$\frac{1}{3}\times 20=\frac{20}{3}=6\frac{2}{3}$ pwt.
$\frac{2}{3}\times 24=16$ gr.

SOLUTION 2D.—$\frac{7}{9}$ of a pound equals $\frac{1}{9}$ of 7 lb. and $\frac{1}{9}$ of 7 lb. we find by dividing is 9 oz. 6 pwt. 16 gr.

OPERATION.

	lb.	oz.	pwt.	gr.
9)	7	0	0	0
		9	6	16

Rule I.—*Reduce the fraction until we reach an integer and a fraction of a lower denomination, set aside the integer and reduce the fraction as before, and thus continue as far as necessary.*

Rule II.—*Regard the numerator as so many units of the given denomination, and divide by the denominator.*

What is the value

2. Of $\frac{2}{3}$ of a £? *Ans.* 13 s. 4 d.
3. Of $\frac{5}{8}$ of a rod? *Ans.* 3 yd. 1 ft. $3\frac{3}{4}$ in.
4. Of $\frac{5}{6}$ of a bushel? *Ans.* 3 pk. 2 qt. $1\frac{1}{3}$ pt.
5. Of $\frac{5}{6}$ of a mile? *Ans.* 266 rd. 11 ft.
6. Of $\frac{8}{9}$ of a year? *Ans.* 10 mo. 2 wk. 4 da. 16 h.
7. Of $\frac{2}{15}$ of a ton? *Ans.* 2 cwt. 66 lb. $10\frac{2}{3}$ oz.
8. Of $\frac{8}{9}$ of an acre? *Ans.* 142 P. 6 sq. yd. 6 sq. ft. 72 sq. in.

CASE III.

453. ***To reduce a denominate decimal to integers of lower denominations.***

1. Reduce £.675 to integers of lower denomination.

SOLUTION.—There are 20 s. in £1, therefore 20 times the number of pounds equals the number of shillings; 20 times .675 equals 13 s. and .5 s.: there are 12 d. in 1 shilling, therefore 12 times the number of shillings equals the number of pence; 12 times .5 equals 6 d. Therefore, £.675 equals 13 s. 6 d.

OPERATION.

$$\begin{array}{r} .675 \\ 20 \\ \hline 13.500 \\ 12 \\ \hline 6.000 \end{array}$$

Rule.—*Reduce the decimal until we reach an integer and a decimal of a lower denomination, set aside the integer and*

reduce the decimal as before, and thus continue as far as necessary.

What is the value

2. Of .9375 of a gal.? *Ans.* 3 qt. 1 pt. 2 gi.
3. Of .1296 of a ton? *Ans.* 2 cwt. 59 lb. 3.2 oz.
4. Of .678$\frac{3}{4}$ lb. Ap.? *Ans.* 8℥ 1ʒ 9.6 gr.
5. Of 1.426 of a day? *Ans.* 1 da. 10 h. 13 min. 26.4 sec.
6. Of .2845 of a year? *Ans.* 103 da. 20 h. 13 min. 12 sec.
7. Of .8469 deg.? *Ans.* 58 mi. 182 rd. 5 yd. .82944 in.

REDUCTION ASCENDING.

CASE I.

454. ***To reduce a common denominate fraction to a common fraction of a higher denomination.***

1. Reduce $\frac{3}{4}$ of a farthing to the fraction of a shilling.

OPERATION.

$\frac{3}{4} \times \frac{1}{4} \times \frac{1}{12} = \frac{1}{64}$.

SOLUTION.—There are 4 farthings in a penny, therefore $\frac{1}{4}$ of the number of farthings equals the number of pence: there are 12 pence in one shilling, therefore $\frac{1}{12}$ of the number of pence equals the number of shillings; hence $\frac{3}{4}$ far. equals $\frac{3}{4} \times \frac{1}{4} \times \frac{1}{12} = \frac{1}{64}$ of a shilling.

Rule.—*Express the division by the required divisors, and reduce by cancellation.*

Reduce

2. $\frac{7}{9}$ of a gill to the fraction of a gallon. *Ans.* $\frac{7}{288}$.
3. $3\frac{1}{18}$ sq. in. to the fraction of a sq. rd. *Ans.* $\frac{5}{64152}$.
4. $8\frac{3}{4}$ in. to the fraction of a mile. *Ans.* $\frac{7}{50688}$.
5. $\frac{2}{3}$ of $\frac{3}{4}$♏ to the fraction of a Cong. *Ans.* $\frac{1}{122880}$.
6. $4\frac{1}{3} \times 6\frac{3}{4}$ cu. in. to the fraction of a cord. *Ans.* $\frac{13}{98304}$.
7. What part of a cable length is 1 link? *Ans.* $\frac{11}{12000}$.

CASE II.

455. ***To reduce a compound number to a common fraction of a higher denomination.***

1. Reduce 9 oz. 6 pwt. 16 gr. to the fraction of a pound.

OPERATION.

oz.	pwt.	gr.
9	6	16 = 4480 gr
		1 lb. = 5760 gr.

$\frac{4480}{5760} = \frac{7}{9}$, Ans.

SOLUTION.—By reduction we find 9 oz. 6 pwt. 16 gr. equals 4480 gr.; and also 1 lb. equals 5760 gr.; 1 gr. equals $\frac{1}{5760}$ of a pound, and 4480 gr. equals 4480 times $\frac{1}{5760}$, which equals $\frac{4480}{5760}$, which reduced to its *lowest terms*, equals $\frac{7}{9}$. Therefore, etc.

Rule.—*Reduce the number to its lowest denomination, and write under it the number of units of this denomination which make a unit of the required denomination; and then reduce the resulting fraction to its lowest terms.*

Reduce

2. 213 rd. 1 yd. $2\frac{1}{2}$ ft. to the fraction of a mile. *Ans.* $\frac{2}{3}$.
3. 13 quires 8 sheets to the fraction of a ream. *Ans.* $\frac{2}{3}$.
4. What part of a bu. of wheat is 52 lb. 8 oz.? *Ans.* $\frac{7}{8}$.
5. What part of a quintal is 85 lb. $11\frac{3}{7}$ oz.? *Ans.* $\frac{6}{7}$.
6. What part of 9 feet square is 9 sq. ft.? *Ans.* $\frac{1}{9}$.
7. What part of 10 ft. square is 4 ft. square and 4 sq. ft.? *Ans.* $\frac{1}{5}$.
8. Reduce 114 P. 8 sq. yd. 5 sq. ft. $113\frac{1}{7}$ sq. in. to the fraction of an acre. *Ans.* $\frac{5}{7}$.
9. What part of a hhd. does a box 7 in. long, $5\frac{1}{2}$ in. wide, and 3 in. deep contain? *Ans.* $\frac{1}{126}$.

CASE III.

456. ***To reduce a compound number to a decimal of a higher denomination.***

1. Reduce 3 qt. 1 pt. 2 gi. to the decimal of a gallon.

SOLUTION.—There are 4 gi. in 1 pt., hence $\frac{1}{4}$ of the number of gills equals the number of pints; $\frac{1}{4}$ of 2 equals .5, which with the 1 pt., equals 1.5 pt.: there are 2 pt. in 1 qt., hence $\frac{1}{2}$ of the number of pints equals the number of quarts; $\frac{1}{2}$ of 1.5 equals .75, which with 3 qt. equals 3.75 qt.: there are 4 qt. in 1 gal., hence $\frac{1}{4}$ of the number of quarts equals the number of gallons; $\frac{1}{4}$ of 3.75 equals .9375; hence 3 qt. 1 pt. 2 gi. equals .9375 gal.

OPERATION.

$$\begin{array}{r|l} 4 & 2.00 \\ 2 & 1.50 \\ 4 & 3.75 \\ \hline & .9375, \textit{Ans.} \end{array}$$

Rule.—I. *Divide the lowest term by the number of units which equals one of the next higher, and annex the decimal quotient to the integer of the next higher denomination.*

II. *Proceed in a similar manner until the whole is reduced to the required denomination.*

NOTE.—It may also be done by reducing to a common fraction, and the common fraction to a decimal.

2. Reduce 13 s. 6 d. to £. *Ans.* £.675.
3. Reduce 2 cwt. $59\frac{1}{5}$ lb. to tons. *Ans.* .1296 ton.

4. Reduce 9℥ 1ʒ 2℈ 8 gr. to ℔. *Ans.* .768$\frac{3}{4}$ ℔.
5. Reduce 12 cwt. 87 lb. 3$\frac{1}{5}$ oz. to tons. *Ans.* .6436 ton.
6. 58 mi. 182 rd. 5 yd. .82944 in. to deg. *Ans.* .8469 deg.
7. 1 da. 10 h. 13 min. 26.4 sec. to days. *Ans.* 1.426 da.
8. 103 da. 20 h. 13 min. 12 sec. to yr. *Ans.* .2845 yr.

ADDITION OF DENOMINATE FRACTIONS.

457. Addition of Denominate Fractions is the process of finding the sum of two or more denominate fractions.

1. **Find the sum of £$\frac{2}{3}$, and $\frac{3}{4}$ of a shilling.**

OPERATION.

£$\frac{2}{3}$ = 13 s. 4 d
$\frac{3}{4}$ s. = 9 d
14 s. 1 d.

SOLUTION.—£$\frac{2}{3}$, we find by Art. 452, equals 13 s. 4 d.; and $\frac{3}{4}$ s. equals 9 d. Taking the sum of 13 s. 4 d. and 9 d. we have 14 s. 1 d. Therefore, etc.

Rule.—*Reduce the fraction to integers, and then add as in addition of compound numbers.*

Find the sum

2. Of $\frac{4}{5}$ bu. and $\frac{2}{3}$ bu. *Ans.* 1 bu. 1 pk. 6 qt. 1$\frac{13}{15}$ pt.
3. Of $\frac{5}{6}$ oz. and $\frac{7}{8}$ lb. *Ans.* 11 oz. 6 pwt. 16 gr.
4. Of $\frac{2}{3}$ mi. and $\frac{7}{8}$ rd. *Ans.* 214 rd. 1 yd. 5$\frac{1}{4}$ in.
5. Of .325 gal. and $\frac{2}{3}$ gal. *Ans.* 3 qt. 1$\frac{14}{15}$ pt.
6. Of \$$\frac{2}{3}$, £$\frac{2}{3}$, and $\frac{2}{3}$ of a franc. *Ans.* \$4.039$\frac{2}{3}$.
7. Of .678 A. and $\frac{8}{9}$ rood.
   Ans. 144 P. 1 sq. yd. 97.92 sq. in.
8. Of $\frac{1}{4}$ wk. $\frac{1}{3}$ da. $\frac{1}{2}$ h. and $\frac{3}{4}$ min.
   Ans. 2 da. 2 h. 30 min. 45 sec.

SUBTRACTION OF DENOMINATE FRACTIONS.

458. Subtraction of Denominate Fractions is the process of finding the difference between two denominate fractions.

1. Subtract $\frac{3}{4}$ of a shilling from $\frac{2}{3}$ of a pound.

OPERATION

£$\frac{2}{3}$ = 13 s. 4 d.
$\frac{3}{4}$ s. = 9 d.
12 s. 7 d.

SOLUTION.—£$\frac{2}{3}$ equals 13 s. 4 d.; $\frac{3}{4}$ of a shilling equals 9 d.; 9 d. subtracted from 13 s. 4 d. leaves 12 s. 7 d. Therefore, etc.

Rule.—*Reduce the fractions to integers and then subtract as in subtraction of compound numbers.*

2. Subtract $\frac{5}{6}$ yd. from $\frac{7}{8}$ of a rd. *Ans.* 3 yd. 2 ft. $11\frac{1}{4}$ in.
3. $\frac{2}{3}$ cwt. from .35 ton. *Ans.* 6 cwt. 33 lb. $5\frac{1}{3}$ oz.
4. $\frac{5}{6}$ ℥ from 1.45 ℔. *Ans.* 1 ℔. 4℥ 4ʒ 1℈ 12 gr.
5. $\frac{5}{6}$ O. from $\frac{13}{16}$ of a Cong. *Ans.* 5 O. 10 f℥ 5 fʒ 20 ♏.
6. $\frac{5}{6}$ cu. ft. from $\frac{7}{8}$ cd. ft. *Ans.* 13 cu. ft. 288 cu. in.
7. $\frac{3}{5}$ mo. from $.\dot{3}\dot{6}$ of a leap year.
Ans. 115 da. 2 h. 10 min. $54\frac{6}{11}$ sec.
8. $.8\dot{3}$ square rod from $.\dot{2}\dot{7}$ acre.
Ans. 42 P. 24 sq. yd. 2 sq. ft. 90 sq. in.

MULTIPLICATION OF DENOMINATE FRACTIONS.

459. Multiplication of Denominate Fractions is the process of finding the product when the multiplicand is a denominate fraction.

1. Multiply £$\frac{3}{4}$ by 7.

SOLUTION.—7 times £$\frac{3}{4}$ equals £$\frac{21}{4}$, which by reduction we find equals £5 5 s. Therefore, etc.

OPERATION.

£$\frac{3}{4}\times 7 =$ £$\frac{21}{4}$ = £5 5 s.

Rule.—*Multiply the denominate fraction by the given multiplier and reduce the product to integers.*

NOTE.—It may also be done by first reducing the fraction to integers and multiplying as in multiplication of denominate numbers.

2. Multiply $\frac{2}{3}$ lb. Troy by 11. *Ans.* 7 lb. 4 oz.
3. Multiply .82 yd. by 13. *Ans.* 10 yd. 1 ft. 11.76 in.
4. Multiply $2\frac{7}{8}$ ℔. by 3.25. *Ans.* 9 ℔. 4℥ 1ʒ.
5. .978 mi. by 3.45. *Ans.* 3 mi. 29 ch. 92 li. 6.336 in.
6. $.08\dot{3}$ A. by .35. *Ans.* 4 P. 20 sq. yd. 1 sq. ft. 72 sq. in.
7. $.\dot{3}2\dot{4}$ A. by $\dfrac{2\frac{1}{3}}{3\frac{1}{2}}$. *Ans.* 34 P. 17 yd. 8 ft. $126\frac{18}{37}$ in.
8. $12.1\dot{6}$ C. by $7.\dot{0}\frac{1}{2}$. *Ans.* 85 C. 10 S. 3° 20′.
9. If a man dig $6\frac{3}{4}$ yd. of ditch in 1 day, how much would he dig in $7\frac{3}{4}$ days? *Ans.* 9 rd. 2 yd. 2 ft. $5\frac{1}{4}$ in.

DIVISION OF DENOMINATE FRACTIONS.

460. Division of Denominate Fractions is the process of finding the quotient when the dividend is a denominate fraction.

461. There are **Two Cases:**

1st. To divide a denominate fraction into equal parts.

2d. To divide a denominate fraction by a denominate integer or fraction.

CASE I.

462. ***To divide a denominate fraction into a number of equal parts.***

1. Divide $\frac{7}{8}$ bu. by 5.

SOLUTION.—$\frac{1}{5}$ of $\frac{7}{8}$ of a bushel equals $\frac{7}{40}$ of a bushel, which, by reduction, we find equals 5 qt. $1\frac{1}{5}$ pt. Therefore, etc.

OPERATION.

$\frac{7}{8}$ bu. $\times \frac{1}{5} = \frac{7}{40}$ bu. $=$ 5 qt. $1\frac{1}{5}$ pt

Rule.—*Divide the denominate fraction by the given divisor, and reduce to integers.*

2. Divide $\frac{1}{2}$ rd. by 3. *Ans.* 2 ft. 9 in.
3. Divide .325 yr. by 8. *Ans.* 14 da. 19 h. 52 min. 30 sec.
4. Divide $3.43\frac{3}{4}$ da. by $.18\frac{3}{4}$. *Ans.* 18 da. 8 h.
5. Divide $6.\dot{3}$ mi. by $4\frac{3}{4}$. *Ans.* 1 mi. 106 rd. 3 yd. 2 ft.
6. Divide $\dot{2}.\dot{7}$ bu. by .436. *Ans.* 6 bu. 1 pk. $\frac{400}{1199}$ pt.
7. Divide $12.1\dot{6}$ wk. by $.\dot{7}$. *Ans.* 3 mo. 3 wk. 4 da. 12 h.
8. Divide 4.35 P. by $1\frac{4}{5}$.
   Ans. 2 P. 12 sq. yd. 5 sq. ft. 63 sq. in.
9. If a bird fly 20.36 mi. in $1.33\frac{1}{3}$ hours, how far does it fly in one hour? *Ans.* 15 mi. 86 rd. 2 yd. 7.2 in.

CASE II.

463. ***To divide a denominate fraction by a denominate integer or fraction.***

1. Divide $1\frac{11}{25}$ cwt. by 16 lb.

SOLUTION.—$1\frac{11}{25}$ cwt. equals 144 lb.; dividing 144 lb. by 16 lb. we have for a quotient 9. Hence the following

OPERATION.

$1\frac{11}{25}$ cwt. $=$ 144 lb.
144 lb. $\div$ 16 lb. $=$ 9, *Ans.*

Rule.—*Reduce both terms to like denominations, and divide as in simple numbers.*

2. Divide $\frac{9}{10}$ oz. by 60 pwt. *Ans.* $\frac{3}{10}$.
3. Divide .625 cwt. by 500 oz. *Ans.* 2.
4. Divide 9.75 h. by $\frac{13}{240}$ da. *Ans.* $7\frac{1}{2}$.
5. Divide $17\frac{1}{80}$ circ. by 1224.9°. *Ans.* 5.
6. Divide $4.8\frac{1}{3}$ cd. by $30\frac{14}{15}$ cd. ft. *Ans.* $1\frac{1}{4}$.
7. Divide $\frac{9}{11}$ yr. by $9.\dot{8}\dot{1}$ mo. *Ans.* 1.
8. Divide $.2\dot{3}\dot{6}$ pwt. by $6.1\dot{0}\dot{9}$ gr. *Ans.* $\frac{13}{14}$.
9. If a farmer raise 50 bu. of wheat on 440 sq. rd. of land, how much can he raise on $20\frac{5}{8}$ acres? *Ans.* 375 bu.
10. If Weston should walk $4\frac{3}{8}$ miles in an hour, how long would it take him to walk 425 miles, walking 10 h. per day? *Ans.* 9 da. 1 h. 4 min. $17\frac{1}{7}$ sec.

THE GREATEST COMMON DIVISOR OF DENOMINATE NUMBERS.

464. The **Greatest Common Divisor** of two or more denominate numbers is the greatest denominate number which will exactly divide them.

1. Find the greatest common divisor of 2 bu. 3 pk. 6 qt and 4 bu. 1 pk. 5 qt.

SOLUTION.—2 bu. 3 pk. 6 qt. equals 94 qt., and 4 bu. 1 pk. 5 qt. equals 141 qt.; the greatest common divisor of 94 qt. and 141 qt. we find, by Art. 179, is 47 qt. which, by reduction, we find equals 1 bu. 1 pk. 7 qt. Hence the following rule:

OPERATION.

2 bu. 3 pk. 6 qt.= 94 qt.
4 bu. 1 pk. 5 qt.= 141 qt.
G. C. D. = 47 qt.
47 qt.= 1 bu. 1 pk. 7 qt., *Ans.*

Rule.—*Reduce the numbers to like denominations; find the greatest common divisor of the results as in simple numbers, and then reduce it to any convenient denomination.*

NOTE.—The greatest common divisor of denominate numbers, though new, will be found interesting and practical.

Find the greatest common divisor

2. Of $1646\frac{2}{5}$ qt. and $.93\frac{1}{3}$ hhd. *Ans.* 58 gal. $3\frac{1}{5}$ qt.
3. Of $\dot{3}37.\dot{0}$ C. and 52000°. *Ans.* 48 C. 1 S. 23° 20′.
4. Of 1 rd. 4 yd. 1 ft. $3\frac{1}{2}$ in. and $1.52\dot{7}$ rd. *Ans.* 2 ft. $3\frac{1}{2}$ in.

5. Of 150 gal. 2 qt. 1 pt. 3.36 gi. and 4220.44 pt.

Ans. 75 gal. 1 qt. 3 68 gi.

6. A farmer has 3 lots of hay weighing respectively $\frac{7}{8}$ of a ton, $44\frac{1}{8}$ cwt. and 5.75 cwt. 65 lb.; what is the heaviest bale into which he can divide them? *Ans.* 3 cwt. 20 lb.

7. A man has a triangular field whose sides are respectively 8 rd. 3 yd. 2 ft. 4 in., 6 rd. 1 yd. 1 ft. 5 in., 7 rd. 2 yd. 2 in. long; what is the greatest length of boards that he can use in fencing it, without cutting the boards? *Ans.* 6 ft. 1 in.

8. What are the largest sized house lots of equal extent into which 3 fields, containing respectively 2 A. 100 P., 3 A. $17\frac{7}{8}$ P., and 2 A. $115\frac{5}{8}$ P. may be divided?

Ans. 15 P. 16 sq. yd. 7 sq. ft. 36 sq. in.

THE LEAST COMMON MULTIPLE OF DENOMINATE NUMBERS.

465. The Least Common Multiple of two or more denominate numbers is the smallest denominate number that is a whole number of times each of them.

1. Find the least common multiple of 2 bu. 3 pk. 6 qt and 4 bu. 1 pk. 5 qt.

SOLUTION.—2 bu. 3 pk. 6 qt. equals 94 qt., and 4 bu. 1 pk. 5 qt. equals 141 qt.; the L. C. M. of 94 qt. and 141 qt. is (Art. 187) 282 qt., which by reduction, we find equals 8 bu. 3 pk. 2 qt. Hence the following rule:

OPERATION.

2 bu. 3 pk. 6 qt. = 94 qt.
4 bu. 1 pk. 5 qt. = 141 qt.
L. C. M. = 282 qt.
282 qt. = 8 bu. 3 pk. 2 qt., *Ans.*

Rule.—*Reduce the numbers to the same denomination, find the least common multiple of the results, as in simple numbers, and reduce it to any convenient denomination.*

NOTE.—The least common multiple of denominate numbers, though new, will be found interesting and practical.

Find the least common multiple

2. Of 40 lb.; 30 lb.; and 26 lb. $10\frac{2}{3}$ oz.

Ans. 2 cwt. 40 lb.

3. Of 30 sq. yd. 6 sq. ft.; 690 sq. ft.; $57\frac{1}{9}$ sq. yd.

Ans. 8 A. 22 P. 21 sq. yd. 1 sq. ft. 72 sq. in,

4. Of 2 bu. $\frac{7}{8}$ pk.; $44\frac{3}{8}$ pk.; 19.96875 bu.

Ans. 99 bu. 3 pk. 3 qt.

5. Of $.3916\frac{2}{3}$ʒ; $.342\frac{17}{24}$ʒ; 117.5 gr. *Ans.* 1℥ 5ʒ 2℈ $2\frac{1}{2}$ gr.

6. Of 10 min. $.\dot{1}$ sec.; 1 h. 10 min. $.\dot{7}$ sec.; 1.5 h. 1 sec.
Ans. 10 h. 30 min. 7 sec.

7. What are the contents of the least possible keg that will hold an exact number of times the contents of each of three kegs, holding respectively $\frac{11}{20}$ of a gallon, 2 gal. 3 qt., and 4 gal. $3\frac{4}{5}$ qt.? *Ans.* 24 gal. 3 qt.

8. What is the area of the smallest square lot that can be enclosed by boards 5 ft. 3 in., 10 ft. 6 in., or 15 ft. 9 in. in length without cutting the boards?
Ans. 3 P. 19 sq. yd. $4\frac{1}{2}$ sq. ft.

MISCELLANEOUS PROBLEMS.

1. A shipper sold 16 bales of hay, each weighing 4 cwt. 96 lb., at $2.$37\frac{1}{2}$ per cwt.; what did it amount to?
Ans. $188.48.

2. A can dig 21 rd. 4 yd. 2 ft. of ditch in 3 days, and B can dig 35 rd. 2 yd. $1\frac{1}{2}$ ft. in 4 days; how much can they together dig in a week? *Ans.* 96 rd. 4 yd. 2 ft. 6 in.

3. Mr. James was born Feb. 29th, 1832, and died May 10th, 1864; how many birthdays did he see and what was his exact age? *Ans.* 9 birthdays; age, 32 yr. 2 mo. 11 da.

4. The Knickerbocker Ice Company have an ice-house 50 ft. long, 40 ft. wide, and 24 ft. high; how many tons of ice will it hold, a cubic foot weighing $58\frac{1}{8}$ lb.? *Ans.* 1395.

5. A druggist bought 3 lb. 8 oz. Av. of drugs for $52.50, and made them up into pills of 5 grains each, which he retailed at 36¢ a dozen; what was his gain? *Ans.* $94.50.

6. The longitude of Cairo is 108° 17′ 6″ east of Washington, and that of St. Joseph 17° 40′ 44″ west; what is the time at St. Joseph when it is midnight at Cairo?
Ans. 3 h. 36 min. $8\frac{2}{3}$ sec. P. M.

7. A lady wished to make tucks $\frac{1}{4}$ of an inch wide, leaving $\frac{1}{8}$ of an inch between the edge of one tuck and the stitching of the next; how many can she make in yard-wide muslin? *Ans.* 41 tucks and $\frac{1}{8}$ inch remaining.

8. I bought 400 tons of coal in England for 3 s. 6 d. a cwt., paid $1.50 a ton for transportation, and sold them in New

York at 62½ cents a cwt.; did I gain or lose, and how much? *Ans.* Lost, \$1813.10.

9. A merchant traded 16 gross and 4 dozen of buttons worth \$5 a gross, for hats worth \$3 apiece, and 3 dozen caps at \$7⅔ a dozen; how many hats did he receive?

Ans. 20.

10. The circumference of the fore wheels of a wagon is 10 ft. 4 in., that of the hind wheels 15 ft. 3 in.; how far must the wagon move that the wheels may hold the same relative position to each other as when it started? *Ans.* 630 yd. 1 ft.

11. What are the dimensions of the least possible pile that can be made either out of scantling 3⅛ ft. long, 2½ in wide, and 1.9 in. deep; 15⅝ ft. long, 12½ in. wide, and 9½ in deep; or 9⅛ ft. long, 7½ in. wide and $5\frac{7}{10}$ in. deep?

Ans. 46⅞ ft.; 37½ in.; 28½ in.

12. A merchant had a cask of vinegar from which there leaked away 26 gal. 3 qt. 1 pt.; he then put in 19 gal. 2 qt., and sold 24 gal. 3 qt., and found there lacked 39 gal. 1 qt. 1 pt of being 60 gal.; how much was in the cask at first?

Ans. 52 gal. 3 qt.

13. A, B, and C started on the morning of the same day to travel round a lake 12½ miles in circumference; A traveled 3 mi. 106⅔ rd., B, 10 miles, and C, 16 mi. 213⅓ rd. a day; how many days must they travel before they will meet again at the place where they started? *Ans.* 3¾ days.

14. G. W. Whitney & Co. bought of Johnson & Co., London, July 15, 1877, 15½ reams Bath post paper @ £1 2 s. 6 d.; 5000 envelopes at 17 s. 9 d. ℔ M.; 20 gross Gillott's steel pens @ 4 s. 3 d.; 5 gross Faber's lead pencils @ 18 s. 9 d.; and 4 dozen diaries at 10 s. a dozen. Make out and receipt Whitney & Co.'s bill. *Ans.* £32 16 s. 3 d.

15. What are the dimensions of the largest possible sticks of timber of equal size that can be used to make 3 piles, respectively 28 ft. long, 17 ft. 6 in. wide and 14 ft. high; 46 ft. 8 in. long, 24 ft. 6 in. wide, and 18 ft. high; and 65 ft. 4 in. long, 31 ft. 6 in. wide, and 22 ft. high?

Ans. 9 ft. 4 in.; 3 ft. 6 in.; 2 ft.

16. An American traveling in a coach going about 10 English miles an hour, inquired the distance to Berlin and was told it was 15 miles; at the end of an hour and a half, seeing no signs of the city, he again inquired how long it would be before they arrived, and was told that it would take about $5\frac{1}{2}$ hours. What was the distance in English miles at the time of the first inquiry? *Ans.* $70\frac{71}{352}$.

17. Two men, A and B, on opposite sides of a pond, which is 97 rd. 2 yd. 1 ft. 6 in. in circumference, start simultaneously to go around it in the same direction. A walks 16 rods in one minute, and B $22\frac{2}{3}$ yards in 15 seconds; how often will B circumambulate the pond before they arrive together at the place from which B started? *Ans.* 17 times.

18. A vessel sailed from Philadelphia, and after being out 30 days, the captain took an observation and found the solar time to be 2 h. 16 min. 24 sec. P. M., the chronometer at the same time marking 11 h. 36 min. 40 sec. A. M.; required the longitude of the vessel, supposing the chronometer to have lost $4\frac{1}{5}$ sec. per day. *Ans.* $35° 44' 35''$ W.

19. Two pedestrians are on a straight road on opposite sides of a gate, and distant from it 2 mi. 120 rd. and 4 mi. 80 rd. respectively, and travel each towards the original station of the other. The first travels 120 rods in 10 minutes, and the second $133\frac{1}{3}$ rods in 10 minutes; how long must they travel before they are equally distant from the gate?

Ans. 1st time $83\frac{13}{19}$ min.; 2d time 450 min.

SECTION VII.

PRACTICAL MEASUREMENTS.

466. The **Applications of Measures** to the farm, the household, the mechanic arts, etc., are so extensive that we now present a distinct treatment of the subject.

467. These **Practical Measurements** include Measures of Surface, Measures of Volume, and Measures of Capacity.

MEASURES OF SURFACE.

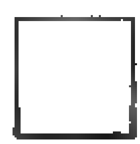
468. A **Surface** is that which has length and breadth without thickness.

469. The **Area** of a surface is expressed by the number of times it contains some other surface used as a *unit of measure.*

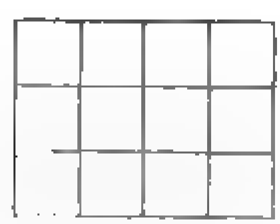
THE RECTANGLE.

470. A **Rectangle** is a plane surface having four sides and four right angles. A slate, a door, the sides of a room, etc., are examples of rectangles.

471. A **Rectangle** has two *dimensions,* length and breadth. A *Square* is a rectangle in which the sides are all equal.



472. The **Area** of a rectangle is the surface included within its sides. It is expressed by the number of times it contains a small square as a *unit of measure.*



Rule I.—*To find the area of a square or rectangle, multiply its length by its breadth.*

For, in the rectangle above, the whole number of little squares is equal to the number in each row multiplied by the number of rows, which is equal to the number of linear units in the length multiplied by the number in the breadth.

Rule II.—*To find either side of a square or rectangle, divide the area by the other side.*

NOTES.—1. The sides multiplied must be of the *same denomination*, and the product will be *square units* of that denomination, which may be reduced, if necessary, to higher denominations.

2. In dividing, the *linear unit* of the side must be of the same name as the *square unit* of the area, and the quotient will be linear units of the same denomination.

EXAMPLES FOR PRACTICE.

1. What is the area of a rectangular lot 150 ft. long and 80 ft. wide?

SOLUTION.—To find the area, we multiply the length by the breadth, and we have $150 \times 80 = 12000$ sq. ft.; reducing this to square yards, we have $1333\frac{1}{3}$ sq. yd.

2. How many square yards in the surface of a blackboard 24 ft. long by $4\frac{1}{2}$ ft. wide? *Ans.* 12 sq. yd.

3. A room 18 feet wide has a floor containing 360 sq. ft; what is its length? *Ans.* 20 ft

4. How many square feet in the walls of a room 25 ft. long, 17 ft. 6 in. wide, and 9 ft. 6 in. high? *Ans.* $807\frac{1}{2}$ sq. ft.

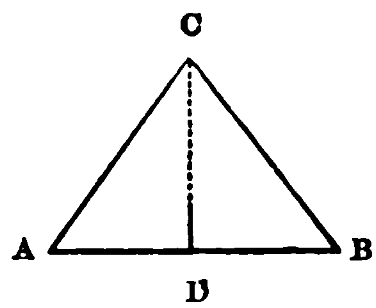
5. What is the surface of a cubical box, each of whose dimensions is 2 feet 9 in.? *Ans.* $45\frac{3}{8}$ sq. ft.

6. How many sq. feet in the surface of a chest 3 ft. 9 in. long, 2 ft. 6 in. wide, and 1 ft. 9 in. high? *Ans.* $40\frac{5}{8}$ sq. ft.

7. A lady wishes to set out tulips in a bed 12 ft. long and 3 ft. wide. How many can be planted at a distance of 9 in. apart and $4\frac{1}{2}$ in. from the edge? *Ans.* 64 tulips.

8. A garden 160 ft. long and 105 ft. wide has a walk around it 7 ft. in breadth; how much ground is contained in the walk? *Ans.* 3514 sq. ft.

THE TRIANGLE.



473. A **Triangle** is a plane surface having three sides and three angles; as ABC.

474. The **Base** is the side upon which it seems to stand; as, AB. The **Altitude** is a line perpendicular to the base, drawn from the angle opposite; as, CD.

475. A triangle which has its three sides equal is called *equilateral;* when two sides are equal it is called *isosceles;* when its sides are unequal it is called *scalene.*

Rule I.—*To find the area of a triangle, multiply the base by one-half of the altitude.*

Rule II.—*To find the base or altitude of a triangle, divide the area by one-half of the other dimension.*

EXAMPLES FOR PRACTICE.

1. What is the area of a triangle whose base is 15 ft. 6 in. and altitude 8 ft. 9 in.?

SOLUTION.—To find the area, we multiply the base by one-half the altitude; $15\frac{1}{2} \times 4\frac{3}{8} = 67\frac{13}{16}$ sq. ft., or 67 sq. ft. 117 sq. in.

2 How many square yards in a triangle whose base is 20 ft. 9 in., and altitude 10 ft. 11 in.?

Ans. 12 sq. yd. 5 sq. ft. $37\frac{1}{2}$ sq. in.

3. Required the area of the gable end of a house 32.5 ft wide, the ridge being 15.25 feet above the wall.

Ans. 27 sq. yd. 4 sq. ft. 117 sq. in.

4. A triangular lot contains 233 sq. yd. 6 sq. ft. 108 sq. in.; its base is 165 ft.; what is its altitude? *Ans.* 25 ft. 6 in.

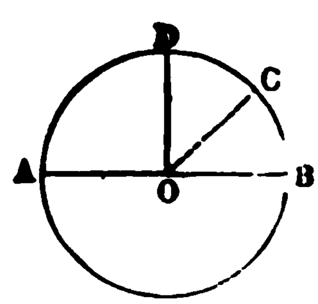
5. I have a triangular flower-bed containing 48 sq. ft. 63 sq. in., whose altitude is 6 ft. 3 in.; what is the base?

Ans. 15 ft. 6 in.

6. The gable end of a house contains 47 sq. yd. 6 sq. ft., the width of the house being 17 yd. 1 ft.; what is the height of the ridge? *Ans.* 16 ft. 6 in.

THE CIRCLE.

476. A **Circle** is a plane figure bounded by a curved line, every point of which is equally distant from a point within, called the *centre.*



477. The **Circumference** of a circle is the bounding line; any part of the circumference, as BC, is an *Arc.* An arc of one-fourth of the circumference is called a *Quadrant.*

478. The **Diameter** is a line passing through the centre

and terminating in the circumference; as, AB. The *Radius* is a line drawn from the centre to the circumference; as, OD.

Rule I.—*To find the circumference of a circle, multiply the diameter by* 3.1416.

Rule II.—*To find the diameter of a circle, multiply the circumference by* .3183.

Rule III.—*To find the area of a circle, multiply the circumference by one-fourth of the diameter, or multiply the square of the radius by* 3.1416.

EXAMPLES FOR PRACTICE.

1. The diameter of a circle is 15 ft. 9 in.; what is its circumference?

SOLUTION.—To find the circumference, we multiply the diameter by 3.1416; $3.1416 \times 15\frac{3}{4} = 49.4802$; hence the circumference equals 49.4802 ft.

2. What is the length of the tire of a carriage wheel 4 ft. 6 in. in diameter? *Ans.* 14.1372 ft.

3. In a square in a certain city is a fountain whose basin is 4 ch. 5 li. in circumference; what is its diameter? *Ans.* 1 ch. 28.9115 li.

4. The end of the minute-hand of a church clock passes over 50 inches in 15 minutes; what is the length of the minute-hand? *Ans.* 31.83 in.

5. How much ground is occupied by a circular light-house, its circumference being 50 ft.? *Ans.* 198.9375 sq. ft.

6. Within a circular plot 50 rods in diameter is a circular pond, whose edge is everywhere 6 rods from the edge of the plot; what is the area of the pond? *Ans.* 1134.1176 P.

7. A walk 3 ft. wide extends around the above mentioned plot; what is the area of the walk? *Ans.* 7803.7344 ft.

MEASUREMENT OF LAND.

479. The **Unit of Measure** of land is the *Acre*, which is sometimes divided into *square rods* and sometimes into *square chains*. Hundredths of an acre are also frequently used

In 1802, Col. Jared Mansfield, Surveyor-General of the North-Western

Territories, adopted a convenient method of laying out Government lands. The country was divided by parallels and meridians 6 miles apart, into squares containing 36 square miles, called *Townships*. The townships are divided into square miles, called *Sections*, and each section into *quarter-sections*. Hence, 640 acres make a *section*, and 160 *acres* a *quarter-section*. The quarter-sections are still further subdivided into *half-quarter-sections*, *quarter-quarter-sections*, and *lots*. Lots are often of irregular form on account of natural boundaries, but contain, as near as may be, a quarter-quarter-section.

A Township in the newer States, laid out as explained above, is a square of 36 miles; but in the older States the townships are irregular in shape and variable in size. A Township is a division of a county made for convenience in holding elections. There must be at least one place of voting in each township.

NOTE.—The pupil will remember that *rods* multiplied by *rods* give *square rods*, *chains* by *chains* give *square chains;* also, that 1 acre = 10 square chains or 160 square rods.

EXAMPLES FOR PRACTICE.

1. I have a rectangular lot 40 ch. long and 36 ch. wide; how many acres does it contain?

SOLUTION.—The area equals 40×36, or 1440 sq. ch. which, reduced to acres by dividing by 10, equal 144 acres.

2. A has a square lot 32 chains on a side; how many acres does it contain? *Ans.* 102 A. 64 P.

3. A rectangular lot contains 144 A. 135 P.; the length of one side is 43 ch.; what is the other? *Ans.* 33.68+ ch.

4. A field is 76 rd. 10 ft. 5 in. long by 44 rd. 7 ft. 9 in. wide; what is its area?

Ans. 21 A. 47 P. 209 sq. ft. 141 sq. in.

5. I wish to fence a quarter-section with hemlock rails $8\frac{1}{2}$ ft. long, lapping 6 inches, the fence being 6 rails high; how many rails will be required, and what will be the cost at $42 ℔ M.? *Ans.* 7920 rails; $332.64.

6. A field 16 chains long contains 16 acres, while another field of the same width contains only 12 acres; what is the length in rods of the latter field? *Ans.* 48 rods

7. How much less will it cost to fence a field 64 rods square than a rectangular field $2\frac{1}{2}$ times as long and $\frac{2}{5}$ as wide, if fencing cost $2.75 a rod? *Ans.* $316.80.

8. In a lot 64 rods square, I planted 4 acres of corn, 350 square rods with potatoes, 20 rods square with vegetables, 6 acres with berries, and the remainder with peach trees; how much was reserved for peaches? *Ans.* 10 A. 146 P.

PLASTERING, PAINTING, AND KALSOMINING.

480. Plastering, Painting, and Kalsomining are usually computed by the square yard.

Allowance is sometimes made for openings in the walls, as doors and windows, but custom varies so much in different localities that no general rule can be given. Half the area of the doors and windows is often deducted.

EXAMPLES FOR PRACTICE.

1. What will be the cost of plastering a room 40 ft. long, 25 ft. 3 in. wide, and 9 ft. 9 in. high, at 38¢ a sq. yd?

SOLUTION.—The surface equals $40 \times 25\frac{1}{4} + (40 + 25\frac{1}{4}) \times 9\frac{3}{4} \times 2 = 2282\frac{3}{8}$ sq. ft. $= 253\frac{43}{72}$ sq. yd.; cost $= \$.38 \times 253\frac{43}{72} = \$96.36\frac{25}{36}$.

2. What is the cost of painting a wainscot 4 ft. 6 in. high in a room 25 ft. by 16 ft. 8 in. at 45¢ a sq. yd? *Ans.* \$18.75.

3. What will it cost to paint the outside of a house 60 ft. long, 40 ft. wide and 25 ft. high, the gable being 16 ft. high, with 2 coats at $9\frac{1}{2}$¢ a sq. yd. each? *Ans.* \$119.06$\frac{3}{4}$.

4. What will it cost to plaster a room 40 ft. long, 22 ft. wide and 9 ft. 6 in. high, at 28¢ a sq. yd., deducting 96 sq. ft. for doors and windows? *Ans.* \$61.04.

5. What will it cost to plaster a house of 10 rooms, 4 of 20×16 ft. and $8\frac{1}{2}$ ft. high; 4 of 20×16 ft. and $9\frac{1}{2}$ ft. high; 2 of 20×15, one being 8 ft. and the other 9 ft. high, 2 halls 32×8 ft., one being $8\frac{1}{2}$ ft. and the other $9\frac{1}{2}$ ft. high, allowance being made for 20 doors, $7\frac{1}{2} \times 3$, and 24 windows $6\frac{1}{2} \times 3$, at 18¢ per sq. yd? *Ans.* \$211.36.

ROOFING, FLOORING, PAVING, ETC.

481. Paving and **ceiling** are estimated by the *square foot* or *square yard*. **Roofing, flooring, partitioning, slating,** etc., are generally reckoned by the *square* of 100 *square feet*, but sometimes by the *square foot* or *yard*.

482. Shingles, which commonly measure 18 in. by 4 in., are estimated by the *thousand* or *bundle*. 1000 shingles are generally allowed to a *square* of 100 sq. ft.

EXAMPLES FOR PRACTICE.

1. What will it cost to slate the roof of my barn 65 ft. long and 24 ft. 6 in. from eaves to ridge, at \$8.75 per square?

SOLUTION.—Surface of roof equals $65\times24\frac{1}{2}\times2=3185$ sq. ft.= 31.85 squares; cost=$\$8.75\times31.85=\$278.68\frac{3}{4}$.

2. What will it cost to pave a street 36 ft. wide and 2240 ft. long, at 30¢ per square foot? *Ans.* $24192.

3. How many slates 12×24, $\frac{1}{3}$ exposed to the weather, will be required to cover a roof 112 ft. long and 40 ft. from eaves to ridge? *Ans.* 13440.

4. What will be the expense of shingling a roof 85 ft. long and 25 ft. from eaves to ridge, the shingles being worth $14.25 ℔ M.? *Ans.* $605.62½.

5. How many shingles will it take to cover a roof 65 ft. long and 30 ft. from eaves to ridge, each shingle being exposed one-third to the weather, and the first course being double? *Ans.* 23790.

6. Which would cost most, a brick sidewalk 7½ ft. wide and 600 ft. long, at $1.25 a sq. yard, or a stone sidewalk of the same dimensions, at 18¢ a sq. ft.? *Ans.* The stone, $178.83⅓.

CARPETING, LINING, ETC.

483. In Carpeting we take into consideration the width of the carpet, the allowance for matching the figures, and whether the strips run lengthwise or crosswise.

1. Carpets are usually 1 yd. or ¾ yd. wide; but matting, oil-cloth and other materials used for covering floors are of various widths.

2. To match the figures we must often turn under or cut off one of the ends. When an exact number of strips is a little too wide for the room, a part of one breadth is turned under.

Rule.—*Find the number of strips required, and multiply the number of yards in each strip by the number of strips.*

NOTE.—In lining divide the whole surface by the surface of a yard of the material.

EXAMPLES FOR PRACTICE.

1. How many yards of carpet ¾ yd. wide will carpet a room 24 ft. 9 in. by 18 ft., the carpet running lengthwise?

SOLUTION.—It will take 8 strips each 24 ft. 9 in. long; hence it will require $24\frac{3}{4}\times8\div3=66$ yd.

2. What will it cost to carpet lengthwise a room 33 ft.

long and 24 ft. wide, with ingrain carpet 1 yd. wide, at 75¢ a yard, allowing 6 in. waste for matching the figures on each piece? *Ans.* $67.

3. A lady wishes to carpet crosswise a parlor 33 ft. by 16 ft. 4 in. with Brussels carpet $\frac{3}{4}$ yd. wide; how many yards must she buy, allowing nothing for matching? *Ans.* $81\frac{2}{3}$ yd.

4. I wish to cover a dining-room 24 ft. long by 21 ft. wide with matting $3\frac{1}{2}$ ft. wide; how many yards will it take running lengthwise? how many crosswise? *Ans.* 48yd.; 49 yd.

5. I have a table 6 ft. long and 3 ft. 4 in. wide, which I wish to cover with a baize cloth hanging down 10 inches on each side; how many square yards do I require, and how many yards in length? *Ans.* $4\frac{7}{27}$ sq. yd.; $2\frac{5}{9}$ yd.

6. My parlor has four windows, to curtain each of which it requires 8 yards of damask $\frac{5}{8}$ yd. wide @ $1.75, lined with silk $\frac{3}{4}$ yd. wide @ 1; also $5\frac{1}{2}$ yds. of trimming @ $1.25, and a cornice @ $4.50; required the number of yards of silk and the whole cost of the curtains. *Ans.* $26\frac{2}{3}$ yd.; 128.16\frac{2}{3}$.

PAPERING.

484. Wall paper is sold only by the roll, and in estimates a part of a roll is reckoned as a whole roll.

485. A roll of American paper is commonly 8 yd. long and $\frac{1}{2}$ yd. wide.

1. Paper is now usually put up in double rolls 16 yd. long.
2. Borders and friezes are sold by the yard, and vary in width from 3 in. to 18 in.
3. On account of waste, the cost of papering a room can only be approximately estimated.

Rule.—I. *Find the entire distance around the room in yards, and multiply this by 2, to fin the number of half-yards or strips.*

II. *Divide the number of strips required for the room by the number of strips that can be cut from a roll; the quotient will be the number of rolls required.*

Since there are 24 feet in a roll, if the length of the strips is 8 feet or less, 3 strips can be cut from a roll; if between 8 ft. and 12 ft., 2 strips, etc. A double roll makes twice as many strips.

EXAMPLES FOR PRACTICE.

1. How many rolls of paper will cover the walls of a room 33 ft. long, 24 ft. wide, and 9 ft. 6 in. high?

SOLUTION.—The distance around the room is $2\times(33+24)=114$ ft. $=38$ yd. $=76$ half yards. Since the height is $9\frac{1}{2}$ ft. we can cut only 2 full strips from a roll of 24 ft. Hence the number of rolls will equal $76\div2=38$ rolls.

OPERATION.

$$2\times(33+24)=114 \text{ ft.}$$
$$=38 \text{ yd.}$$
$$=76 \text{ half yd.}$$
$$76\div2=38 \text{ rolls.}$$

2. What will be the cost of papering the above room at 25¢ a roll, including also a gilt moulding around the top of the walls at 6¢ a ft? *Ans.* $16.34.

3. What will be the cost of papering the walls and ceiling of a room 25 ft. long, 15 ft. 8 in. wide, and $9\frac{1}{2}$ ft. high, at $1.75 a double roll, deducting 3 rolls for doors and windows? *Ans.* $24.50.

4. Required the cost of papering the walls of a parlor 35 ft. long, 18 ft. wide, 10 ft. high, with base-board 1 ft. wide, at $1.25 a double roll, having also a border 18 in. wide, at 45¢ per yard, the price including the cost of putting on the paper and border. *Ans.* $30.90.

NOTE.—The width of base-board and border must be deducted from height of room to obtain length of strip.

5. What will be the cost of papering a room 25 ft. long, 16 ft. wide, and 9 ft. high, the base board being 6 in. wide, with paper at 45¢ the double roll, having a border 6 in. wide, at 9¢ per yard, the cost of putting on the paper being $4? *Ans.* $10.96.

MEASURES OF VOLUME.

486. A **Volume** is that which has length, breadth, and thickness or height. These three elements are called *dimensions*. A volume is also called a *solid*.

487. By the **Contents** of a volume we mean the amount of space it contains. The contents are expressed by the number of times it contains a *cube* as a *unit of measure*.

THE CYLINDER.

488. A **Cylinder** is a round body of uniform size, with equal and parallel circles for its ends. The two circular ends are called *bases*.

489. The **Altitude** of a cylinder is the distance from the centre of one base to the centre of the other.

490. The **Convex Surface** of a cylinder is the surface of the curved part.

Rule I.—*To find the convex surface of a cylinder, multiply the circumference of the base by the altitude.*

Rule II.—*To find the contents of a cylinder, multiply the area of the base by the altitude.*

EXAMPLES FOR PRACTICE.

1. What is the convex surface of a cylinder, the diameter of whose base is 10 inches, and whose altitude is 18 inches?

SOLUTION.—The circumference of the base equals 10 in.$\times$3.1416, which is 31.416 inches; multiplying by the altitude, 18, we have 565.488 square inches, the convex surface.

2. What is the surface of a marble column 20 ft. high, and 24 inches in diameter? *Ans.* 125.664 sq. ft.

3. What is the length of a log of wood 18 inches in diameter, whose convex surface is 47.124 square feet? *Ans.* 10 ft.

4. How many cubic feet of water will a cistern hold whose depth is $7\frac{1}{2}$ ft. and diameter $5\frac{1}{2}$ ft.? *Ans.* 178.1876.

5. A cistern is to be dug in a place where its diameter can only be 6 ft., but is to contain 420 cu. ft. of water; what must be the depth? *Ans.* 14.85+ft.

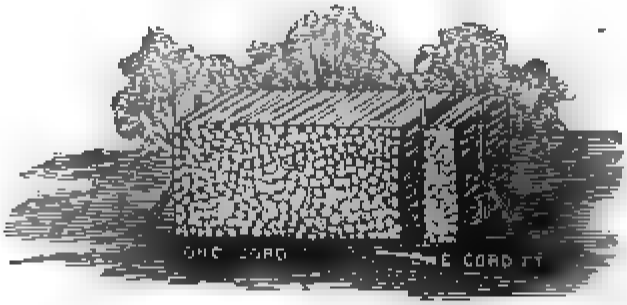
6. What will it cost to line a cylindrical cistern with tin, at 50 cents a square foot, the diameter being 6 ft. and depth 8 ft.? *Ans* $89.54.

WOOD MEASURE.

491. The **Measure of Wood** is the *cord*, which is divided into *cord feet*, etc.

492. A Cord of wood is a pile 8 feet long, 4 feet wide, and 4 feet high. It contains 8 cord feet, or 128 cubic feet.

493. A Cord Foot is a part of this pile 1 foot long. It is thus 1 foot long, 4 feet wide, and 4 feet high, and contains 16 cubic feet.



Rule.—*To find the number of cords in a pile of wood, find the number of cubic feet and reduce to cord feet and cords.*

EXAMPLES FOR PRACTICE.

1. How many cords in a pile of wood 32 ft. long, 8 ft. high, and 6 ft. wide?

Solution.—The number of cubic feet equals $32 \times 8 \times 6$, which equals 1536; dividing by 16, to reduce this to cord feet, we have 96 cord feet; dividing by 8 to reduce this to cords, we have 12 cords.

2. How many cords of wood in a pile 28 ft. long, 12 ft wide, and 6 ft. high? *Ans.* 15 cd. 6 cd. ft.

3. If a pile of wood is 10 ft. high and 6 ft. wide, how long must it be to contain 12 cords? *Ans.* 25 ft. $7\frac{1}{5}$ in.

4. A man bought 50 cords of wood, $3\frac{1}{2}$ ft. long, and proposes to put it in a pile 12 ft. high; how long will the pile be? *Ans.* $152\frac{8}{21}$ ft

5. How much will a pile of wood weigh, 12 ft. long, 4 ft. wide, and 6 ft. high, $\frac{3}{4}$ of which is white oak, and the rest white pine, provided a cubic foot of white oak weighs 55 lb., and of white pine 30 lb.? *Ans.* 7 tons 40 lb.

BOARDS AND TIMBER.

494. Boards and **Timber** are usually estimated in what are called *board feet*, instead of in cubic feet.

495. A Board Foot is 1 foot long, 1 foot wide, and 1 inch thick. A *cubic foot*, therefore, contains 12 *board feet*. Hence, *board feet* may be reduced to *cubic feet* by dividing by 12; and *cubic feet* to *board feet* by multiplying by 12.

496. A Standard Board, in commerce, is 1 inch thick,

many perches of stone have I, and how many perches of masonry can I build from it? *Ans.* 480; 540.

5. Mr. Nelson has a cellar 45 ft. long, 27 ft. wide, and 8 ft. deep, which he contracts to have walled at a cost of \$3.87½ a perch, the wall to be 2 ft. thick, and *one-half* to be allowed for corners; what will be the cost? *Ans.* \$350.71.

6. Mr. James, having contracted to build a house, agrees to pay John Newman 45¢ a load for digging the cellar, and Samuel Forman \$3.65 a perch for walling it with rough stone to the surface, and with cut stone above ground at 25¢ per square foot; the cellar to be 54 ft. 9 in. long, 35 ft. wide and 5½ ft. deep, and the wall 1¾ ft. thick and 3 ft. high above ground; what will the work cost? *Ans.* \$565.07.

BRICKWORK.

500. Brickwork is generally estimated by the *thousand bricks*, but sometimes in *cubic feet*.

The average size of bricks is 8 in. × 4 × 2, but Philadelphia and Baltimore bricks are $8\frac{1}{4}$ in. × $4\frac{1}{8}$ × $2\frac{3}{8}$; Maine bricks, $7\frac{1}{2}$ in. × $3\frac{3}{8}$ × $2\frac{3}{8}$; North River bricks, 8 in. × $3\frac{1}{2}$ × $2\frac{1}{4}$; and Milwaukee bricks; $8\frac{1}{2}$ in. × $4\frac{1}{8}$ × $2\frac{3}{8}$.

To build one *square foot* of wall 1 brick, or 4 inches, thick, requires about 7 common bricks; 2 bricks, or 9 in., thick, 14 bricks; 3 bricks, or 13 in., thick, 21 bricks. Hence the following rule:

Rule I.—*To find the number of common bricks required for a wall, multiply the number of square feet in the wall by* 7, *if the wall is* 1 *brick thick; by* 14, *if* 2 *bricks thick; by* 21 *if* 3 *bricks thick.*

To find the number of any kind of bricks required to build 1 square foot of wall 1 brick thick, we add ¼ of an inch to the length and thickness of the brick for mortar, and divide 144 by the product of these two sums. Hence the rule:

Rule II.—*To find the number of any kind of bricks required for a wall, multiply the number of bricks in a square foot of wall,* 1 *brick thick, by the number of bricks in the thickness, and this product by the number of square feet in the wall.*

NOTE.—An old rule was—*Deduct* $\frac{1}{10}$ *of the solid contents for the mortar and divide the remainder by the contents of one brick.* We may also *find the contents of a brick with the mortar surrounding it, and divide a cubic foot by this quantity, to find the number of bricks in a cubic foot.*

EXAMPLES FOR PRACTICE.

1. How many bricks will be required to build a house in Baltimore 25 ft. front, 80 ft. deep, the wall being 34 ft. high and 3 bricks thick, allowing 352 ft. for doors and windows?

OPERATION.

$8\frac{1}{2}\times2\frac{5}{8}=22\frac{5}{16}$ sq. in. surface 1 brick.
$144\div22\frac{5}{16}=6\frac{54}{119}$, No. bricks in sq. ft.
$(25+80)\times2\times34=7140$ sq. ft.
$\frac{13}{12}\times4\times34=147\frac{1}{3}$ sq. ft.
$7140-(147\frac{1}{3}+352)=6640\frac{2}{3}$ sq. ft.
$6640\frac{2}{3}\times6\frac{54}{119}\times3=128572\frac{4}{17}$.

SOLUTION.—Since the house is in Baltimore, the outer surface of a brick, after adding $\frac{1}{4}$ inch, equals $8\frac{1}{2}\times2\frac{5}{8}$ or $22\frac{5}{16}$ sq. in.; dividing 144 by $22\frac{5}{16}$, we have $6\frac{54}{119}$, the number of bricks in 1 square foot of wall 1 brick thick. The outer surface of the walls equals $(25+80)\times2\times34=7140$ sq. ft. As we are estimating for material we deduct the corners, which, reckoning a wall 3 bricks thick as 13 in. thick, equal $\frac{13}{12}\times4\times34=147\frac{1}{3}$ sq. ft. Hence the wall equals $7140-(147\frac{1}{3}+352)=6640\frac{2}{3}$ sq. ft., which, multiplied by $6\frac{54}{119}$ and by 3, the number of bricks in the thickness, gives $128572\frac{4}{17}$ bricks.

2. How many Maine bricks will build a house 24 ft. front, 63 ft. long, 30 ft. high, and 3 bricks thick, having 3 doors each 7×3 ft., and 21 windows each 6×3 ft.? *Ans.* 98722—.

3. Mr. Johnson wishes to build a house of rough stone faced with Philadelphia brick, the brick wall being 80 ft. long, 28 ft. wide and 20 ft. high, and 1 brick thick, and 354 ft. being allowed for openings; how many bricks will be required? *Ans.* 25418+.

4. What will it cost to build, of average bricks, a house 50 ft. long, 22 ft. wide, and 23 ft. high, the wall being 13 in. thick, there being two doors, each 7×3½ ft., 1 door 6×3½ ft., and 18 windows, each 6×3 ft., the brick costing $9 ℔ M. and laying $2 50 ℔ M.? *Ans.* $706.55.

5. What will be the cost of building a house 36 ft. square, the wall being 24 ft. high and 3 bricks thick, of Milwaukee bricks, 216 sq. ft. being allowed for doors, and the brick costing $10.50 ℔ M. and the laying $2.75 ℔ M.? *Ans.* $798.07.

MEASURES OF CAPACITY.

501. Measures of Capacity are volumes used to determine the quantity of fluids and many dry substances.

502. The **Principal Measures** of capacity are the *gallon* for liquid substances, and the bushel for dry substances.

CAPACITY OF CISTERNS, ETC.

503. The **Capacity of Cisterns**, etc., is usually expressed in *gallons* or *barrels*.

504. The **Standard Liquid Gallon** of the United States contains 231 cubic inches, and is equal to about $8\frac{1}{3}$ lb. Avoirdupois of pure water.

505. The **Barrel** of $31\frac{1}{2}$ gallons and the *hogshead* of 63 gallons, are used in measuring the capacity of cisterns, vats, tanks, etc. When used as the names of vessels, these terms express no definite quantity.

The Imperial Gallon of Great Britain contains 277.274 cubic inches, and is equal to about 1.2 U. S. gallons. The beer gallon contains 282 cubic inches, but is now seldom used. A cubic foot of pure water weighs 1000 oz. Avoirdupois, or very nearly $62\frac{1}{2}$ lb.

Rule I.—*To find the capacity of a cistern or vessel in gallons, divide the contents in cubic inches by* 231.

Rule II.—*To find the cubic inches in a given number of gallons, multiply the given number of gallons by* 231.

EXAMPLES FOR PRACTICE.

1. How many gallons of water are contained in a tank 9 ft. long, 5 ft. wide, and 4 ft. deep?

SOLUTION.—The contents of the tank equal $9 \times 5 \times 4$, which are 180 cubic feet; multiplying by 1728 to reduce to cubic inches, and dividing by 231, the number of cubic inches in a gallon, we have $1346\frac{38}{77}$ gal.

2. A trough 8 ft. long, 5 ft. wide, and 3 ft. deep, will hold how many gallons of water? *Ans.* $897\frac{51}{77}$ gal.

3. A vat 12 ft. long, 8 ft. wide, and 6 ft. deep, will contain how many gallons of beer? *Ans.* $3529\frac{25}{47}$.

4. A cistern whose length was 10 ft. and width 7 ft., contained 50 hogsheads; what was its depth? *Ans.* $6\frac{1}{64}$ ft.

5. How many hogsheads of water can be contained in a well whose diameter within the curb is $3\frac{1}{2}$ ft., and depth 15 ft.? *Ans.* 17.136 hhd.

6. The diameter of a cylindrical cistern is 6 ft., and it contains 40 barrels of water; what is the depth of the water? *Ans.* 5.957 + ft.

7. A cistern 18 ft. 3 in. long and 7 ft. 4 in. wide is full of

water; how many gallons must be drawn off to lower the surface 1 foot? *Ans.* $1001\frac{1}{7}$ gal.

8. How many imperial gallons in a cistern 8 ft. 4 in. long, 3 ft. 3 in. wide, and 2 ft. 6 in. deep? *Ans.* 421.965+gal.

9. A cistern 6 ft. long, 3 ft. 4 in. wide, and 4 ft. 9 in. deep, is emptied by a pipe in 1 h. 45 min.; how many gallons are discharged in 1 minute? *Ans.* $6\frac{414}{539}$ gal.

10. If 40 gal. 3 qt. of water flow through a pipe in $2\frac{1}{2}$ hours, how long will it take to fill a cistern 6 ft. long, 3.5 ft. wide, and $8.66\frac{2}{3}$ ft. deep? *Ans.* 3 da. 11 h. 31 min. 29+sec.

11. What will be the expense of pumping out a reservoir 24 ft. long, 18 ft. wide, and 8 ft. deep, at 10 cents a hogshead, the reservoir being half full? *Ans.* $20.52—.

12. A man wishing to construct a tank in his attic, found that it would not be safe to place there a weight of more than 4500 lb. of water; what length can he make the tank with a width of 4 ft., and a depth of 3 ft.? *Ans.* 6 ft.

CAPACITY OF BINS, ETC.

506. The **Capacity of Bins**, etc., is usually expressed in *bushels*.

507. The **Standard Bushel** of the United States is a cylindrical measure $18\frac{1}{2}$ in. in diameter and 8 in. deep, containing 2150.42 cubic inches.

The bushel *heaped measure* is the standard bushel heaped in the form of a cone $19\frac{1}{2}$ in. in diameter and at least 6 in. high, and containing 2747.7167 cu. in., while the even measure is called *stricken measure*. Grains, seeds, and small fruit are sold by *stricken measure;* but potatoes, corn in the ear, coarse vegetables, large fruits, coal, and other bulky articles are sold by heaped measure. In practice we may call 5 *stricken measures* equal to 4 *heaped measures*.

Grain is shipped from New York by the *quarter* of 480 lb. (8 U. S. bu.), or by the *Ton* of $33\frac{1}{3}$ U. S. bushels. The *Imperial Bushel* of Great Britain contains 2218.192 cu. in., and the English Quarter contains 8 Imperial or $8\frac{1}{4}$ U. S. bushels.

Rule I.—*To find the capacity of a bin in bushels, divide the contents in cubic inches by* 2150.42.

Rule II.—*To find the cubic feet in a given number of bushels, multiply the number of bushels by* 2150.42, *and divide by* 1728.

EXAMPLES FOR PRACTICE.

1. How many bushels of grain are contained in a bin 9 ft. long, 4 ft. wide, and $2\frac{1}{2}$ ft. deep?

SOLUTION.—The contents equal $9\times4\times2\frac{1}{2}$, or 90 cubic feet, which equals 155520 cubic inches; dividing by 2150.42, the number of cubic inches in a bushel, we have 72.32 bushels.

2. How many bushels of grain are contained in a bin 10 ft. long, 5 ft. wide, and 3 ft. deep? *Ans.* 120.53+ bu.

3. What must be the width of a bin which is 8 ft. long and 3 feet deep, in order to contain 96 bushels? *Ans.* 4.97 ft.

4. An elevator containing 5000 bushels is 32 ft. long and 16 ft. wide; what is its depth? *Ans.* 12.15 ft.

APPROXIMATE MEASUREMENTS.

While the previous rule gives exact results, the following approximate values are often used, being more easily reckoned and sufficiently exact in most cases.

Since 2150.42 is to 1728 as 5 to 4, nearly, a bushel is nearly equal to $1\frac{1}{4}$ cu. ft. Therefore, for practical purposes, $\frac{4}{5}$ *of the number of cubic feet will equal the number of bushels, and* $\frac{5}{4}$ *of the number of bushels will equal the number of cubic feet.*

Coal.—Coal is bought and sold in large quantities by the *ton*, in small quantities by the *bushel*, 28 heaped bushels, or about 43.5 cu. ft., being considered equal to a ton.

Ordinary *anthracite coal* measures from 36 to 40 cu. ft. to the *ton*; *bituminous coal*, from 36 to 45 cu. ft. to the ton; Lehigh white-ash coal, egg size, measures about $34\frac{1}{2}$ cu. ft. to the ton; Schuylkill white-ash, 35 cu. ft., and pink, gray, or red ash, 36 cu. ft., to the ton.

Hay.—Hay, when loose or in loads, or upon a scaffold, measures about 500 cu. ft. to the ton; on a mow, 400 cu. ft.; and in a large well-settled stack, 10 cu. yd.

EXAMPLES FOR PRACTICE.

1. I have a bin $4\frac{1}{2}$ ft. long, 3 ft. 6 in. wide, and 2 ft. deep; how many bushels of oats will it hold?

SOLUTION.—The contents equal $4\frac{1}{2}\times3\frac{1}{2}\times2=31\frac{1}{2}$ cu. ft. Since oats are sold by *stricken measure*, the number of bushels will be $\frac{4}{5}$ of $31\frac{1}{2}$, or $25\frac{1}{5}$ bu.

2. A bin 11 ft. 3 in. long, 6 ft. 6 in. wide, and 4 ft. 9 in. deep, was filled with wheat; what is it worth at \$1.875 a cental, reckoning 60 lb. to a bushel? *Ans.* \$312.61.

3. The box of a wagon is 12 ft. 3 in. long, 3 ft. 2 in. wide, and 2 ft. 3 in. deep; how many bushels of oats and how many of potatoes will it hold? Ans. 69.825; 55.86.

4. A bin 12 ft. 7 in. long, 6 ft. 3 in. wide, and 3 ft. 9 in deep, is filled with wheat; how many barrels of flour will it make if one bushel of wheat makes 48 lb. of flour?

Ans. 57.78 bbl.

5. A dealer has a bin of coal 80 ft. long, 10 ft. wide, and 8 ft. deep; how many tons of Lehigh white ash does it contain, and how many of Schuylkill gray ash?

Ans. 185.5+; 177.77+.

6. A shed 8 yd. long, $6\frac{1}{4}$ yd. wide, and $7\frac{1}{2}$ feet high, is $\frac{2}{3}$ full of Schuylkill white ash coal; what is its value, at $4.50 a ton? *Ans.* $289.29.

7. I have a haymow 22 ft. long, 17 ft. 6 in. wide, and 11 ft. 8 in. high; what is its value, when filled, at $11.50 a ton? *Ans.* $129.14.

8. Mr. Barr had a rectangular stack of hay 9 ft. long, $7\frac{1}{2}$ ft. wide, and $6\frac{1}{2}$ ft. high, which he sold at $13.25 a ton; what did he receive for his hay? *Ans.* $21.53.

9. A crib filled with corn in the ear measures on the inside 17 ft. 2 in. in length by 6 ft. 9 in. in width, and 8 ft. 3 in. in height; what is the value of the corn when shelled at 85¢ a bushel, if 2 bushels of ears make 1 bushel of shelled corn? *Ans.* $260.02.

COMPARISON OF MEASURES OF CAPACITY.

508. The **Dry Gallon,** or *half peck,* contains 268.8 cubic inches; hence 6 dry gallons equal nearly 7 liquid gallons.

NOTE.—The pupil will remember that the liquid gallon contains 231 cu. in., and the old beer gallon 282 cu. in.

EXAMPLES FOR PRACTICE.

1. Reduce 4 gal. 3 qt. wine measure to the old beer measure. *Ans.* 3 gal. 3 qt. $1\frac{6}{47}$ pt.

2. Reduce 57 gal. 3 qt. dry measure to wine measure.

Ans. 67 gal. $1\frac{3}{5}$ pt.

3. Reduce 336 gal. wine measure to dry measure.

Ans. 288 gal. 3 qt.

4. Reduce 8 bu. 3 pk. 2 qt. dry measure to the old beer measure. *Ans.* 67 gal. $1\frac{3}{5}$ pt.

5. A farmer has a bin which contains 924 bu. of grain; how much water would it hold? *Ans.* 8601.68 gal.

6. If a man should buy 28 qts. of milk at 5 cts. a quart, beer measure, and sell it at $6\frac{1}{2}$ cts. a quart, wine measure, what would he gain? *Ans.* $82\frac{2}{11}$ cts.

7. If a milkman bought 141 gal. milk, wine measure, at 25 cents a gallon, and sold it at 8 cts. a quart, beer measure, what did he gain? *Ans.* $1.71.

8. A grocer bought 33 bushels of berries, at $2.25 per bushel, and sold them by mistake by wine measure at 8 cents a quart; what was his gain or loss? *Ans.* Gain, $24.055.

MISCELLANEOUS PROBLEMS.

1. How much alloy must be mixed with 2 lb. 1 oz. 12 pwt. 15 gr. of pure gold, that the mixture may be 18 carats fine? *Ans.* 8 oz. 10 pwt. 21 gr.

2. What costs the excavation for a cellar $6\frac{1}{4}$ ft. deep under the main building of a dwelling-house 40×32 ft., and an excavation for the walls of an L 20 ft. square, $1\frac{1}{4}$ ft. wide and 2 ft. deep, at 50¢ per cu. yd? *Ans.* $150.81.

3. James Morton & Co., New York, bought of a firm in Chicago 15600 bu. of wheat @ $1.25, delivered in New York, and shipped the same to Liverpool, the freight being $\$1.37\frac{1}{2}$ per quarter, where they sold the whole cargo at 60 s. per English quarter; what was the gross gain in U. S. money? *Ans.* $5425.08.

4. Required the cost of the cellar and brickwork of a dwelling-house in the form of an L, the main building being 40×30 ft., and the L projecting 24 ft., and being 14 ft. wide, the walls of the main building being 24 ft. high, and 13 in. thick, and those of the L 14 ft. high and 9 in. thick; chimneys and gables being reckoned as 400 sq. ft., 13 in. thick, and equal to windows and doors; the estimate being as follows: 50¢ per cubic yard for excavating a cellar 5 ft. deep; cellar wall, $1\frac{1}{2}$ ft. thick, $2.25 a perch, for lower part of wall, 5 ft. high, and 15¢ per sq. ft. for cut stone 2 ft. high; brick $10 ℔ M., and laying $2 ℔ M. *Ans.* $1270.08.

SECTION VIII.

PERCENTAGE.

509. Percentage is the process of computation in which the basis of comparison is a *hundred.*

510. The **Term** *per cent.*—from *per*, by, and *centum*, *a hundred*—means *by* or *on the hundred;* thus, 6 per cent. of any quantity means 6 of every hundred of the quantity.

511. The **Symbol of Percentage** is %. The per cent. may also be indicated by a common fraction or a decimal; thus, $6\% = \frac{6}{100} = .06$.

512. The **Quantities** considered in percentage are the *Base*, the *Rate*, the *Percentage*, and the *Amount* or *Difference.*

513. The **Base** is the number on which the percentage is computed.

514. The **Rate** is the number of hundredths of the base which are to be taken.

515. The **Percentage** is the result obtained by taking a certain per cent. of the base.

516. The **Amount** or **Difference** is the sum or difference of the base and percentage. They may both be embraced under the general term *Proceeds.*

NOTE.—In computation the rate is usually expressed as a decimal. For the difference between *Rate* and *rate per cent.*, see *Brooks's Philosophy of Arithmetic.*

EXPRESSION OF THE RATE.

1. Express 5% as a decimal and a common fraction.

SOLUTION.—Since per cent. is so many on a hundred, 5% of a quantity is .05 of it; or, as a common fraction, $\frac{5}{100}$, or $\frac{1}{20}$ of it.

OPERATION.

$5\% = .05 = \frac{5}{100} = \frac{1}{20}$

Express

2. 6%.	*Ans.* .06, or $\frac{3}{50}$.	5. 10%.	*Ans.* 10, or $\frac{1}{10}$.
3. 8%.	*Ans.* .08, or $\frac{2}{25}$.	6. $12\frac{1}{2}\%$.	*Ans.* $.12\frac{1}{2}$, or $\frac{1}{8}$.
4. 4%.	*Ans.* .04, or $\frac{1}{25}$.	7. $16\frac{2}{3}\%$.	*Ans.* $.16\frac{2}{3}$, or $\frac{1}{6}$.

8. $11\frac{1}{9}\%$. *Ans.* $.11\frac{1}{9}$, or $\frac{1}{9}$.
9. $37\frac{1}{2}\%$. *Ans.* $.37\frac{1}{2}$, or $\frac{3}{8}$.
10. $8\frac{3}{4}\%$. *Ans.* $.08\frac{3}{4}$, or $\frac{7}{80}$.
11. $42\frac{6}{7}\%$. *Ans.* $.42\frac{6}{7}$, or $\frac{3}{7}$.
12. $\frac{1}{2}\%$. *Ans.* .005, or $\frac{1}{200}$.
13. $.004\frac{6}{11}$. *Ans.* $\frac{5}{11}\%$.
14. .028. *Ans.* $2\frac{4}{5}\%$.
15. .0025. *Ans.* $\frac{1}{4}\%$.

517. Cases.—There are **Three Cases**, as follows:

1. Given, the rate and the base, to find the percentage or the proceeds.

2. Given, the rate and the percentage or the proceeds, to find the base.

3. Given, the base and the percentage or the proceeds, to find the rate.

NOTES.—1. Authors usually present the subject in five or six cases, but it is thought that the method here adopted is to be preferred, on account of its logical accuracy and practical convenience.

2. A percentage deducted from the price of goods is called a *Discount*. *Successive discounts*, called *Trade Discounts*, are often taken off, as "10 and 5% off," meaning 10% off and 5% off of the remainder.

CASE I.

518. ***Given, the base and the rate, to find the percentage or the proceeds.***

1. What is 25% of $480?

SOLUTION.—Since 25% of a number equals .25 of the number, 25% of $480 equals .25 times $480, which, by multiplying, we find to be $120.

OPERATION.
$$\begin{array}{r} 480 \\ .25 \\ \hline \$120.00 \end{array}$$

2. What is the amount of $480 increased by 25% of itself?

SOLUTION.—A number increased by 25%, or .25 times itself, equals 1.25 times itself; 1.25 times $480 equals $600.

OPERATION.
$$\begin{array}{r} \$480 \\ 1.25 \\ \hline \$600.00 \end{array}$$

Rule I.—*Multiply the base by the rate, to find the percentage.*

Rule II.—*Multiply the base by 1 plus the rate, to find the amount; or by 1 minus the rate, to find the difference.*

NOTES.—1. The method of solving by reducing the rate to a common fraction is simpler when the rate gives a small common fraction.

2. The *amount* equals the base *plus* the percentage; the *difference* equals the base *minus* the percentage.

What is

3. 8% of $500? *Ans.* $40.
4. 25% of $960? *Ans.* $240.
5. $\frac{7}{8}\%$ of $75? *Ans.* $\$.65\frac{5}{8}$.
6. $33\frac{1}{3}\%$ of 54? *Ans.* 18.

7. $12\frac{1}{2}\%$ of \$900? *Ans.* \$112.50.
8. 35% of \$248? *Ans.* \$86.80.
9. $66\frac{2}{3}\%$ of 596 lb.? *Ans.* $397\frac{1}{3}$ lb.
10. $\frac{4}{5}\%$ of 627 yd.? *Ans.* 5.016 yd.
11. $42\frac{6}{7}\%$ of 343 acres? *Ans.* 147 A.
12. $45\frac{5}{11}\%$ of \$165? *Ans.* \$75.

13. I sold a lot of envelopes marked \$7.20 ℔ M., at 10 and 10% off; what did I receive? *Ans.* \$5.83.

14. William bought a lot of base-balls at \$12 a dozen and sold them for 20 and 5% on; what did he receive? *Ans.* \$15.12.

15. What is the difference between 20% off and 10 and 10% off; between 15% on and 10 and 5% on? *Ans.* 1%; $\frac{1}{2}\%$.

16. A clerk's salary is \$2500 a year; if he pays 15% for board, 8% for clothing, 5% for books, and 12% for incidentals, how much will he save in a year? *Ans.* \$1500.

17. A bought a house for \$2500, and paid $62\frac{1}{2}\%$ of the price in cash, and gave a mortgage for the remainder; what was the amount of the mortgage? *Ans.* \$937.50.

18. A man contracted a debt of \$570; he paid $33\frac{1}{3}\%$ of it the first quarter, 25% of the remainder the second quarter, and $16\frac{2}{3}\%$ of what was still due the third quarter; how much remained unpaid? *Ans.* \$237.50.

19. Mr. Martin deposited \$1250 in bank; he drew out 15% of it the first month, 20% of the remainder the next month, and having realized $18\frac{3}{4}\%$ on what he had drawn, deposited it; what was his bank deposit then? *Ans.* \$925.

20. If 48% of whisky is alcohol, how much alcohol does a man swallow in 35 years who drinks half a gill of whisky 6 times a day? *Ans.* 574.875 gal.

21. A mechanic contracts to supply dressed stone for a church for \$87,560, if the rough stone cost him 18 cents a cubic foot; but if he can get it for 16 cents a cubic foot, he will deduct 5% from his bill; required the number of cubic feet, and the charge for dressing the stone.

Ans. 218900 cubic feet; charge 22 cents per cu. ft.

CASE II.

519. ***Given, the rate and the percentage or the proceeds, to find the base.***

1. 75 is 5% of what number?

SOLUTION.—If 75 is 5% of some number, then .05 times *some number* equals 75; if .05 times *some number* equals 75, the *number* equals 75÷.05, which is 1500.

OPERATION.
$75 \div .05 = 1500$

2. What number, increased by 25% of itself, equals 450, or diminished by 25% of itself, equals 270?

SOLUTION.—A number, increased by 25% or .25 of itself, equals 1.25 times the number; and if some number multiplied by 1.25 equals 450, the number equals 450÷1.25, or 360. A number diminished by 25% of itself, equals .75 times the number; and if some number multiplied by .75 equals 270, the number equals 270÷.75, or 360.

OPERATION.
$450 \div 1.25 = 360$

Rule I.—*Divide the percentage by the rate, to find the base.*

Rule II.—*Divide the amount by* 1 *plus the rate, or the difference by* 1 *minus the rate, to find the base.*

3. 96 is $56\frac{1}{4}$% of what number? *Ans.* $170\frac{2}{3}$.

4. 101 is $68\frac{3}{4}$% of what number? *Ans.* $146\frac{10}{11}$.

5. $375\frac{3}{4}$ is $81\frac{1}{4}$% of what number? *Ans.* $462\frac{6}{13}$.

6. 784 is $83\frac{1}{3}$% of what number? *Ans.* 940.8.

7. Bought 2 doz. cast steel riveting hammers @ $$32.06\frac{1}{4}$ at 25 and 5% off; what was the marked price? *Ans.* $45.

8. Sold 200 carriage bolts @ $2.91 ℔ C. for 10 and 5% on; what was the cost? *Ans.* $2.52.

9. The fraction $\frac{9}{11}$ is 5% more or 4% less than what fractions? *Ans.* $\frac{60}{77}$, or $\frac{75}{88}$.

10. I draw 45% of my bank deposit to pay a note of $5670; what did I have at first? *Ans.* $12600.

11. A man collecting $75 of the money due him, increases his funds $16\frac{2}{3}$%; how much had he at first? *Ans.* $450.

12. In $99\frac{2}{3}$ gal. of alcohol the water is $8\frac{1}{3}$% of the spirits; how many gallons are there of each? *Ans.* 92; $7\frac{2}{3}$.

13. A farmer's crop of oats this year is $7\frac{1}{2}$% greater than his crop of last year; what was last year's crop if in the two years he raised 725 bushels? *Ans.* $349\frac{33}{83}$ bushels.

14. A lady's cloak cost \$40; the making cost $33\frac{1}{3}\%$ less than the cloth, and the trimmings 25% more than the cloth; what did each cost?

Ans. Cloth, $\$13\frac{5}{7}$; making, $\$9\frac{1}{7}$; trimmings, $\$17\frac{1}{7}$.

15. A and B together have 1320 acres of land, $31\frac{1}{4}\%$ of A's equaling $37\frac{1}{2}\%$ of B's, and $56\frac{1}{4}\%$ of B's equaling $66\frac{2}{3}\%$ of C's; how much land has C? *Ans.* $506\frac{1}{4}$ acres.

16. Mr. Howard drew 75% of his money from bank, and paid $87\frac{1}{2}\%$ of it for a house worth \$5600; how much money had he remaining in bank? *Ans.* $\$2133.33\frac{1}{3}$.

17. In an engagement 5% of an army were killed, $12\frac{1}{2}\%$ of the remainder were wounded, and $16\frac{2}{3}\%$ of the wounded died; there were 290 more killed than mortally wounded; how many men were in the army? *Ans.* 9600.

18. A, wishing to sell a cow and horse to B, asked 150% more for the horse than the cow; he then reduced the price of the cow 25%, and the horse $33\frac{1}{3}\%$, at which price B took them, paying \$290; what was the price of each?

Ans. Cow, \$90; horse, \$200.

19. In building a church, the trustees paid three times as much for material as for labor; had they paid $4\frac{1}{3}\%$ more for material and 7% more for labor, the church would have cost \$14700; what was its cost? *Ans* \$14,000.

CASE III.

520. ***Given, the base and the percentage or the proceeds, to find the rate.***

1. 24 is what per cent. of 96?

SOLUTION.—If 24 is some per cent. of 96, then 96 multiplied by *some rate* per cent. equals 24; if 96 multiplied by *some rate* equals 24, the *rate* equals 24 divided by 96, which is .25 or 25%.

OPERATION.
$24 \div 96 = .25$

Rule I.—*Divide the percentage by the base, to find the rate.*

Rule II.—*Divide the difference between the proceeds and base by the base, to find the rate.*

NOTE.—The rate may also be found by dividing the proceeds by the base and taking the difference between 1 and the quotient.

2. What % of 280 is 112? *Ans.* 40%

3. What % of 960 is 160? *Ans.* $16\frac{2}{3}$%

4. What % of 1470 yd. is 980 yd.? *Ans.* $66\frac{2}{3}$%.

5. What % of \$3606 is \$450$\frac{3}{4}$? *Ans.* $12\frac{1}{2}$%.

6. 25% of $\frac{4}{5}$ of an article is how many % of $\frac{3}{4}$ of it? *Ans.* $13\frac{1}{3}$%

7. The base is 28 35 gal., the percentage 5.3156$\frac{1}{4}$ gal.; what is the rate? *Ans.* $18\frac{3}{4}$%.

8. A cask containing 25 gal. 2 qt. leaked so that there escaped 9 gal. $2\frac{1}{4}$ qt.; what % leaked out? *Ans.* $37\frac{1}{2}$%.

9. Standard gold or silver of the United States is 9 parts pure and 1 part alloy; what % is pure? *Ans.* 90%.

10. A pound of English standard silver contains 18 pwt. of alloy; what % is pure? *Ans.* $92\frac{1}{2}$%.

11. English standard gold is 22 carats fine; what % of alloy is there in a sovereign? *Ans.* $8\frac{1}{3}$%.

12. The dry gallon contains 268.8 cubic inches; how many % larger is it than the wine gallon, or smaller than the old beer gallon? *Ans.* $16\frac{4}{11}$% larger; $4\frac{32}{47}$% smaller.

13. I bought a quantity of valentines, wholesale, at a discount of 50, 50, and 25%; what is the rate of discount? *Ans.* $81\frac{1}{4}$%.

14. A manufacturer sold a quantity of slates at 60, 10, 10, 10, 10, and 5% discount; what was the rate of discount? *Ans.* 75.0682%.

15. What is the difference between a discount of 40% and 10% taken 4 times? between 40% and 20% taken twice? *Ans.* 5.61%; 4%.

16. A person deposited \$6000 in bank, checked out $33\frac{1}{3}$% of it, deposited \$6000 more and checked out 15% of what was then in; what per cent. of his deposit remains in bank? *Ans.* $70\frac{5}{6}$%.

17. Mr. Johnson drew $33\frac{1}{3}$% of his money from the bank, and paid $62\frac{1}{2}$% of it for a horse worth \$125, and then deposited the remainder; what per cent. of his entire deposit was the sum then remaining in bank? *Ans.* $79\frac{1}{6}$%.

GENERAL FORMULAS.

521. Formulas.—These methods and rules may all be presented in general formulas. Let b represent the *base*, r the *rate*, p the *percentage*, A the *amount*, D the *difference*, and we have the following:

CASE I.	CASE II.	CASE III.
1. $b \times r = p$.	1. $p \div r = b$.	1. $p \div b = r$.
2. $b \times (1+r) = A$.	2. $A \div (1+r) = b$.	2. $A \div b = 1+r$.
3. $b \times (1-r) = D$.	3. $D \div (1-r) = b$.	3. $D \div b = 1-r$.

522. The second and third formulas of each case may be united in one; thus, using P for *proceeds*, $P = b \times (1 \pm r)$; $b = P \div (1 \pm r)$; $r = P \div b - 1$, or $r = 1 - P \div b$.

NOTE.—These formulas apply to all the cases in practical applications, and may be used instead of the rules, or with them, as the teacher prefers.

APPLICATIONS OF PERCENTAGE.

523. The **Applications of Percentage** are extensive, owing to the great convenience of reckoning *by the hundred* in business transactions.

524. The **Method of Treating** the cases of the Applications of Percentage is the same as in Percentage itself.

525. These Applications of Percentage are of two classes; those not involving time and those involving time. The following are the most important of these applications:

1ST CLASS.	2D CLASS.
1. Profit and Loss.	1. Simple Interest.
2. Commission.	2. Partial Payments.
3. Stocks, Dividends, etc.	3. True Discount.
4. Premium and Discount.	4. Discounting and Banking.
5. Brokerage.	5. Exchange.
6. Stock Investments.	6. Compound Interest.
7. Taxes.	7. Annuities.
8. Duties or Customs.	8. Insurance.

NOTES.—1. In the different cases of the application of percentage, care should be taken to see clearly the *base* upon which the percentage is reckoned.

2. The subject of percentage has been greatly extended by the fact of our money system reckoning a *hundred cents* to a dollar. Pupils should remember, however, that *per cent*. and *cents* are two distinct things.

PROFIT AND LOSS.

526. Profit and **Loss** are terms which denote the gain or loss in business transactions.

527. The **Quantities** considered are: 1. The *Cost;* 2. The *Rate of Profit* or *Loss;* 3. The *Profit* or *Loss;* 4. The *Proceeds* or *Selling Price.*

NOTES.—1. Profit and Loss are not always estimated upon things bought and sold.

2. In marking goods it is customary to take one or more words or a phrase or sentence, consisting of ten different letters, and let each letter in succession represent one of the Arabic figures. The prices marked thus can only be read by those who have the key.

CASE I.

528. ***Given, the cost and the rate of profit or loss, to find the profit or loss, or the selling price.***

1. A house was bought for $5780, and sold at a gain of 12%; what was the gain?

SOLUTION.—If a house was bought for $5780 and sold at a gain of 12%, the gain was .12 times $5780, which is $693.60.

OPERATION.

$$\begin{array}{r} \$5780 \\ .12 \\ \hline \$693.60 \end{array}$$

Rule I.—*Multiply the cost by the rate, to find the profit or loss.*

Rule II.—*Multiply the cost by 1 plus the rate of profit, or by 1 minus the rate of loss, to find the selling price.*

2. I bought fish at $4.50 a quintal, and sold the same at a gain of 8%; what was my gain? *Ans.* $0.36.

3. A furrier sold a set of furs which cost $87.50, at a gain of 12½%; what did he receive for them? *Ans.* $98.43¾.

4. A train of cars was running 24 miles an hour, when the conductor, to make up lost time, increased the speed 25%; how fast did he then run? *Ans.* 30 miles.

5. The price of a certain lot of drugs is $96; if I buy at 10% off and sell at 25% on, what do I gain? *Ans.* $33.60.

6. My key for marking my goods is "Charleston;" if I buy cassimere @ $3.75, what will be the mark for the selling price if I intend to gain 15%? *Ans.* $r.ac\frac{c}{r}$.

7. Bought 50 yards of paper muslin @ 8¢ and marked it

at a profit of 25%; what will be my profit if I sell at $12\frac{1}{2}$% less than the selling mark? *Ans.* $37\frac{1}{2}$¢.

8. A gentleman bought a yacht for \$3500, sold it at a loss of 20%, and the buyer sold it at a gain of 25%; what did the latter receive for it? *Ans.* \$3500.

9. A drover bought 75 cows at \$$24\frac{1}{4}$ a head; if 9 of them were killed by an accident, how must he sell the remainder to gain 20%, the expenses being \$75? *Ans.* \$$34.43\frac{2}{11}$.

10. A cistern containing 230 barrels of water, receives by one pipe $7\frac{3}{4}$% of its contents in an hour, and loses by another $16\frac{2}{3}$%; how much water is in the cistern at the end of an hour? *Ans.* $209.49\frac{1}{6}$ bar.

11. A merchant marks down some old-fashioned goods $12\frac{1}{2}$%; how should he mark to the nearest half-cent those selling @ $12\frac{1}{2}$¢, $18\frac{3}{4}$¢, $62\frac{1}{2}$¢, 75¢, \$$1.62\frac{1}{2}$, \$$1.87\frac{1}{2}$, $2.37\frac{1}{2}$?
Ans. 11¢, $16\frac{1}{2}$¢, $54\frac{1}{2}$¢, $65\frac{1}{2}$¢, \$1.42, \$1.64, \$2.08.

12. I buy 7 lots of English prints averaging 75 yd. in a lot, marked 10¢, at a discount of 10, $12\frac{1}{2}$, 15, 10 and 5, 20, 25, and 20 and 20%, and sell them all at 7% below marked price; what is my clear profit? *Ans.* \$6.30.

13. A began business with \$25,000; he cleared 25% the first year, and added it to his capital; the 2d year he cleared 25% and added it to his capital; the 3d year he did the same; what was his entire gain? *Ans.* \$$23,828.12\frac{1}{2}$.

CASE II.

529. ***Given, the rate and the profit or loss, or the selling price, to find the cost.***

1. A man sold a house for \$870 above cost, and gained 25%; required the cost.

SOLUTION.—At a gain of 25%, .25 times the *cost* equals the gain, which is \$870; if the *cost* multiplied by .25 equals \$870, the *cost* equals \$870 divided by .25, or \$3480.

OPERATION.
\$870 ÷ .25 = \$3480

Rule I.—*Divide the profit or loss by the rate, to find the cost.*

Rule II.—*Divide the selling price by 1 plus the rate of profit, or by 1 minus the rate of loss, to find the cost.*

2. I sell chintzes @ 25¢ and gain 25%, and also @ 20¢ and lose 20%; what was their cost? *Ans.* 20¢; 25¢.

3. A drover lost $2\frac{1}{2}$% of his cattle by disease, and $3\frac{1}{4}$% by accident, losing altogether 46; how many were in the drove at first? *Ans.* 800.

4. I sold my horse at a gain of $16\frac{2}{3}$%, and with the proceeds bought another which I sold for $180.32, at a loss of 8%; what did each horse cost? *Ans.* 1st, $168; 2d, $196.

5. Two newsboys invested, during the year, equal sums of money in papers; the one gained $6\frac{2}{5}$% and the other $8\frac{1}{3}$%; what amount did each invest, if the gain of the second was $14.50 more than that of the first? *Ans.* $750.

6. A merchant bought some water-proof cloth @ $1, and marked it so that he could fall 5% on his asking price, and gain 25% on cost; how did he mark it? *Ans.* $1.32.

7. A merchant bought a lot of alpaca @ 30¢; what must the goods be marked that he may throw off 25% from the marking price, and still make 25% profit? *Ans.* 50¢.

8. Brown sold Jones some goods for $585 and lost $2\frac{1}{2}$%, and Jones sold them to Robinson and made $2\frac{1}{2}$%; did Robinson pay more or less than Brown? *Ans.* 0.37\frac{1}{2}$ less.

9. I purchased a lot of ingrain carpets from the manufacturer, and marked them $1.25 retail, which is $11\frac{1}{9}$% above the rate at which I actually sold them; if I gained $28\frac{4}{7}$%, what was the cost? *Ans.* $87\frac{1}{2}$¢.

10. My gain this year was $1413, which was $78\frac{1}{2}$% of my gain last year, and that was $112\frac{1}{2}$% of my gain the year before; required my gain last year and the year before.

Ans. $1800 last; $1600 before.

11. Mr. Brenner offered his house for sale at an advance of 20%, but afterwards sold it for $5250, which was $12\frac{1}{2}$% less than his original offer; what was the first cost of the house? *Ans.* $5000.

12. A man's capital increased 25% for each of 4 years, on what he had at the beginning of each year, and at the end of the time he was worth $12207.03; what was his capital? *Ans.* $5000.

13. A sold two city lots, which cost the same price, to B, at a loss of 15%; B sold them to C, gaining 20% on one, and losing 25% on the other; what did each cost A, if B received $765 more for one than the other? *Ans.* $2000.

14. Mr. Brown bought a horse and carriage, the horse costing twice as much as the carriage, but afterwards sold the carriage for $33\frac{1}{3}$% more, and the horse for 10% less than he gave, and received for both $705; what was the cost of each?
Ans. Carriage, $225; horse, $450.

15. Mr. Oakley, when stocking his farm, spent equal sums in cows, sheep, and hogs; on the sale of which he made 2% on the cows, lost 9% on the sheep, and gained $12\frac{1}{2}$% on the hogs; if he received for all $1833, and bought 120 sheep, 15 cows, and 150 hogs, what did he pay per head?
Ans. Sheep, $5; cows, $40; hogs, $4.

16. A man invested a certain sum in several different stocks; on $\frac{5}{9}$ of his investment he gained 25%, and on the remainder lost 15%; his whole profit was $260; had he gained 15% on $\frac{5}{9}$ and lost 25% on the remainder, would he have gained or lost, and how much? *Ans.* Lost $100.

CASE III.

530. ***Given, the cost and the profit or loss, or the selling price, to find the rate.***

1. A man bought a boat for $560, and sold it at a gain of $140; what was the gain per cent.?

SOLUTION.—Since $560, the base, multiplied by the rate, equals $140, the rate must equal $140 divided by $560, which we find to be .25, or 25%.

OPERATION.
$140÷$560=.25

Rule I.—*Divide the profit or loss by the cost, to find the rate.*

Rule II.—*Divide the difference between the cost and the selling price by the cost, to find the rate.*

2. Calico was bought at $18\frac{3}{4}$ cents a yard, and sold for 25 cents a yard; what was the gain per cent.? *Ans.* $33\frac{1}{3}$%.

3. If I buy at 10% off and sell at 15% on, what % profit do I make? what if I buy at 20% off and sell at 20% on?
Ans. $27\frac{7}{9}$%; 50%.

4. I marked calicoes 15¢ and sold them at $12\frac{1}{2}$¢; what do I lose per cent., if I marked them 20% above cost?

5. A street car fare in Philadelphia was 7 cents, but 4 tickets were sold for a "quarter;" what per cent. did I save by buying a package? *Ans.* $10\frac{5}{7}$%.

6. If $\frac{5}{6}$ of the cost equals the selling price, what is the loss per cent.; and if $\frac{5}{6}$ of the selling price equals the cost, what is the gain per cent.? *Ans.* Loss, $16\frac{2}{3}$%; gain, 20%.

7. A grocer bought melons for 25% more than 12 cents each, and sold them for 25% less than 12 cents each; what was the loss per cent.? *Ans.* 40%.

8. A farm was offered for sale at 25% advance on its cost, but finding no purchasers at that price, it was finally sold at 25% less than was first asked; what was the gain or loss per cent.? *Ans.* Loss, $6\frac{1}{4}$%.

9. I marked plaid Nainsooks $62\frac{1}{2}$¢, and gained at that rate 25%; if I sell them for 6 cents more than the marked price, what is my gain per cent.? *Ans.* 37%.

10. Mr. Walker was offered $3500 for his house, but declined to sell, as he would thereby lose $12\frac{1}{2}$%; a few months later he sold for $4800; did he gain or lose, and what per cent.? *Ans.* Gained 20%.

11. Mr. Fawcett bought a quantity of broadcloth @ $5.25, and marked it *r.bw*, his key being "be smart now;" what was the gain per cent. at the marked price? *Ans.* $16\frac{4}{21}$%.

12. Mr. Howard bought 25 building lots for $11250 and sold 20 of them for what he paid for all of them; what was his gain per cent. on the investment? *Ans.* 25%.

13. Miss Martin sold her diamond breastpin for $560, and thereby cleared $12\frac{1}{2}$% of this money; what would she have gained per cent. by selling it for $600? *Ans.* $22\frac{2}{49}$%.

14. Mr. Bechtel sacrificed $37\frac{1}{2}$% on his library by selling it for $1500; if he had sold it for 20% more than he did, what would he have lost per cent.? *Ans.* 25%.

15. If a druggist gives me a lb. Troy of washing soda instead of a lb. Avoirdupois, what is my loss and his gain per cent.? *Ans.* Loss, $17\frac{5}{7}$%; gain, $21\frac{19}{36}$%.

16. I ride on two street railway lines, making exchanges, and get an exchange ticket for 9 cents; what % more would it cost to use a ticket on each car at 4 for 25 cts., and what % more to pay single fares at 7 cts.? *Ans.* $38\frac{8}{9}\%$; $55\frac{5}{9}\%$.

17. A news agent bought 50 copies *N. Y. Tribune*, and 60 copies *N. Y. Herald* @ $3\frac{1}{4}$¢, sold @ 4¢; 50 *Harpers' Weekly* at $7\frac{1}{2}$¢, sold @ 10¢, 12 *Independent* @ $6\frac{1}{2}$¢, sold @ 10¢; 12 *Lippincott's Magazine* @ 27¢, sold @ 35¢; 8 *Harpers' Magazine* @ 28¢, sold @ 35¢; what was the average per cent. of profit? *Ans.* $29\frac{1}{2}\frac{37}{47}\%$.

COMMISSION.

531. **Commission** is a percentage paid to an agent for the transaction of business.

532. An **Agent** is a person who transacts business for another, and is often called a *Commission Merchant;* a *Factor*, etc.

533. The **Base** in commission is the actual *amount* of the *sale, purchase, collection*, or *exchange*.

534. The **Net Proceeds** is the sum left after the commission and charges have been deducted from the amount of a sale or collection.

535. The **Entire Cost** is the sum obtained by adding the commission and charges to the amount of a purchase.

536. The **Quantities** considered are: 1. The *Amount sold, bought,* etc.; 2. The *Rate of Commission;* 3. The *Commission;* 4. The *Entire Cost* or *Net Proceeds.*

The goods forwarded to be sold on commission are called a *consignment;* the person sending them is called the *consignor;* and the person to whom they are sent, the *consignee*, or *factor*. An agent residing at a great distance from his employer, is often called a *correspondent;* the person for whom an agent does business is called the *principal.*

CASE I.

537. ***Given, the base and the rate, to find the commission, or the net proceeds, or the entire cost.***

1. An agent sold goods to the amount of \$2560; required his commission at $2\frac{1}{4}\%$.

SOLUTION.—The commission is $.02\frac{1}{4}$ times \$2560, which equals \$57.60.

OPERATION.
$\$2560 \times .02\frac{1}{4} = \$57.60$

Rule I.—*Multiply the base by the rate, to find the commission.*

Rule II.—*Multiply the base by* 1 *minus the rate, to find the net proceeds; or by* 1 *plus the rate, to find the entire cost.*

2. I sold on commission 540 barrels of flour, @ \$8.25, and 75 barrels of cider, 35 gallons each, @ $37\frac{1}{2}$ cents; what is my commission at $2\frac{1}{2}\%$? *Ans.* \$135.98.

3. A lawyer, having a debt of \$6500 to collect, compromises for $97\frac{1}{2}\%$; his commission is $2\frac{1}{4}\%$; how much does he remit to his employer? *Ans.* \$6194.91.

4. A speculator sells \$7452.50 worth of dry goods through a factor who charges $2\frac{1}{2}\%$ commission and $2\frac{1}{2}\%$ for insuring payment; what does the factor remit? *Ans.* $\$7079.87\frac{1}{2}$.

5. An agent sold a consignment of flour for \$5657.25, and afterward bought 2500 bushels of grain at 87¢ a bushel; his commission is $2\frac{1}{4}\%$; what sum did he remit to his employer? *Ans.* \$3306.02.

6. I sell on commission 1500 bushels of clover seed @ 75¢, through an agent to whom I pay $1\frac{1}{4}\%$; how much do I clear, my commission being $2\frac{1}{2}\%$? *Ans.* $\$14.06\frac{1}{4}$.

7. Frantz, Herr & Co. bought 600 tons of railroad iron at \$30 a ton, through their agent in Pittsburgh, and shipped it to Omaha, where it was sold by an agent at \$38 a ton; how much did they clear, transportation costing \$250, and commission being $2\frac{1}{4}\%$? *Ans.* \$3632.

8. An agent bought 580 yards of woolen goods at an average price of 75¢ a yard; paid \$50 storage and \$25 transportation, and sold it for \$1 a yard; how much should he remit to his principal, commission $2\frac{1}{4}\%$? *Ans.* \$47.16.

CASE II.

538. ***Given, the rate and the commission or the net proceeds or the entire cost, to find the base.***

1. An agent's commission in one year for collecting money at $3\frac{1}{4}\%$ is \$2600; how much did he collect?

Solution.—At a commission of $3\frac{1}{4}\%$, $.03\frac{1}{4}$ times the cost of the goods equals the commission, which is \$2600; hence the *cost* equals *\$2600 divided by* $.03\frac{1}{4}$, which we find is \$80,000.

OPERATION.

$$\frac{\$2600}{.03\frac{1}{4}} = \$80,000$$

Rule I.—*Divide the commission by the rate, to find the base.*

Rule II.—*Divide the net proceeds by* 1 *minus the rate, or the entire cost by* 1 *plus the rate, to find the base.*

2. An agent bought 2400 bu. of wheat, his commission at $2\frac{3}{4}\%$ was \$68.75, and charges for storage, freight, etc., \$191.25; what was the actual cost a bushel? *Ans.* \$1.15.

3. A lawyer collects a debt for a client, takes $3\frac{1}{4}\%$ for his fee, and remits the balance, \$19,350; what was the debt and fee? *Ans.* Debt, \$20,000; fee, \$650.

4. An agent buys goods on commission at $1\frac{1}{8}\%$, and pays \$35 for freight; the whole amount was \$4080; what was the amount expended? *Ans.* \$4000.

5. Sold 200 barrels of pork, commission $2\frac{1}{2}\%$, guaranty $1\frac{5}{8}\%$, net proceeds due consignor, \$2876.25; what did I receive per barrel for the pork? *Ans.* \$15.

6. My agent bought 40 horses, and paid \$105 for their transportation, which, with his commission, $3\frac{1}{2}\%$, amounted to \$6315; what did the horses cost apiece? *Ans.* \$150.

7. I sent orders to my agent to buy a certain quantity of woolen goods, allowing him $2\frac{1}{4}\%$ commission and paying $1\frac{1}{4}\%$ for delaying payment a month; his bill was \$2898; how many yards did he buy at \$1.12 a yard? *Ans.* 2500 yd.

8. I sold a consignment of cotton, commission 4%; invested the net proceeds in flour, commission 2%; my whole commission was \$180; required the value of the cotton and flour. *Ans.* Cotton, \$3060; flour, \$2880.

9. I sold a consignment of grain and invested the proceeds in sugar, deducting my commissions, $4\frac{1}{2}\%$ for selling and $2\frac{1}{2}\%$ for buying; the sugar cost \$7640; what did the grain sell for, and what were my commissions?

Ans. Grain, \$8200; 1st com. \$369; 2d com. \$191.

10. I sold a consignment of goods through a factor who charged me $1\frac{1}{4}\%$; I was allowed $2\frac{1}{4}\%$ commission and $3\frac{1}{4}\%$ for insuring payment, and I cleared \$68; what was my commission and the sum remitted to the consignors?

Ans. Sum remitted, \$1512; com., \$88.

11. I received \$4850 and a consignment of 2000 barrels of flour from Badeau & Co., which I sold at \$7.50 a barrel, and invested the net proceeds and cash in cotton; how much did I invest in cotton, my commission being 3% for selling and 1½% for buying, and the expenses for storage and freight \$350? *Ans.* \$18768.47.

CASE III.

539. ***Given, the base and the commission or the net proceeds or the entire cost, to find the rate.***

1. A commission merchant collected \$8650, and his commission was \$302.75; what was the rate of commission?

SOLUTION.—The commission, \$302.75, equals the base, \$8650, multiplied by the *rate;* hence, the *rate* equals \$302.75 divided by \$8650, which we find is .03½, or 3½%.

OPERATION.

$$\frac{302.75}{8650} = .03\tfrac{1}{2}$$

Rule I.—*Divide the commission by the base, to find the rate.*

Rule II.—*Divide the difference between the base and the net proceeds or the entire cost, by the base, to find the rate.*

2. A factor in Iowa sold some land for me, and after retaining his commission, \$90, remitted me \$1910; what rate of commission did he charge? *Ans.* 4½%.

3. A commission merchant sells 5980 pounds of tea, at \$1.12½ a pound, and remits the net proceeds, \$6559.31¼; what is his rate of commission? *Ans.* 2½%.

4. I sold a consignment of goods through an agent for \$5745; my commission was \$201.07½, and I paid the agent \$86.17½; what was the rate of commission of each?

Ans. Mine, 3½%; agent's, 1½%.

5. My agent bought 40 horses at \$150 each, paid \$25 for their keeping, and \$80 for transportation; he drew on me for \$6315; what was his commission? *Ans.* 3½%.

6. A commission merchant in Philadelphia having received a consignment from Chicago of 1850 bushels of wheat, sells it at \$1.25 a bushel; having deducted \$35 for freight, and \$25 for storage, and his commission, he remits to Chicago \$2171.56¼; what was his rate of commission?

Ans. 3½%.

STOCKS AND DIVIDENDS.

540. A **Company** is an association of individuals for the transaction of business. It may or may not be *incorporated.*

541. A **Corporation** is a company regulated in its operation by a general law or a special charter.

542. The **Stock** of a company is the capital invested in the business. The owners of stock are called *Stockholders.*

543. A **Share** is one of the equal parts into which the stock is divided. A share is usually \$50 or \$100.

544. **Scrip** or **Certificates of Stock** are the papers issued by a corporation to its stockholders, as evidence of the number of shares belonging to each respectively.

545. **Stocks** is a general name applied to the scrip or bonds of a corporation, and to government bonds and public securities.

546. **Bonds** are written or printed obligations to pay certain sums of money at or before a specified time.

547. **State Stocks** or **United States Stocks** are bonds of a State, or of the United States, payable at some future time, with interest at a fixed rate.

548. An **Installment** is a sum required of stockholders as a payment on their subscription.

549. A **Dividend** is a sum paid to stockholders out of the gains of the company.

550. An **Assessment** is a sum required of stockholders to meet the expenditures or losses of the company.

551. The **Base** upon which dividends and assessments are estimated is the original or par value of the stock.

552. The **Quantities** considered are as follows: 1. The *Stock;* 2. The *Rate;* 3. The *Dividend* or *Assessment.*

The capital stock of any corporation is limited by the charter. In general, only a certain percentage is paid at the time of subscription, the remainder being reserved for future expenses.

When the capital is all paid up, if more money is needed, it may be obtained by *loans,* secured by mortgage upon property of the company.

The *bonds* issued for these loans entitle the holder to a certain fixed interest, while the stockholders participate in the profits of the company in proportion to the stock they hold. The increase of the stock of a company by the issue of new shares is termed "watering" the stock.

CASE I.

553. ***Given, the stock and the rate of dividend or assessment, to find the dividend or assessment.***

1. A man owns 65 shares of bank stock at $50 a share; the bank declares a dividend of 7%; what does he receive?

SOLUTION.—If one share of stock is worth $50, 65 shares are worth 65 times $50, or $3250, and 7% of $3250 is .07 times $3250, or $227.50.

OPERATION.

$65 \times 50 = 3250$
$\$3250 \times .07 = \$227.50$

Rule.—*Multiply the par value of the stock by the rate, to find the dividend or assessment.*

NOTE.—It is often convenient to find the result by multiplying the dividend or assessment on one share by the number of shares.

2. A man owns 98 shares of Pennsylvania R. R. stock at $50 each; if the company declares a semi-annual dividend of 5%, what is his dividend? *Ans.* $245.

3. A telegraph Co., whose stock consists of 5780 shares at $25 each, declares a quarterly dividend of $2\frac{1}{2}$%; what sum was divided among the stockholders? *Ans.* $3612.50.

4. The Lancaster Gas Co., with a capital of $65,000, declares a dividend of 8%, and has a surplus of $800; how much has it earned to do this? *Ans.* $6000.

5. The net earnings of the Manor Turnpike Company are $9000, and the stock $157,000; the company declared a 5% dividend; what was the surplus? *Ans.* $1150.

6. A owns 72 shares, at $100 each, in a mutual insurance company, which on account of losses, requires an assessment of $3\frac{1}{3}$%; how much does A pay? *Ans.* $240.

7. B has 135 shares of Farmers' Bank stock ($50); the bank declares a dividend of 8%; how many shares of stock could he buy with his dividend? *Ans.* 10; $40 surplus.

8. Mr. Wilson owns 80 shares of N. Y. Central R. R. stock ($100); if the company declares a dividend of 5%, payable in stock, how many shares will he then own? *Ans.* 84.

9. *If a company* "water" its stock by issuing 20% of

new stock, and their original capital was \$5,000,000, what would be the amount of a dividend of 8% before the watering, and what afterwards? *Ans.* \$400,000; \$480,000.

10. The capital stock of a Western railroad is \$1,750,000, and its debt is \$675,000; its gross earnings for the year 1873 were \$565,000, and expenses \$384,500; after deducting interest on the debt at 6%, what dividend would a stockholder receive on 20 shares at \$100 each? *Ans.* \$160.

CASE II.

554. ***Given, the rate and the dividend or assessment, or the result of increase or decrease of stock, to find the stock.***

1. A bank divides \$4875 among its stockholders, being a 5% dividend; required the whole amount of stock.

SOLUTION.—If \$4875 is 5% of the stock, then .05 times the stock equals \$4875, the dividend, hence the stock equals \$4875, the dividend, divided by .05, the rate, which is \$97500.

OPERATION.

$$\frac{4875}{.05} = 97500.$$

Rule I.—*Divide the dividend or assessment by the rate, to find the stock.*

Rule II.—*Divide the result of increase by 1 plus the rate, or the result of decrease by 1 minus the rate, to find the stock.*

2. A company divides \$15,000 among its stockholders, as the result of a $7\frac{1}{2}$% dividend; what is A's stock, provided he owns $\frac{1}{10}$ of the entire stock? *Ans.* \$20,000.

3. A man pays an assessment of \$275, at $5\frac{1}{2}$%, on his insurance stock; how many shares does he own, the shares being valued at \$50 each? *Ans.* 100.

4. A gas company whose annual expenses are \$2980, and gross earnings \$5780, pays a dividend of 5% semi-annually; required the value of the stock. *Ans.* \$28,000.

5. Mr. B received \$240 payable in stock, as his share of an 8% dividend; how many shares had he at first, and how many has he now, shares at \$50? *Ans.* 60; 64, and \$40.

6. Henry received a 6% stock dividend, and then had 93 shares, \$50 each, and \$14 of another share; how many shares had he at first? Ans. 88.

7. Mr. D received 12 shares and $7.50 in money as his share of a $7\frac{1}{2}\%$ dividend; how many shares, at $50 each, did he then own? *Ans.* 174 shares.

8. I received two dividends in the stock of the Pittsburgh Gas Company, one at 8%, another at 10%, and I then had 297 shares ($100); how many shares had I at first? *Ans.* 250.

9. I received a stock dividend of 5% in a Nevada mining company, in April, and a similar dividend of 10% in December; I then owned 231 shares at $50; how many shares had I at first? *Ans.* 200.

CASE III.

555. ***Given, the stock and the dividend or assessment, or the result of increase or decrease of stock, to find the rate.***

1. My dividend on $6500 worth of bank stock was $975; what was the rate of dividend?

SOLUTION.—Since the dividend is some per cent. of the stock, the base, $6500, multiplied by the *rate*, equals $975; hence the *rate* equals the dividend, $975, divided by the stock, $6500, which equals .15, or 15%.

OPERATION.
$975 \div 6500 = .15$

Rule I.—*Divide the dividend or assessment by the stock, to find the rate.*

Rule II.—*Divide the difference between the stock and the result of increase or decrease, by the stock, to find the rate.*

2. A owns 85 shares of railroad stock, at $100 a share, and receives a dividend of $680; what was the rate of dividend? *Ans.* 8%.

3. An oil company, whose stock is $100,000, clears $7850; required the largest integral rate of dividend that they can declare, and the surplus. *Ans.* 7%; surplus, $850.

4. After receiving a stock dividend, I had 93 shares ($100) and $28 toward another share; what was the rate of the dividend, if I had 88 shares at first? *Ans.* 6%.

5. I have 250 shares in a Chicago Gas Company ($100); I received two stock dividends, the first amounting to $2000 and the second to $2700; what were the rates of the dividends? *Ans.* 8% and 10%.

6. The capital stock of a railroad is \$895,750; the passenger earnings in 1 yr. were \$74,537.50, and the freight earnings \$94,567.50; the disbursements were \$107,963; what rate of dividend can be declared? *Ans.* 6%; \$7397 surplus.

7. A company whose capital was \$6,000,000, "watered" it by the issue of 25% of new stock; if it divided \$450,000 among the stockholders after the watering, what will be the rate per cent. of the dividend? *Ans.* 6%.

8. The charter of a new railroad company fixes the capital stock at \$1,000,000, of which three installments of 25%, 35%, and 30%, respectively, have already been called in; the amount already expended is \$750,000, and it is estimated that \$450,000 more will be required to finish the road. After the last installment is called in, what must be the rate per cent. of the assessment on the stockholders to make up the deficiency? *Ans.* 20%.

PAR, PREMIUM, AND DISCOUNT.

556. **Capital** is property consisting of *Money, Bonds, Stocks, Drafts,* etc.

557. **Drafts, Checks,** and **Bills of Exchange** are written orders for the payment of money at some definite place.

558. The **Par Value** of capital is the value marked on its face, called the *nominal value* or *face.*

559. The **Real Value** or **Market Value** of capital is what it will sell for.

560. Capital is **At Par** when it sells for its nominal value or its face.

561. Capital is **Above Par,** or at a *Premium,* or at an *Advance,* when it sells for more than its nominal value.

562. Capital is **Below Par,** or at a *Discount,* when it sells for less than its nominal value.

Stocks are often named from the rate of interest they draw; thus, we have 5's, 6's, 7-30's, etc. The time to run or date when due sometimes gives the name: as, 4's of '97.

The Stock of a company will generally be above par when the company is doing a lucrative business, and below par when it is doing a poor

business. The stock of a town, city, etc., varies according to the confidence in its security, the fluctuations of the money-markets, etc.

If the paper currency of a country becomes depreciated in value, *gold* becomes an object of investment, the same as stocks. The value of gold being fixed, its fluctuations in price indicate the changes in the value of the currency. Thus, when gold is said to be at a *premium*, currency is really at a *discount*.

563. The **Base** upon which premium and discount are estimated is the *par value*.

564. The **Quantities** considered are four: 1. The *Par Value;* 2. The *Rate;* 3. The *Premium or Discount;* 4. The *Real Value.*

NOTE.—The problems under this subject are solved without brokerage—the sales and exchanges being regarded as direct without the aid of a broker.

CASE I.

565. ***Given, the par value and the rate of premium or discount, to find the premium or discount, or the real value.***

1. A bought 45 shares of stock ($50) at 7% premium; required the premium and cost, or real value.

SOLUTION.—The par value of 45 shares at $50 each is $50×45= $2250; and the premium at 7% is .07 times $2250, or $157.50, and this added to the par value, equals $2407.50, the real value.

OPERATION.

$$
\begin{array}{rl}
\$50\times45=\$2250 & = \text{par value.} \\
.07 & \\
\hline
157.50 & = \text{premium.} \\
2250 & \\
\hline
\$2407.50 & = \text{real value.}
\end{array}
$$

Rule I.—*Multiply the par value by the rate, to find the premium or discount.*

Rule II.—*Multiply the par value by* 1 *plus the rate of premium, or by* 1 *minus the rate of discount, to find the real value.*

2. A broker bought 27 shares of Penn. R. R. stock ($50) at 45, and sold it at 55; how much did he gain? *Ans.* $270.

3. A banker bought $960 in gold at a premium of $12\frac{1}{2}$%, and sold it at a premium of $17\frac{1}{4}$%; how much did he make by the operation? *Ans.* $45.60.

4. Bought 172 shares of Reading R. R. stock ($50) for $46\frac{5}{8}$, and gave in payment a draft on St. Louis for $7500 at $\frac{1}{8}$% discount, and the balance in cash; how much cash did I pay? *Ans.* 585.12\frac{1}{2}$.

5. A exchanged 82 shares of bank stock ($50) at $3\frac{1}{2}\%$ premium for 106 shares of Camden and Atlantic R. R. stock, at $47\frac{1}{2}$, paying the balance in cash; how much cash did he pay? *Ans.* $791.50.

6. If I buy $6800 U. S. 6's at 108, and $7200 Cin. 7.30's, at 102; what is the market value of each? *Ans.* $7344.

CASE II.

566. ***Given, the rate and the premium or discount or the real value, to find the par value.***

1. A sold some drafts at $3\frac{1}{2}\%$ premium, and gained $210 on the par value; what was the par value?

SOLUTION.—If the premium at $3\frac{1}{2}\%$ is $210, then $.03\frac{1}{2}$ times the *par value* equals $210; hence the *par value* equals $210 divided by $.03\frac{1}{2}$, which we find is $6000.

OPERATION.
$\$210 \div .03\frac{1}{2} = \$6000$

Rule I.—*Divide the premium or discount by the rate, to find the par value.*

Rule II.—*Divide the real value by* 1 *plus the rate of premium, or by* 1 *minus the rate of discount, to find the par value.*

2. When State 6's are 96%, what is the par value of the amount that can be bought for $4992? *Ans.* $5200.

3. A speculator sold 68 shares of bank stock at a premium of $4\frac{1}{2}\%$, and received $3553; what was the par value of a share? *Ans.* $50.

4. A broker receives $51,000 to invest in Illinois 6's standing at 85; how many $1000 bonds can he buy? *Ans.* 60.

5. Required the face of a draft at $\frac{1}{2}\%$ discount, which will buy 75 shares of United Companies of New Jersey R. R. stock ($100), selling at $123\frac{7}{8}$. *Ans.* $9337.31.

6. How many shares of canal stock ($50) at 95% must I sell in order that the proceeds, + $9.50, invested in Cin. 7-30's at 108%, may have a par value of $1900? *Ans.* 43.

7. During a commercial panic, a merchant, wishing to raise $20,000, was obliged to sell certain stocks; the market value being $77\frac{1}{2}$, how many shares must he sell, the par being $100? *Ans.* 258 shares, and add $5 besides.

8. I exchanged $56,000 in drafts at $1\frac{1}{4}$% discount, and $2500 Mo. 6's at 110, for New Jersey Central (100) at 95; how many shares did I buy? *Ans.* 611, with $5 rem.

CASE III.

567. ***Given, the par value and the real value or the premium or discount, to find the rate of premium or discount.***

1. A sold some stock whose face value was $6000, at a discount of $420; what was the rate of discount?

SOLUTION.—Since the premium equals the par value multiplied by the rate, $6000 multiplied by the *rate* equals $420; hence the *rate* equals $420 divided by $6000, which we find is .07.

OPERATION.

$420 ÷ $6000 = .07

Rule I.—*Divide the premium or discount by the par value, to find the rate.*

Rule II.—*Divide the difference between the real value and the par value by the par value, to find the rate.*

2. A man bought 106 shares of railroad stock ($50) for $4902.50; what was the rate of discount? *Ans.* $7\frac{1}{2}$%.

3. If the above mentioned 106 shares were sold for $5591.50, what is the rate of premium and rate of gain?

Ans. Premium, $5\frac{1}{2}$%; gain, $14\frac{2}{37}$%.

4. Bought $2000 City 6's at $102\frac{1}{2}$, and sold them at a gain of $65; at what rate were they sold? *Ans.* $105\frac{3}{4}$.

5. In 1872 I gave $250 in currency for $200 in gold; what was the premium on gold and the discount on currency? *Ans.* 25% premium; 20% discount.

6. Bought $1000 of stock, premium $12\frac{1}{2}$%, and sold it for $1200; what was the premium on the sale, and what the gain per cent.? *Ans.* 20% premium; $6\frac{2}{3}$% gain.

7. B bought 84 shares of canal stock ($50) at 20% premium, and gave in payment a draft on New York for $5000; what was the rate of premium of the draft? *Ans.* $\frac{4}{5}$%.

8. Sold 82 shares of North Pennsylvania R. R. stock for $4438.25, and bought with the proceeds, +$3, 95 shares Lehigh Navigation; what were the premium and discount of the two stocks? *Ans.* $8\frac{1}{4}$% premium; $6\frac{1}{2}$% discount.

BROKERAGE.

568. Brokerage is a percentage charged by brokers for the transaction of business.

569. A **Broker** is a person who buys or sells money, stocks, bills of exchange, real estate, etc., for others.

570. A **Stock Broker** is one who deals in stocks; he is generally called simply a *Broker*. The operations of a stock broker are called *Stock-Jobbing*.

For convenience in carrying on their business, the brokers of large cities associate themselves in a "Board," which meets at the "Stock Exchange." At the meeting, the stocks on the list are called in a certain order, and those wishing to buy or sell, bid for and offer the stock as it is reached. If the sale is for *cash*, the certificate for the stock sold is delivered that day; if it is *regular*, the certificate is delivered and the money paid the next day.

An operator "sells short" when he sells stock that he does not hold, borrowing it for delivery, and hoping to buy at a lower rate. If stocks should fall before he is obliged to replace what he has borrowed, he makes a profit; but if they should rise, he will have to "cover" at an advance, and lose the difference. An operator is "long" when he buys stock and holds it in expectation of a rise. Those operators who try to depress prices are called *bears;* those who try to raise prices are called *bulls.*

Contracts are sometimes made, in consideration of a certain sum, to take or deliver, within a specified time, so many shares of a certain stock at a certain price. These contracts are known as "calls" and "puts."

571. Stock Quotations are reports of sales of stocks. Stocks are quoted either at the price of one share, or at the price of $100 of par value of the stock, whatever be the par value of a share. The former method is used in Philadelphia; the latter in New York.

572. Various **Abbreviations** are used in stock quotations; among the principal ones are the following:

Coup. and *Reg.* are abbreviations used for *coupon* and *registered; ex coup.—without coupon*—is used when a coupon for interest just due has been cut off before the sale of the bond; *ex int.—without interest*—when interest just due is not to be paid to the purchaser. Thus, "20,000 U. S. 4's coup. 1907, 128¼" signifies that 4% coupon bonds, due in 1907, whose par value was $20,000, sold at $128¼ for $100.

1st m.—first mortgage; conv.—convertible—means that the bonds may be exchanged for the stock of the company. Thus, "6000 Erie 1st m. 100 conv.," signifies that bonds of the Erie Railroad, secured by a first mortgage, whose par value was $6000, and which may be exchanged for stock of the company, were selling at par.

Pref.—Preferred stock—is stock issued by some companies which has advantages over the common stock in regard to dividends; *ex. div., without dividend*—means that the stock was sold without entitling the purchaser to a dividend about to be paid. Thus, "100 Camden and Atlantic Pref. ex. div. $52\frac{1}{4}$," means that 100 shares of the preferred stock of the Camden and Atlantic Railroad, without dividend, were sold at $\$52\frac{1}{4}$ for a share whose par value was \$50, or at $4\frac{1}{2}\%$ premium.

*b*3, *b*20, *b*60, etc., mean that the *buyer* must take the stock at the end of 3, 20, 60, etc., days, or sooner at his option. If the stock should rise he would "call" it before the expiration of the time, but should it decline he would claim the whole time. He pays interest at 6% for the time he holds the contract, but is entitled to any dividend or interest falling due within that time. *s*3, *s*20, *s*60, etc., indicate that the *seller* must deliver or "put" the stock at the end of 3, 20, 60, etc., days, or sooner, at his option. If the stock should fall he would probably "put" it on the buyer before the expiration of the time, but otherwise would probably hold it as long as possible. The difference in these two transactions is, that the "option" rests in one case with the buyer, and in the other with the seller, the contract otherwise being the same. *bc.—between calls*, —signifies that the stock was sold between the times at which it was called at the board; *c.—cash*—that cash was paid for the stock.

573. The **Base** upon which the commission for the purchase and sale of bonds and stocks is estimated, is their par value.

574. The **Rate** is usually $\frac{1}{4}\%$, and will be so understood when no other rate is mentioned. In New York the rate is $\frac{1}{8}\%$ for both bonds and stocks.

575. The **Quantities** considered are: 1. The *Par Value of the amount sold, bought*, etc.; 2. The *Rate of Brokerage;* 3. The *Brokerage;* 4. The *Market Value of* \$100, *or of* 1 *share;* 5. The *Entire Cost, or Net Proceeds.*

CASE I.

576. ***Given, the par value, the rate, and the market value, to find the brokerage, the net proceeds, or the entire cost.***

1. A broker bought for a party 19 shares of Minehill R. R. (\$50), rate of brokerage being $\frac{1}{4}\%$; required the brokerage.

SOLUTION.—The par value was $19 \times \$50$, or \$950. The brokerage was $.00\frac{1}{4}$ times \$950, which equals $\$2.37\frac{1}{2}$.

OPERATION.

$19 \times \$50 = \$950$

$\$950 \times .00\frac{1}{4} = \$2.37\frac{1}{2}$

Rule I.—*Multiply the par value by the rate, to find the brokerage.*

Rule II.—*Multiply the par value by the market value minus the rate, to find the net proceeds; or by the market value plus the rate, to find the entire cost.*

NOTE.—It is often shorter to multiply the brokerage on one share by the number of shares. When the par is $50, one-half the rate should be used.

2. A broker bought for me 75 shares of Penn. R. R. stock ($50); required the brokerage at $\frac{1}{4}\%$. *Ans.* $9.37½.

3. I sold, through a broker, 86 shares of Lehigh Navigation stock ($50) at 43½; what was the brokerage and what did I receive? *Ans.* Brok., $10.75; proceeds, $3730.25.

4. My broker bought 65 shares of Bank of Commerce of New York at 112, and sold them at 120; required the broker's commission and my profit. *Ans.* $32.50; $487.50.

5. Shall I gain or lose if I buy 25 shares of stock ($100) at $12\frac{1}{2}\%$ advance, and after receiving a 6% dividend, sell them for 5% less than they cost me, brokerage in both cases being $\frac{1}{4}\%$, interest not considered? *Ans.* $3.125 loss.

6. Sold through a broker 150 shares Second and Third Street Passenger Railway stock ($50) at 60¼, and bought 75 shares of United Companies of New Jersey ($100) at 121; shall I gain or lose by the transaction, the brokerage in both cases being $\frac{1}{4}\%$? *Ans.* Lose $75.

7. Bought 45 shares Norristown R. R. stock ($50), and having received a 3% quarterly dividend, I sold them at 88 and bought $4000 of Missouri 6's at 95, brokerage in each case being $\frac{1}{4}\%$; did I gain or lose by the transaction, interest on money not regarded? *Ans.* Gained $211.87½.

CASE II.

577. ***Given, the rate, the brokerage, or the net proceeds or entire cost, and the market value, to find the par value.***

1. A broker received $75 at $\frac{1}{4}\%$ for selling Northern Pacific bonds for me; what was their par value?

SOLUTION.—At a rate of $\frac{1}{4}\%$, .00¼ times the par value of the bonds equals the brokerage, which is $75; hence the *par value* equals $75 divided by .00¼, which we find is $30,000.

OPERATION.

$$\frac{\$75}{.00\frac{1}{4}} = \$30,000$$

Rule I.—*Divide the brokerage by the rate, to find the par value.*

Rule II.—*Divide the net proceeds by the market value, minus the rate, or the entire cost by the market value plus the rate, to find the par value.*

2. I paid a broker $75 for selling Tennessee 6's at $\frac{3}{4}$% brokerage; what was their par value? *Ans.* $10,000.

3. A broker received $4062.50 to invest in Union Pacific 7's at 81, deducting his commission of $\frac{1}{4}$%; what was the par value of the bonds? *Ans.* $5000.

4. I sent a Philadelphia broker a draft on Drexel & Co. for $2933, directing him to invest the money in Pennsylvania R. R. stock ($50), and deduct his commission of $\frac{1}{4}$%; how many shares shall I receive, the stock being worth 52$\frac{1}{4}$? *Ans.* 56.

5. My broker sold $3000 Central Pacific Gold Bonds at 94$\frac{3}{4}$, and invested the proceeds in N. Y. Central R. R. stock ($100) at 100$\frac{7}{8}$; how many shares did he buy, brokerage on each transaction being $\frac{1}{4}$%? *Ans.* 28 and $3.50 rem.

6. I bought Northern Central R. R. stock ($50) at 31, and sold it at 34$\frac{1}{8}$; after paying the brokerage on each transaction, I have a profit of 112.12\frac{1}{2}$; how many shares did I buy? *Ans.* 39 shares.

7. A merchant, wishing to raise $25,000 in the panic of '73, directed a broker to sell North Pennsylvania R. R. stock sufficient to produce the required sum and also pay the brokerage; if the stock was selling at 31$\frac{1}{2}$, how many shares must he sell and what would be his surplus?

Ans. 797 shares; 5\frac{7}{8}$ surplus.

8. Sold $3000 of Philadelphia 6's at 101$\frac{7}{8}$, and invested the proceeds in Farmers' and Mechanics' Bank stock ($100) at 121; what is my actual investment after deducting brokerage on both transactions? *Ans.* 25 shares; $17.50 surplus.

9. I have a balance of $500 in bank and sell through my broker 15 shares of Philadelphia National Bank ($100) at 158$\frac{1}{4}$, and request him to buy as many shares of Lehigh

Valley R. R. at 61, as my bank account will allow; how many shares can he buy, and what is my balance in bank?

Ans. 46 shares; $58.25 balance.

CASE III.

578. ***Given, the par value, and the brokerage, or the net proceeds or entire cost, and the market value, to find the rate.***

1. A broker buys Camden and Amboy 6's, whose par value is $9500, and his charge was $23.75; what was the rate of brokerage?

SOLUTION.—The brokerage, $23.75, equals the par value, $9500, multiplied by the *rate;* hence, the rate equals $23.75 divided by $9500, which we find is .00¼ or ¼%.

OPERATION.

$$\frac{23.75}{9500} = .00\tfrac{1}{4}$$

Rule I.—*Divide the brokerage by the par value, to find the rate.*

Rule II.—*Divide the difference between the real value of the stock, and the net proceeds or entire cost, by the par value, to find the rate.*

2. A broker buys 54 shares of turnpike stock at 95; the brokerage was $13.50; what was the rate? *Ans.* ¼%.

3. I send a broker $11,225 for which he buys 200 shares of Pittsburgh Gas Co. stock ($50) at a premium of 12%, retaining the brokerage; what was the rate? *Ans.* ¼%.

4. My broker having purchased, according to order, 18 shares of Reading R. R. stock ($50) at 57, informs me that the entire cost is $1028.25; what is the rate? *Ans.* ¼%.

5. I gave a broker $15900 to invest in "Camden and Amboy" ($100); he bought 124 shares at 127½, and remitted me the balance, $28; what rate of brokerage did he charge? *Ans.* ½%.

6. A broker sold 56 shares of a certain gas company ($25) at 31, and deducting his commission on both transactions, bought 26 shares Kensington Bank stock ($50) at 65, with a surplus of $39 25; what was the rate of brokerage?

Ans. ¼%.

INCOME FROM INVESTMENTS.

579. Investments in bonds, stocks, etc., may be made either for interest on the money or for increase of capital.

580. Bonds are written or printed obligations to pay certain sums of money at or before a specified time, insuring the holder an income, yearly, semi-yearly, or quarterly, at a given rate per cent.

581. The **Denomination** of a bond is its face, or the amount it promises to pay. The most usual denomination is $100, but other denominations are also common.

582. The **Rate of Interest** of a bond is the annual income it pays on $100. Thus bonds pay 6%, 5%, 4½%, 3%, etc.

Bonds are often designated by the rate of interest; as, New York 7's, Phila. 6's, Penna. 5's, Cincinnati 7–30's, U.S. 4½'s, etc.

583. The **Classes of Bonds** most common are those of *Corporations, Cities, States*, the *United States*, etc.

Bonds are issued by Corporations, Towns, Cities, States, etc., for the purpose of raising money to pay their indebtedness, make improvements, etc. Thus, a city often raises money to erect school buildings by the issue of bonds.

Bonds are valued as an investment in accordance with their security and rate of interest. Thus the bonds of a State or of the United States are so secure that even when the rate of interest is low they sell at a premium.

584. Bonds are distinguished as *Registered* and *Coupon Bonds*. The **Registered Bonds** are payable to the order of the holder, and cannot be transferred without being indorsed.

585. The **Coupon Bonds** have *coupons*, or certificates of interest, attached to them, which may be cut off and the interest collected when due.

586. The principal bonds of the United States, called Government Bonds, are the following:

The 4's of 1907, which are 4% bonds due in 1907. These bonds are both coupon and registered, and the interest is payable quarterly.

The 4½'s of 1891, which are 4½% bonds due in 1891, interest payable quarterly, both coupon and registered.

The currency 6's, issued to aid in constructing several railroads to the Pacific. There are several series of these bonds, maturing respectively in 1895, 6, 7, 8, and 9.

Government Bonds are so secure that they are much sought after *for investment*, and thus command a premium. This premium

usually becomes smaller each year, as the bonds approach maturity.

When the interest on bonds is payable in gold, and gold is at a premium, the income in currency is equal to the income in gold, plus the premium.

587. The two principal classes of private securities are *Mortgages* and *Ground-Rents.*

588. A **Mortgage** is a conditional conveyance of property as security for the payment of a debt.

Should the interest not be promptly paid, the mortgage may be *foreclosed*, and the property is then sold by the sheriff to the highest bidder, and the mortgage paid off from the proceeds. Property is usually not mortgaged beyond a certain part of its value, in order that the mortgagee may be secure from loss. It is sometimes the case, however, that a second mortgage is given, but this cannot be paid, in case of foreclosure, till the first is fully paid, and hence may not be a very good security.

589. A **Ground-Rent** is a fixed rent paid for ground, generally used for building purposes.

It is a common practice in Philadelphia, when a person wishes to build one or more houses, instead of *buying* the ground required, to agree to pay the interest on its value as rent, the contract to continue in force as long as the rent is regularly paid. A ground-rent may be *redeemable* or *irredeemable.* Some cities, as Philadelphia, prohibit the issue of any more irredeemable ground-rents.

Mortgages and *ground-rents* are not bought and sold at the Stock Exchange, but conveyancers are generally employed in the transaction, as the title and condition of the property must be examined, and the necessary papers drawn up. Well-secured mortgages and ground-rents are in such high esteem as safe investments, that they are among the securities in which trust funds may be legally invested.

590. The **Quantities** considered are: 1. The *Amount Invested;* 2. The *Rate of Dividend or Interest;* 3. The *Income;* 4. The *Market Value of* $100, *or of one share;* 5. The *Rate of Income.*

NOTE.—In changing from one investment to another, it is often the case that there is a little more realized from the sale of the first than will procure an exact number of shares of the second. In such cases the income will be calculated on the number of shares, without noticing the surplus. Brokerage is not to be reckoned unless mentioned.

CASE I.

591. ***Given, the amount of an investment, the market value, and the rate of dividend or interest, to find the income.***

1. If a person invests $8075 in 6% bonds, at 95, what will be his annual income?

SOLUTION.—Since for 95 cents you can buy \$1 worth of stock, for \$8075 you can buy as many dollars' worth of stock as \$0.95 is contained times in \$8075, or \$8500. The annual income on this is \$8500×.06 = \$510.

OPERATION.
\$8075÷.95=\$8500
\$8500×.06=\$510

Rule.—I. *Divide the amount invested by the market value, to find the par value.*

II. *Multiply the par value by the rate, to find the income.*

NOTE.—In case of stocks, we may divide the market value by the par value for the number of shares, and multiply the dividend on each share by the number of shares, but this rule will not apply to mortgages and ground-rents, which are not divided into shares, but taken as one whole, and therefore is not general.

2. What annual income would I receive by investing \$2750 in State 6's at 110? *Ans.* \$150.

3. If I invest \$4920 in United Companies of New Jersey stock at 120, dividend 10%, what will be my income?
Ans. \$410.

4. If I invest \$5100 in 6¼% bonds at 102, what is my annual income from this investment? *Ans.* \$325.

5. Mr. Wilkins sold on ground-rent a lot 150 ft. front by 210 ft. deep. valued at \$56.25 per foot front; what would be the ground-rent per foot front at 6%? *Ans.* \$3.375.

6. A conveyancer sold a lot 50 ft. front and 125 ft. deep, on ground-rent redeemable on payment of \$4500; what is the ground-rent at 6% per annum? *Ans.* \$270.

7. A man had \$9500 in 6% bonds, selling at 118; would his annual income have been greater or less if he had exchanged them for 5% bonds, selling at 106? *Ans.* Less by \$42.50.

8. A lady receiving a legacy of \$3000, bought \$1500 of 5% bonds at 97½, invested as much of the remainder as possible in 6% bonds (lowest denominations \$50) at 105; what surplus remained, and what was her annual income?
Ans. \$15 surplus; income \$162.

9. A man sold \$5500 of U. S. 6% bonds at 120, and invested the proceeds in Lehigh Valley R. R. stock (\$50) at 61, paying 10% dividend, brokerage being ¼% for both buying and selling; how much did he gain or lose annually by the exchange?
Ans. Gain, \$205.

10. Mr. Bennett conveyed a lot on a 6% ground-rent, payable in gold, redeemable on payment of $7500; what was its value in currency, when gold was 115, and what was the ground-rent in gold, and what in currency?

Ans. $8625; gold rent, $450; currency, $517.50.

11. A capitalist holding bonds of the N. Y. Central and Hudson R. R. to the amount of $20,000, exchanged them for the stock of the same company at 98¾%. The bonds drew 6% interest, while on the stock two semi-annual dividends were declared, the first 3% and the second 4%; how much did he gain annually by the exchange? *Ans.* $214.

12. A man having $6000 of Philadelphia 6's at 102, decided to sell them and invest ½ in West Philadelphia Pass. Railway at 67, dividend 10%, ⅓ in Norristown R. R. at 87, dividend 12%, and the remainder, including the surplus from the other investments, in Green and Coates St. Railway at 47, dividend 8%; what will be the increase of his income, deducting ¼% for brokerage on each transaction? *Ans.* $91.

CASE II.

592. ***Given, the income, the rate of dividend, and the market value, to find the amount invested.***

1. When 6% bonds are selling at 108, how much must be invested in them to secure an annual income of $672?

Solution.—Since $1 of stock gives an income of $0.06, to give an income of $672 it will require 672÷.06, or $11200; $11200 of stock at 108% will cost $11200×1.08= $12096.

OPERATION.

$672 \div .06 = 11200$

$\$11200 \times 1.08 = \$12096$

Rule.—I. *Divide the income by the rate, to find the par value.*

II. *Multiply the par value by the market value of 1 share, to find the amount invested.*

2. A brick dwelling subject to a ground rent of $54 at 6%, was sold for $4500; what was its value? *Ans.* $5400.

3. What sum must I invest in Government 6's of '97, at 112¼, to secure an annual income of $600, brokerage ¼%?

Ans. $11,250.

4. When Michigan Central 7's are selling at 98½, what sum must be invested in them to yield $343 a year, brokerage ⅛%? *Ans.* $4832.62½.

5. If I sell $10,000 U. S. 6's of '99 at 121, and buy sufficient Union Pacific 7's at 81 to yield $770 income, what shall I have left, deducting brokerage? *Ans.* $3137.50.

6. If I invest a certain sum in 6's at 85, and the same sum in 7's at 95, and receive $5 more a year from the latter investment, how much do I invest in each? *Ans.* $1615.

7. What must I pay for Missouri 6's to realize 7% on the investment? What must I pay for Lehigh Valley 7's to yield 6½%? *Ans.* 85$\frac{5}{7}$; 107$\frac{9}{13}$.

8. When gold was selling at 130, how much must be paid for 6% gold bearing bonds in order to realize 7% on the investment? *Ans.* 111$\frac{3}{7}$.

9. Mr. Francis bought a lot of 150 ft. front and 148 ft. deep, at a ground-rent of $1.50 per foot front; what would be the purchase money for the whole property, the ground-rent being 6% of it? *Ans.* $3750.

10. How many shares of Nesquehoning Valley R. R. ($50) at 54½, must be sold in order that the proceeds, invested in Allegheny City 6's at 95, may yield an income of $750, not considering brokerage? *Ans.* 218 shares, $6 surplus.

11. A man sold $16,000 New York 6's at 106, and invested a part of the proceeds in U. S. 6's at 112, sufficient to yield an annual income of $540, and spent the remainder for a lot; what did the lot cost? *Ans.* $6880.

12. What must have been the price of gold so that a person investing in gold-bearing 6's at 110, could realize 7%? What must have been the price of gold so that a person could realize 6% from investing in gold-bearing 5's at 98? *Ans.* 128⅓; 117⅗.

13. I invested through my broker, equal sums in Rhode Island 6's and Illinois 6's, the former costing 105, and the latter 98; the income from both was $522; how much was invested in each? *Ans.* $4410.

14. I hold Alabama 8's at 70 which give an income of $504, but preferring an investment nearer home, I exchange

them for Pittsburgh 7's at 101; how much must I add to my investment to secure the same income? *Ans.* \$2862.

CASE III.

593. ***Given, the market value, and the income or rate of dividend, to find the rate of income on the investment.***

1. What per cent. of his money will a man realize by purchasing 6% stock at 90?

SOLUTION.—\$1 of stock will cost \$.90, and pays \$.06; if on \$.90 the gain is \$.06, on \$1 it is as many per cent. as .06÷.90, or $6\frac{2}{3}$%.

OPERATION.
$.06 \div .90 = .06\frac{2}{3}$

Rule.—*Divide the annual income or dividend of the stock, by its market value, to find the rate of income.*

2. What is the rate of income on Union Pacific 7's bought at 81? *Ans.* $8\frac{52}{81}$%.

3. Bought an irredeemable ground-rent of \$25 per annum for \$375; what per cent. do I realize? *Ans.* $6\frac{2}{3}$%.

4. I buy a ground rent of \$360 for \$5875; what interest do I receive on my investment? *Ans.* $6\frac{6}{47}$%.

5. The Fidelity Trust and Safe Deposit Company, as trustee, invests \$28,000 in a ground-rent of \$1500; what interest do they realize? *Ans.* $5\frac{5}{14}$%.

6. Which is the better investment, 5% bonds at 106, or 6% bonds at 112; and how much? *Ans.* The latter; $\frac{475}{742}$%.

7. When U. S. 5's (gold bearing) were at 95 and gold at 140, what % in currency did these bonds yield? *Ans.* $7\frac{7}{19}$%.

8. Which is the more profitable investment, Missouri 6's at 95, or Iowa 7's at 108? *Ans.* Iowa 7's.

9. A man invested in R. R. 7–30's at $105\frac{1}{2}$, and afterwards exchanged them for City 6's at $94\frac{1}{2}$; which was the better investment? *Ans.* The former.

10. Desiring to make a permanent investment, I find three classes of bonds equally secure, 5's at 70, 6's at 80, and 7's at 95; which is the best investment? *Ans.* 6's at 80.

11. Buy at Philadelphia Exchange \$25,000 Congress Hall, Cape May, N. J., coupons, 7%, at 85; what rate of income do I realize? *Ans.* $8\frac{4}{17}$%.

12. I bought at "exchange," \$2500 (\$50) Reading and Columbia Railroad 1st mortgage coupons, 7%, \$1000 at 94 and \$1500 at $93\frac{1}{2}$; what will be the average rate of income from the investment? *Ans.* $7\frac{441}{937}\%$.

GENERAL TAXES.

594. A **Tax** is a sum of money assessed on persons or property for public purposes.

595. **Taxes** are assessed by the national government, a state, county, or town.

596. A **Property Tax** is a tax upon property. Property is of two kinds; *Real Estate* and *Personal Property.*

597. **Real Estate** is immovable property; as land, buildings, etc. *Personal Property* is movable property; as money, stock, furniture, etc.

598. A **Poll Tax** is a tax on the person. It is assessed in some states on each male citizen not exempt by law.

599. An **Assessment Roll** is a list or schedule containing the names of persons taxed, the valuation of their property, and the amount of their taxes.

600. An **Assessor** is an officer who appraises the property, and prepares the assessment roll.

601. The **Quantities** to be considered are: 1. The *Taxable Property;* 2. The *Rate of Taxation;* 3. The *Amount of Tax.*

Real estate is usually assessed by the proper officer for not more than $\frac{1}{2}$ or $\frac{2}{3}$ of its real value. The value of personal property may be given in by the owner under oath, or if he neglects to do this, it is valued by the officer.

The term *poll* is from the German *polle*, the head. A poll tax is a *capitation tax*, from the Latin *caput*, the head. In some States, the *income* from a person's occupation is assessed at a small sum and taxed. Money on interest secured by bond and mortgage is taxed in some States.

After the taxes have been assessed, each person receives a notice of his taxation, stating the day of appeal, when he may appear before the proper officers and show reasons for correcting any mistakes that have been made.

NOTE.—Government Taxes are taxes levied by the general government. They will be considered under *Duties* and *Customs*.

CASE I.

602. ***Given, the taxable property and the rate of taxation, to find the amount of tax.***

1. The taxable property of a town is $578,000, and the rate of taxation $.005 on a dollar; what is the rate?

Solution.—If the tax is $.005 on $1, on $578,000 it will be 578000 times $.005, or $2890.

OPERATION.

578000 × $.005 = $2890

Rule.—*Multiply the amount of taxable property by the rate, to find the tax.*

Note.—If there is a poll tax, the sum produced by it should be added to the property tax to give the whole tax.

2. The real estate of a town is valued at $250,896, and the personal estate at $356,729; there are also 350 polls at $1.25 each; what is the whole tax, the rate being 4 mills on a dollar? *Ans.* $2868.

603. Table.—In the assessment of taxes upon a town, city, etc., a table is usually constructed by which the labor of calculation is greatly facilitated. The following table is based on the rate of $.005 to the dollar:

Prop.	Tax.	Prop.	Tax.	Prop.	Tax.	Prop.	Tax.	Prop.	Tax.
$1	$.005	$10	$.05	$100	$0.50	$1000	$5	$10,000	$50
2	.010	20	.10	200	1.00	2000	10	20,000	100
3	.015	30	.15	300	1.50	3000	15	30,000	150
4	.020	40	.20	400	2.00	4000	20	40,000	200
5	.025	50	.25	500	2.50	5000	25	50,000	250
6	.030	60	.30	600	3.00	6000	30	60,000	300
7	.035	70	.35	700	3.50	7000	35	70,000	350
8	.040	80	.40	[illegible]	4.00	8000	40	80,000	400
9	.045	90	.45	900	4.50	9000	45	90,000	450

3. Find A's tax, whose property is assessed at $9540, if he pays for 3 polls at $1.25 each?

Solution.—We find from the table the tax on $9000, then on $500, then on $40, then calculate the tax on three polls, and take the sum of the results, which will be the entire tax.

OPERATION.

Tax	on	$9000 =	$45.00
"	"	500 =	2.50
"	"	40 =	.20
"	"	3 polls =	3.75
		Whole tax =	$51.45

4. A's property is assessed at $25,090 and his sister's at $22,850; what tax will they together pay? Ans. $239.70.

5. My property is assessed at \$12,500; I pay for 2 polls @ \$1.25, and $.1\frac{1}{4}\%$ on the income from my occupation, assessed at \$1200; what was my entire tax? *Ans.* \$66.50.

6. A town wishes to raise a tax of $7\frac{1}{2}$ mills on the dollar. There are 350 polls at \$1.25 each, the personal property is valued at \$230,000 and the real estate at \$900,000; make out the assessor's table, find the whole tax, and also the tax of D, whose personal property is valued at \$7,540, real estate at \$15,700, and who pays for 3 polls. *Ans.* Last, \$178.05.

7. I find I have been assessed as follows: Real estate, \$20,000; personal property, \$2700; money at interest, \$20,000; income from occupation, \$2000; and 2 gold watches. I obtain an abatement of $\frac{2}{5}$ on the real estate, $\frac{1}{8}$ on personal property, \$1500 on money at interest, $\frac{3}{5}$ for occupation, and 1 gold watch; how much does this lessen my tax, the rate being $\$.004\frac{1}{4}$, and \$1 for each watch?

Ans. \$50.30.

CASE II.

604. ***Given, the rate of taxation and the tax, or the amount left after payment of tax, to find the amount assessed.***

1. What is the assessed value of property, taxed \$75.12 at 6 mills on a dollar?

SOLUTION.—At 6 mills on the dollar, .006 times the amount assessed equals the tax, which is \$75.12; hence the *amount* equals \$75.12 divided by .006, which we find is \$12,520.

OPERATION.

$$\frac{\$75.12}{.006} = \$12{,}520$$

Rule I.—*Divide the tax by the rate, to find the amount assessed.*

Rule II.—*Divide the amount left after payment of tax by* 1 *minus the rate.*

2. A tax of \$2100 is raised in a town containing 1200 polls, each taxed \$.75; the property tax is .25%; required the value of the taxable property. *Ans.* \$480,000.

3. The expense of building a town hall is \$6550, to be defrayed by a tax upon the property-holders; the rate was $.5\frac{1}{2}\%$, and the collector's commission $2\frac{1}{2}\%$; what was the valuation of the property? *Ans.* \$1,221,445 22.

4. A man sold property for $241,367.875, which included the tax for the year at .5½% and 3½% for collecting it; what was the price of the property? *Ans.* $240,000.

5. Mr. Fish paid one year .35% township tax, .2½% county tax, .56% school tax, and $3.50 poll tax; his whole tax amounted to $409.50; what was the amount of his property? *Ans.* $35,000.

CASE III.

605. *Given, the assessed value and the tax, to find the rate.*

1. A tax of $8375 is to be assessed in a town; the real estate is valued at $960,000, and the personal property $580,000; there are 450 polls, each of which is taxed $1.50; what is the rate of taxation?

OPERATION.

$1.50×450 = $675.
$8375 − $675 = $7700
$7700 ÷ $1,540,000 = .005

SOLUTION.—We multiply the tax on 1 poll by the number of polls, which gives $675 as the poll tax; subtracting this from the whole tax, we have remaining $7700, the property tax; dividing $7700 by $1,540,000, the amount of property, we have 5 mills, the tax on $1.

Rule.—*Divide the property tax by the amount of taxable property; the quotient will be the rate of taxation.*

NOTE.—If there is a poll tax, subtract it from the whole tax before dividing.

2. A town whose property was assessed at $602,880, built a school-house which cost $1890.78, the collector's commission being 3½%; what was the rate of taxation? *Ans.* .3¼%.

3. The trustees of a school expended $800 for salary of the teacher, $36.50 for fuel, and $55.75 for apparatus; the school fund amounted to $175.50, and the rest of the expenses were paid by a rate bill; if the attendance was 6540 days, what was A's tax, who sent 3 pupils 120 days each? *Ans.* $39.45.

4. In a town whose taxable property is $2,560,000, the estimated expenses for a year are $7675, the balance in the treasury is $585, and there are 3800 polls to be assessed $.35 each. What is my tax if my property is assessed at $45,600, I pay for 4 polls, also a state tax of ¼ of a mill on a dollar, and a county tax of 1¼ mills on a dollar? Ans. $172.40.

SIMPLE INTEREST.

606. Interest is money charged for the use of money for a certain time.

607. The **Principal** is the sum for which interest is charged. Interest is reckoned as a percentage of the principal.

608. The **Rate** of interest is the rate per cent. on $1 for a certain time. The usual time is one year.

609. The **Time** is the period during which the money is on interest.

610. The **Amount** is the sum of the principal and the interest.

611. Simple Interest is the interest on the principal only. **Compound Interest** is interest also on the interest.

612. Legal Interest is interest at the rate fixed by law. It varied in different States in 1894 as follows:

STATES.	RATE %		STATES.	RATE %		STATES.	RATE %	
Alabama	8	8	Kentucky	6	6	North Dakota	7	12
Alaska	8	10	Louisiana	5	8	Ohio	6	8
Arizona	7	*	Maine	6	*	Oklahoma	7	12
Arkansas	6	10	Maryland	6	6	Oregon	8	10
California	7	*	Massachusetts	6	*	Pennsylvania	6	6
Colorado	8	*	Michigan	6	8	Rhode Island	6	*
Connecticut	6	6	Minnesota	7	10	South Carolina	7	8
Delaware	6	6	Mississippi	6	10	South Dakota	7	12
Dist. Columbia	6	10	Missouri	6	8	Tennessee	6	6
Florida	8	10	Montana	10	*	Texas	6	10
Georgia	7	8	Nebraska	7	10	Utah	8	*
Idaho	10	18	Nevada	7	*	Vermont	6	6
Illinois	5	7	New Hampshire	6	6	Virginia	6	6
Indiana	6	8	New Jersey	6	6	Washington	8	*
Indian Ter	6	10	New Mexico	6	12	West Virginia	6	6
Iowa	6	8	New York	6	6	Wisconsin	6	10
Kansas	6	10	North Carolina	6	8	Wyoming	12	*

The first column gives the legal rate; second column, the rate that may be collected if agreed upon in writing; the * indicates no limit to the rate.

613. Usury is interest at a rate greater than the law allows. Various penalties are attached to the taking of usury in the different States.

The legal rate in England and France is 5%; and in Ireland, Canada, and Nova Scotia, 6%.

In notes, contracts, accounts, mortgages, etc., when no rate is specified *the legal rate* is understood.

Notes draw interest after they become due, though interest is not mentioned in them; and interest is reckoned on book accounts after the expiration of the term of credit.

614. The **Quantities** in simple interest are five: 1. The *Principal;* 2. The *Interest;* 3. The *Rate;* 4. The *Time;* 5. The *Amount.*

NOTE.—In computing interest it is customary to reckon a month as $\frac{1}{12}$ of a year, and a day as $\frac{1}{30}$ of a month. In dealing with the U. S. Government and foreign correspondents, each day is $\frac{1}{365}$ of a year.

CASE I.

615. ***Given, the principal, the rate per cent., and the time, to find the interest or the amount.***

COMMON METHOD.

1. What is the interest of \$3600 for 6 yr. 7 mo. 15 da. at 7%?

SOLUTION.—By reduction we find that 6 yr. 7 mo. 15 da. equals $6\frac{5}{8}$ yr. If the interest of \$1 for 1 yr. is 7 cts. the interest of \$3600 for 1 yr. is 3600 times 7 cts., which is \$252, and for $6\frac{5}{8}$ yr. it is $6\frac{5}{8}$ times \$252, which by multiplying we find to be \$1669.50. Hence the following

OPERATION.

$$\begin{array}{r} \$3600 \\ .07 \\ \hline 252.00 \\ 6\frac{5}{8} \\ \hline \$1669.50 \end{array}$$

Rule I.—*Multiply the principal by the rate, and that product by the time expressed in years, to find the interest.*

II. *Add the interest to the principal, to find the amount.*

Required the interest

2. Of \$380 for 3 yr. 4 mo. 12 da. at 6%. *Ans.* \$76.76.
3. Of \$495 for 8 yr. 3 mo. 9 da. at 6%. *Ans.* \$245.76$\frac{3}{4}$.
4. Of \$85.85 for 5 yr. 7 mo. 16 da. at 8%. *Ans.* \$38.65.
5. Of \$387 for 5 yr. 10 mo. 15 da. at 7%. *Ans.* \$159.15$\frac{3}{8}$.
6. Of \$795.87$\frac{1}{2}$ for 7 yr. 9 mo. 18 da. at 10%. *Ans.* \$620.78$\frac{1}{4}$.

SIX PER CENT. METHOD.

616. The **Six Per Cent. Method** is so called because the process is based upon that rate.

1. What is the interest of \$360 for 8 yr. 10 mo. 18 da. at 6%?

SOLUTION.—The Int. of \$1 for 1 yr. is \$0.06, and for 8 yr. it is 8 times \$0.06, or \$0.48.

The Int. of \$1 for 1 mo., or $\frac{1}{12}$ of a year, is $\frac{1}{12}$ of 6¢, or $\frac{1}{2}$ of a cent, and the Int. for 10 mo. is 10 times $\frac{1}{2}$ of a cent, or \$0.05.

The Int. of \$1 for 1 mo., or 30 da., is $\frac{1}{2}$ of a cent, or 5 mills, and for 1 day it is $\frac{1}{30}$ of 5 mills, or $\frac{1}{6}$ of a mill, and for 18 da., it is 18 times $\frac{1}{6}$ of a mill, or \$0.003.

OPERATION.

$$\begin{array}{r} .06\times 8=\$0.48 \\ \tfrac{1}{2}\times 10= \quad .05 \\ \tfrac{1}{6}\times 18= \quad .003 \\ \hline \$0.533 \\ 360 \\ \hline \$191.88 \end{array}$$

Adding these results, we have \$0.533 as the Int. of \$1 for 8 yr. 10 mo. 18 da., and on \$360 the Int. is 360 times \$0.533, or \$191.88.

Rule.—I. *Multiply the rate, .06, by the number of years; take $\frac{1}{2}$ of the number of months as cents, and $\frac{1}{6}$ of the number of days as mills; their sum will be the interest of \$1 for the given time at 6%.*

II. *Multiply this sum by the principal, the product will be the interest at 6 per cent. For any other rate, take as many sixths of it as that rate is of six.*

NOTES.—1. Another "6 per cent. method" is to reduce the years to months, and take *half* the number of *months* for *cents*, etc., as before.

2. Another method is to take the *number* of *months* as *cents* and *one-third* of the number of *days* as *mills*, and multiply the sum by *half the principal.*

3. The *method for days*, popularly expressed, is "*Multiply dollars by days, and divide by 6000.*"

In applying this method, cut off three decimal figures, multiply by the number of days, and divide by 6.

REMARK.—Require the pupils to solve the following problems by each of the above methods.

Required the interest

2. Of \$560 for 3 yr. 8 mo. 12 da. at 6%. *Ans.* \$124.32.
3. Of \$750 for 7 yr. 7 mo. 24 da. at 6%. *Ans.* \$344.25.
4. Of \$35.60 for 5 yr. 9 mo. 15 da. at 5%. *Ans.* \$10.31.
5. Of \$45.50 for 6 yr. 3 mo. 9 da. at 4%. *Ans.* \$11.42.
6. Of \$75.35 for 8 yr. 5 mo. 14 da. at 6%. *Ans.* \$38.227.
7. Of \$60.75 for 4 yr. 5 mo. 21 da. at 7%. *Ans.* \$19.029.
8. Of \$756.25 for 3 yr. 7 mo. 11 da. at $6\frac{1}{2}$%.
Ans. \$177.645.
9. Of \$831.56 for 9 yr. 11 mo. 17 da. at $6\frac{3}{4}$%.
Ans. \$559.276.
10. Of \$753.33$\frac{1}{3}$ for 8 yr. 9 mo. 11 da. at $7\frac{3}{4}$%.
Ans. \$512.638.
11. Of £155 10 s. 8 d. for 5 yr. 9 mo. 13 da., at $4\frac{1}{2}$% ?
Ans. £40 9 s. 11$\frac{1}{4}$d.+.

THE 60 DAY METHOD.

617. The **Sixty Day** method of interest is the simplest and best for short periods of time.

Principle.—At 6% a year, the rate for 2 mo., or 60 da., is 1%; hence *for 60 da.,* $\frac{1}{100}$ *of the principal equals the interest.* From this we have the following rule:

Rule.—*Point off two places in the principal for the interest for 60 days, and take multiples or aliquot parts of this interest for any other number of days.*

1. What is the interest of $480 at 6% for 66 da.? for 96 da.?

SOLUTION.—Pointing off two places we have $4.80, the Int. for 60 da.; then take $\frac{1}{10}$ of $4.80, we have $0.48, the Int. for 6 da.; then the sum of these interests, or $5.28, is the interest for 66 da.

OPERATION.

$4.80=	Int. for	60 da.
.48=	" "	6 da.
$5.28=	" "	66 da.

SOLUTION.—Pointing off two places we have $4.80, the Int. for 60 da.; take $\frac{1}{2}$ of $4.80 for the Int. for 30 da.; and $\frac{1}{10}$ of $4.80 for the Int. for 6 da.; their sum is the Int. for 96 da.

OPERATION.

$4.80=	Int. for	60 da.
2.40=	" "	30 da.
.48=	" "	6 da.
$7.68=	" "	96 da.

2. Find the Int. of $720, at 6%, for 6 mo. 12 da. For 128 da.

OPERATION.

$7|20=	Int. for	60 da.
21|60=	" "	6 mo. (3×2 mo.)
1|20=	" "	10 da. ($\frac{1}{6}$ of 60 da.)
|24=	" "	2 da. ($\frac{1}{5}$ of 10 da.)
$23.04=	Int. for	6 mo. 12 da.

OPERATION.

$7|20=	Int. for	60 da.
14|40=	" "	120 da.
|72=	" "	6 da.
|24=	" "	2 da.
$15.36=	" "	128 da.

Required the interest

3. Of $4500 for 4 mo. at 6%. *Ans.* $90.
4. Of $1560 for 10 mo. at 6%. *Ans.* $78.
5. Of $1920 for 2 mo. 9 da. at 6%. *Ans.* $22.08.
6. Of $2520 for 33 da. at 6%. *Ans.* $13.86.
7. Of $1600 for 63 da. at 6%. *Ans.* $16.80.
8. Of $450 for 135 da. at 6%. *Ans.* $10.12½.
9. Of $1680 for 3 mo. 28 da. at 6%. *Ans.* $33.04.
10. Of $1250 for 4 mo. 29 da. at 7%. *Ans.* $36.21+.
11. Of $2500 for 1 yr. 9 mo. 12 da. at 7%. *Ans.* $312.08+.
12. Of $1750 for 2 yr. 3 mo. 24 da. at 8%. *Ans.* $324.33+.
13. Of $275.50 for 2 yr. 8 mo. 15 da. at 5%. Ans. $37.31—.
14. *Of* $360.50 for 5 yr. 4 mo. 16 da. at 4½%. Ans. $87.24+.

15. On a 3 mo. note for $4800 dated April 15, 1876, at 6%, allowing 3 da. of grace, exact time.

EXACT TIME.	
In April,	15 da.
In May,	31 da.
In June,	30 da.
In July,	15 da.
	91 da.+3 da.=94da.

OPERATION.			
$48|00=	Int. for	60 da.
24|00=	" "	30 da.
3|20=	" "	4 da.
$75.20=	" "	94 da.

16. On a 3 mo. note for $3600, dated Jan. 20, 1887, at 6%, exact time. *Ans.* $54.

17. On a note for $2550, running from May 12 to Oct. 20, at 6%, exact time. *Ans.* $68.43.

18. On a note of $1440 running from June 16 to Nov. 10, at 7%, exact number of days. *Ans.* $41.16.

19. On a note for $1140, dated Nov. 24, 1887, and due Feb. 10, 1888, at 6%, allowing 3 da. of grace. *Ans.* $15.39.

20. On a note for $1050, dated Oct. 16, 1887, and due Mar. 25, 1888, at 6%, allowing 3 da. of grace. *Ans.* $28.70.

METHOD BY CANCELLATION.

618. The following Six Per Cent. Method by *Cancellation* will be found convenient and practical:

Rule for Months.—*Point off two places in the principal, divide by 2, and multiply by the number of months.*

For the Int. of 2 mo. is $\frac{1}{100}$ of the principal, and for 1 mo. it is $\frac{1}{2}$ of $\frac{1}{100}$ of the principal; hence for any number of months, the Int. is $\frac{1}{2}$ of $\frac{1}{100}$ of the principal multiplied by the number of months.

Rule for Days.—*Point off three places in the principal, divide by 6, and multiply by the number of days.*

For the Int. of 2 mo. for 60 days is $\frac{1}{100}$ of the principal, and for 6 da. it is $\frac{1}{10}$ of $\frac{1}{100}$, or $\frac{1}{1000}$ of the principal, and for 1 da. it is $\frac{1}{6}$ of $\frac{1}{1000}$ of the principal; hence for any nuuber of days the Int. is $\frac{1}{6}$ of $\frac{1}{1000}$ of the principal multiplied by the number of days.

1. What is the Int. of $3600 for 38 mo. at 6%? at 7%?

SOLUTION.—We point off two places in $3600 to divide by 100, then divide by 2, and multiply by 38; cancelling and multiplying, we have $684.

OPERATION.

$$\frac{\$36.00\times38}{2}=\$684.$$

SOLUTION.—We divide by 6 to find the Int. at 1%, and multiply by 7 to find the Int. at 7%; reducing by cancellation, *we have $798.*

OPERATION.

$$\frac{\$\overset{3}{\cancel{36}}.00\times38\times7}{\cancel{2}\times\cancel{6}}=\$798.$$

2. What is the Int. of $4800 for 63 da. at 6%? at 4%?

SOLUTION.—The Int. for 1 da. is $\frac{1}{6}$ of $\frac{1}{1000}$ of the Prin.; hence we point off three places in $4800, divide by 6 to find the Int. for 1 day, and multiply by 63 to find the Int. for 63 days; cancelling and multiplying we have $50.40.

OPERATION.

$$\frac{\$\cancel{4}.\cancel{800}\times 63}{\cancel{6}} = \$50.40. \quad \textit{Ans.}$$

(with .800 above)

NOTE.—To find the Int. at 4½%, divide by 6 and multiply by 4½, expressing the work and cancelling.

EXAMPLES FOR PRACTICE.

Required the interest

3. Of $120 for 2 yr. 9 mo. at 6%. *Ans.* $19.80.
4. Of $245 for 3 yr. 4 mo, at 6%. *Ans.* $49.00.
5. Of $1360 for 75 days at 6%. *Ans.* $17.00.
6. Of $2240 for 2 mo. 12 da. at 6%. *Ans.* $26.88.
7. Of $2520 for 3 mo. 14 da. at 7%. *Ans.* $50.96.
8. Of $1265 for 4 mo. 24 da. at 5%. *Ans.* $25.30.
9. Of $5000 for 5 mo. 12 da. at 4½%. *Ans.* $101.25.
10. Of $18.27 for 93 days at 7%. *Ans.* $0.33+.
11. Of $2480 for 124 days at 7½%. *Ans.* $64.07—.
12. Of $3600 for 3 yr. 3 mo. at 4½%. *Ans.* $526.50.

13. On a 3 mo. note for $3600, dated Dec. 20, 1887, at 6%, exact time.

EXACT TIME.

In Dec. 11 da.
In Jan. 31 da.
In Feb. 29 da.
In Mar. 20 da.
91 da.

OPERATION.

$$\frac{\$\cancel{3}.\cancel{600}\times 91}{\cancel{6}} = \$54.60.$$

14. On a 3 mo. note for $3600, dated Dec. 26, 1888, at 6%, exact time. *Ans.* $54.

15. On a 90 da. note for $5600, dated Dec. 20, 1888, at 6%, allowing 3 days of grace. *Ans.* $86.80.

16. On a note for $4800, dated Dec. 16, 1887, and due Sept. 14, 1888, at 8%. *Ans.* $291.20.

METHOD OF EXACT INTEREST.

619. Exact Interest is that which is obtained by reckoning 365 days to the year.

Exact Interest is reckoned by the United States Government, and is growing in favor with business men.

The common method of interest gives a little too much, since it regards a day as $\frac{1}{360}$ of a year, while it is actually $\frac{1}{365}$.

The *Interest Tables* used by bankers and other business men are often calculated to exact interest.

1. What is the exact interest of \$840 from August 15 to November 12, at 7%?

SOLUTION.—From August 15 to November 12 there are 89 days; the interest of \$840 for 1 year of 365 days, at 7%, is \$58.80, and for 89 days it is $\frac{89}{365}$ of \$58.80, which is \$14.34—.

OPERATION.

$$\begin{array}{r} \$840 \\ .07 \\ \hline 58.80 \\ 89 \\ \hline 365)5233.20 \\ \hline \$14.33\frac{55}{73} \end{array}$$

Rule.—I. *Multiply the principal by the rate, and this product by the integral number of years.*

II. *Multiply the interest for* 1 *year by the exact number of days, and divide by* 365.

III. *Take the sum of the two results for the entire interest.*

NOTES.—1. When the time is less than 1 year, we may find the interest by the common method, and deduct $\frac{1}{73}$ of it for a common year or $\frac{1}{61}$ for a leap year.

2. What is the interest, at 6%, of \$425.60, from April 6, 1864, to July 17, 1868? *Ans.* \$109.28.

3. A man had \$1800 on interest at $6\frac{1}{2}$% from Oct. 16, 1862, to April 1, 1867; what was the interest? *Ans.* \$521.53.

4. What is the common interest on £521 3 s. 6 d. for 3 yr 7 mo. 15 da. at 4%? *Ans.* £75 11 s. 4 d. $3\frac{1}{2}$ qr.

5. What is the interest on £456 7 s. 9 d. from Jan. 16, 1860, to Sept. 14, 1860, at 5%? *Ans.* £15 2 s. 7 d.+.

6. Find the interest of \$100 from Jan. 16, 1864, to Dec. 10 of the same year. *Ans.* $\$5.40\frac{60}{73}$.

7. Difference between common and exact int. of \$600, from June 18, 1863, to April 13, 1866, at $6\frac{1}{2}$%. *Ans.* $\$.44\frac{163}{438}$.

8. If I borrow \$7500 in Pennsylvania and loan it in Alabama, at legal rates, what is my annual gain? *Ans.* \$150.

9. If I receive \$8000, the property of a minor, who is 16 yr. 5 mo. 15 da. old, at $5\frac{1}{2}$%, common interest, what should I pay him when he comes of age? *Ans.* $\$9998.33\frac{1}{3}$.

CASE II.

620. ***Given, the time, the rate, and the interest or the amount, to find the principal.***

1. What principal will in 3 yr. 6 mo. at 6%, give $49.14 interest?

SOLUTION.—We find the interest of $1 for 3 yr. 6 mo. at 6 per cent., is $0.21. If $1 gives an interest of $0.21, to give $49.14 interest it will require as many dollars as $0.21 is contained times in $49.14, which are $234. Hence the following

OPERATION.
3 yr. 6 mo = 42 mo.
$\frac{1}{2}$ of 42 - $0.21
$\frac{\$49.14}{.21} = \$234.$

Rule.—*Divide the given interest by the interest of* $1 *for the given rate and time; or, divide the amount by the amount of* $1.

2. What principal will, in 7 yr. 8 mo. 5 da. at 7%, amount to $1005.62? *Ans.* $654.

3. What principal will, in 5 yr. 8 mo. 11 da. at 6%, give 437.54\frac{2}{3}$ interest? *Ans.* $1280.

4. What principal will, in 7 yr. 5 mo. 19 da., at 7%, amount to $2307.13? *Ans.* $1515.

5. What principal at 4$\frac{1}{2}$% will amount to $1570.963 in 57 days, exact interest? *Ans.* $1560.

CASE III.

621. ***Given, the principal, the rate, and the interest or the amount, to find the time.***

1. In what time will $860 at 5%, give $247.25 interest?

SOLUTION.—The interest of $860 at 5% for 1 yr. is $43. If in *one* year the principal gains $43 interest, to gain $247.25 interest it will require as many times 1 year as $43 is contained times in $247.25, which are 5$\frac{3}{4}$ yr., or 5 yr. 9 mo. Hence we have the

OPERATION.
$860
.05
43.00 Int. for 1 yr.
$\frac{247.25}{43.00} = 5\frac{3}{4}$ yr.
5 yr. 9 mo.

Rule.—*Divide the given interest by the interest of the principal at the given rate for one year.*

NOTE.—When the amount is given, subtract the principal from the amount to find the interest, and then proceed as before.

2. In what time will $240, at 5%, give $64 interest? *Ans.* 5 yr. 4 mo.

3. In what time will \$72.50, at 6%, give \$14.68$\frac{1}{8}$ interest?

Ans. 3 yr. 4 mo. 15 da.

4. In what time will \$13.25, at 6%, give \$7.062$\frac{1}{4}$ interest?

Ans. 8 yr. 10 mo. 18 da.

5. In what time will \$1515, at 7%, give \$791.84 interest?

Ans. 7 yr. 5 mo. 18 da.

6. The amount of a principal for a certain time, at 4%, is \$2838.33$\frac{1}{3}$, and for the same time at 9% is \$3261.25; required the principal and time. *Ans.* \$2500; 3 yr. 4 mo. 18 da.

7. A sum of money on interest amounts at 4$\frac{1}{2}$% for a certain time to \$5208.92, and at 9% for the same time to \$6092.84; required the principal and time. *Ans.* \$4325.

CASE IV.

622. ***Given, the principal, the time, and the interest, or the amount, to find the rate.***

1. At what rate will \$750 in 2 yr. 4 mo. give \$105 interest?

SOLUTION.—We find that the interest of \$750 for 2 yr. 4 mo. at *one* per cent. is \$17.50. If the principal in the given time, at *one* per cent., gives \$17.50 interest, to give \$105 interest, it will require as many times 1 per cent. as \$17.50 is contained times in \$105, or 6 per cent. Hence the following

OPERATION.

$$\begin{array}{r} \$750 \\ .01 \\ \hline 7.50 \\ 2\frac{1}{3} \\ \hline 17.50 \text{ Int. at } 1\%. \end{array}$$

$$\frac{105.00}{17.50} = 6\%.$$

Rule.—*Divide the given interest by the interest of the principal for the given time at one per cent.*

NOTE.—When the amount is given, subtract the principal from the amount to find the interest, and then proceed as before.

At what rate

2. Will \$480, in 6 yr. 3 mo. 18 da., give \$211.68 interest?

Ans. 7%.

3. Will \$960, in 1 yr. 1 mo. 1 da., give \$52.13$\frac{1}{3}$ interest?

Ans. 5%.

4. Will \$13.50, in 10 yr. 8 mo. 29 da., give \$26.56 amount?

Ans. 9%.

5. Will \$26 50, in 8 yr. 9 mo. 11 da., give \$17.45 interest?

Ans. 7$\frac{1}{2}$%.

6. The amount of a certain principal for 7 yr. 5 mo. 18 da. at a certain rate is \$2306.84, and for 5 yr. 4 mo. 21 da. at the same rate \$2086.78$\frac{2}{3}$; required the principal and rate.

Ans. Principal, \$1515; rate, 7%.

INTEREST ON DAILY BALANCES.

623. Interest is allowed by some bankers on daily balances left in their hands, making a settlement at the end of each month or quarter. Exact interest at 4% is frequently allowed.

As each daily balance is entitled to one day's interest, the sum of the balances is entitled to one day's interest. If, however, any of the balances remain unchanged for several days, they may be multiplied by the respective number of days and these products added in the sum of balances instead of writing them for each day separately.

1. I deposited in bank \$500, Aug. 1; \$150, Aug. 7; drew out by check \$200, Aug. 15; deposited \$350, Aug. 20, and drew \$240, Aug. 27; what interest was due Sept. 1, at 4%?

OPERATION.

\$500 × 6 =	\$3000
650 × 8 =	5200
450 × 5 =	2250
800 × 7 =	5600
560 × 5 =	2800
Sum of balances =	\$18850

$$\tfrac{1}{365} \times \tfrac{4}{100} = \tfrac{1}{9125}$$

$$18850 \div 9125 = 2.065+$$

SOLUTION.—If \$500 was deposited Aug. 1, and no change made in the balance till Aug. 7, we have 6 daily balances of \$500 each, which is equivalent to \$3000 for one day; in the same manner \$650 for 8 days is equivalent to \$5200 for one day; and we proceed thus till all the balances for the month are found, when we find their sum to be \$18,850; but if the interest for one year is $\frac{4}{100}$ of the principal, for one day it is $\frac{1}{365}$ of $\frac{4}{100}$, or $\frac{1}{9125}$ of the principal; and $\frac{1}{9125}$ of \$18,850 equals \$2.065. Hence the following

Rule.—*Divide the sum of the daily balances by* 9125 *to find the exact interest at* 4%.

NOTE.—If we take the year as 360 days we find the interest at 4% by dividing by 9000, since $\frac{1}{360}$ of $\frac{4}{100} = \frac{1}{9000}$; at 6% by dividing by 6000, for $\frac{1}{360}$ of $\frac{6}{100} = \frac{1}{6000}$; at 5% by dividing by 7200, for $\frac{1}{360}$ of $\frac{5}{100} = \frac{1}{7200}$.

2. What is due March 1, 1872, to a person who deposited \$1500 Feb. 1, \$750 Feb. 12, \$950 Feb. 19, and \$2000 Feb. 28; and drew out \$575 Feb. 5, \$800 Feb. 14, and \$1000 Feb. 23; int. 4%?

Ans. \$4.08.

3. What would be the balance due in the previous example, taking 360 days to the year, at 4%; what at 3%; what at 5%?

Ans. \$4.14; \$3.11 —; \$5.18 —.

INTEREST ON PROMISSORY NOTES.

624. A **Promissory Note** is a written promise of the payment of a certain sum of money on demand, or at a specified time.

625. The **Face** of a note is the sum whose payment is promised. It is written in *words* in the body of the note, and in figures at the top or bottom.

626. The **Maker** of a note is the party who promises to make the payment. The *maker* of a note *signs* it.

627. The **Payee** is the party to whom it is made payable. A note is said to be "made in favor of" the *payee*. The owner of a note is called the *Holder*.

628. A **Time Note** is one made payable at a specified time. When no time is specified, the note is due on *demand*.

629. A **Joint Note** is a note signed by two or more persons who are jointly liable for its payment.

630. A **Joint and Several Note** is a note signed by several persons who are both jointly and singly liable for its payment.

631. A **Principal and Surety Note** is one in which another person becomes security for the payment of the note.

A *surety note* should be made payable to the order of the *surety*, who should indorse it on the back to the order of the creditor. It is held that a note made in favor of the creditor and indorsed by the surety, does not bind the latter to the payment of the debt.

632. A **Negotiable Note** is a note that can be transferred from one party to another. A note is negotiable when made payable to "bearer," or to the "order" of the payee.

633. The **Indorser** of a note is the party who puts his name on its back as security for its payment. The writing of the name on the back of a note is called an *indorsement*.

It is customary in raising money on notes, to obtain one or more responsible indorsers as security for its payment. If the maker refuses to pay the note, when due, each indorser is liable for its whole amount in the order of signing, unless he writes above his name "without recourse," or unless there is an agreement between two or more indorsers to share the loss.

When the maker fails to pay a note, it is usual for the holder to *make his demand* on the last *liable* indorser, who pays the note and

then gets the amount from the preceding indorser, and so on, up to the first indorser. The holder, however, has the option of collecting the amount from *any* liable indorser, and when so collected, all *subsequent* indorsers are released, the indorser who pays becomes the holder, and may collect from any *prior* liable indorser, and so on up to the first.

634. A note payable to *bearer* is negotiable without indorsement. A note payable to *order* must be indorsed by the payee before it is negotiable. A note not negotiable may be transferred by *assignment.*

635. An **Indorsement in Blank** is simply the name written on the back of the note. A *special indorsement* is an order for the note to be paid to a particular person.

A note should contain the words "value received," otherwise the holder may be required to prove that value was received. The words "without defalcation" are inserted in Pennsylvania to make a note negotiable; in New Jersey, "without defalcation or discount;" and in Missouri, "negotiable and payable without defalcation or discount."

If a note reads "with interest," it draws interest from date; otherwise it draws interest from the time of maturity until paid. A note may draw interest from a particular time after date, if so specified in the note. When no rate is mentioned the legal rate of the State is understood.

636. The **Maturity** of a note is its becoming legally due at the expiration of the time.

637. **Days of Grace** are the three days allowed in some States by law for the payment of a note after the expiration of the time specified in the note.

In several of the States, as New York, Pennsylvania, New Jersey, Illinois, etc., days of grace are abolished by law.

When grace is allowed the note matures on the *last day of grace.* When no grace is allowed it matures at the expiration of the time specified. If a note is payable *on demand,* it is legally due when presented.

If a note becomes legally due on Sunday or a legal holiday, it must be paid in most States on the day preceding. In Pennsylvania it is to be paid the next business day following. In Connecticut three days grace is allowed on notes for $35 or more, but not on notes for a less amount; if the last day is a legal holiday falling on Sunday, the note is due on Monday. In Maine and Nebraska, if the last day is a legal holiday falling on Monday, the note is payable on Tuesday; and in New York a note maturing on a legal holiday, or Monday observed as such holiday, is payable the following day. The following notation indicates when a note is nominally and legally due: May 6 | 9, 1875.

When the time of a note is stated in months, calendar months are meant. A note for 4 months dated October 29, 30 or 31, would expire on the last day of February, and be legally due March 3.

638. A **Protest** is a written declaration made by a *notary public,* that the maker of a note has failed to pay it.

The neglect to protest a note on maturity releases an indorser from all obligation to pay it, unless the words "waiving demand and notice" appear above the indorser's signature.

There are two modes of estimating the time between different dates. The first is by compound subtraction, which is still generally used in partial payments. The second is by determining the number of entire years, if any, and then reckoning the exact number of days left. This latter method is now generally adopted by merchants in finding interest on items in an account, and for calculations for short periods, and will be used in the following examples.

639. The **Principal Kinds** of notes will now be given, and the calculation of the interest upon them required.

DEMAND NOTE.

$195.75. MILLERSVILLE, PA., MAY 9, 1884.

For value received, I promise to pay David M. Sensenig, or order, on demand, One Hundred and Ninety-five $\frac{75}{100}$ Dollars, without defalcation. E. O. LYTE.

1. ### TIME NOTE.

$750. LANCASTER, PA., MARCH 15, 1884.

Sixty days after date, I promise to pay N. C. Fetter, or bearer, Seven Hundred and Fifty Dollars, with interest, for value received, without defalcation. FRANK ALBERT.

What will be due at maturity? *Ans.* $757.50.

2. ### PRINCIPAL AND SURETY NOTE.

$4500. ST. LOUIS, MO., JUNE 23, 1885.

Two months after date, I promise to pay H. S. Snyder, or order, Forty-five Hundred Dollars, with interest, value received, negotiable and payable without defalcation or discount. GEO. W. HULL.

Surety, H. S. SNYDER.

What will be due at maturity? *Ans.* $4548.

3. ### JOINT NOTE.

$675.75. NEWARK, N. J., JAN. 11, 1885.

On demand, for value received, we promise to pay John Arnold, or order, Six Hundred and Seventy-five $\frac{75}{100}$ Dollars, with interest, without defalcation or discount.

DANIEL SHOTWELL.
EDWARD UNDERHILL.

What will be due April 17, 1885? *Ans.* $686.56.

NOTE.—Days of grace are reckoned on Ex. 2.

4. **JOINT AND SEVERAL NOTE.**

$250.25. PHILADELPHIA, APRIL 9, 1888.

Six months after date, we jointly and severally promise to pay Margaret Wilson, or order, Two Hundred and Fifty $\frac{25}{100}$ Dollars with interest, at 7%, for value received, without defalcation.

SARAH E. HUDSON.
ANNA E. HARTMAN.

What will be due at maturity? *Ans.* $259.15.

5. **COMPANY NOTE PAYABLE AT A BANK.**

$575. PHILADELPHIA, JUNE 11, 1887.

Ninety days after date, we promise to pay to J. B. Lippincott & Co., or order, at the Philadelphia National Bank, Five Hundred and Seventy-five Dollars, for value received, without defalcation.

SOWER, POTTS & CO.

What is this note worth, Nov. 1, 1887? *Ans.* $580.08.

6. A 60-day note for $350, without interest, was paid in 90 days; what was the amount due? *Ans.* $351.75.

7. A 30-day note for $525, with interest from date, was paid in 80 days; what was the amount due? *Ans.* $532.

8. What is the difference in interest between a note for $500, given Jan. 1, 1885, due 2 months after date, and one given at the same date for the same amount, due 60 days after date? *Ans.* 0.08\frac{1}{3}$.

ANNUAL INTEREST.

640. Annual Interest is the simple interest of the principal and of each year's interest from the time of its accruing until settlement.

641. Annual Interest is sanctioned by some States when the note is written "with interest payable annually."

Simple Interest is not due and cannot be collected until the principal is due, unless the note reads "with interest payable annually." *Annual Interest* allows interest on the *unpaid interest* of a debt as well as upon the debt itself.

Annual Interest differs from *Compound Interest*, since in *Compound Interest* each year's interest is added to the principal, and the sum forms a new principal for the succeeding year.

The neglect to collect the annual interest on a note drawn "with interest payable annually," is, in some States, regarded as a waiving of the *contract* requiring it.

1. What is the amount due on a note of $500, at 6%, for 3 yr. 6 mo., interest payable annually?

SOLUTION.—The interest on $500 for one year is $30, and for 3 yr. 6 mo. is $105; the first year's interest is on interest 2 yr. 6 mo. giving an interest of $4.50; the second year's interest draws interest for 1 yr. 6 mo. amounting to $2.70; the third year's interest is on interest 6 mo., drawing $0.90; adding the interest on the principal, the interest on each year's interest, and the principal, we have $613.10 as the amount due. Hence the

OPERATION.

$\$500 \times .06 =$	$30,	int. for 1 yr.
$\$30 \times 3\frac{1}{2} =$	$105,	int. for $3\frac{1}{2}$ yr.
$\$30 \times .15 =$	4.50,	int. on 1st int.
$\$30 \times .09 =$	2.70,	int. on 2d int.
$\$30 \times .03 =$	.90,	int. on 3d int.
	500.00,	principal.
	$613.10	

Rule.—I. *Find the interest on the principal for the given time and rate; also find the interest on each year's interest for the time it has remained unpaid.*

II. *The sum of these interests will be the annual interest, and this, added to the principal, will be the amount due.*

NOTE.—The work may be shortened by calculating the interest for the sum of the times during which the different interests remain unpaid.

2. What is the interest due on a note for $750, dated Sept. 3, 1883, interest payable annually, if no payments are made until March 1, 1887? *Ans.* $169.50.

3. How much is due Sept. 1, 1889, on a note for $720, dated May 13, 1885, interest payable annually at 7%, if the yearly interest has been regularly paid? *Ans.* $735.54.

4. $875.00. DETROIT, MAY 9, 1885.

For value received, I promise to pay to R. T. Cornwell, Esq., or order, on demand, Eight Hundred and Seventy-five Dollars, interest payable annually.

J. WILLIS WESTLAKE.

What was the amount of the note, Jan. 18, 1889, no payments having been made? *Ans.* $1085.66.

5. $1000. TRENTON, JAN. 11, 1879.

For value received, I promise to pay to the order of Charles Parker, on demand, without defalcation or discount, One Thousand Dollars, with interest annually.

SAMUEL DECOU.

What was due on this note March 17, 1883, if the interest *was paid* up for the first two years? *Ans.* $1135.73.

PARTIAL PAYMENTS.

642. Partial Payments are payments in part of notes or other obligations bearing interest.

643. An **Indorsement** is an acknowledgment of a payment, written on the back of the obligation, stating the time and amount of the payment.

The term *Indorsement* is used in different business papers, in each case meaning a *writing on the back*, from the Latin *dorsum*, the back.

1. The writing of the name on the back of a check, draft, note, etc., is called a *General Indorsement*, or an *indorsement in blank*.
2. A *Special Indorsement* directs the obligation to be paid to some particular person or to his order.
3. An acknowledgment of the payments on a note, written on the back of it, is also an indorsement. The person holding the obligation signs his name to this statement as a receipt.

644. The Supreme Court of the United States, and nearly all the States, adopt the following rule for partial payments, called

THE UNITED STATES RULE.

I. *Find the amount of the principal to the time of the first payment; if the payment equals or exceeds the interest, subtract the payment from the amount and treat the remainder as a new principal.*

II. *If the payment is less than the interest, find the amount of the same principal to the time when the sum of the payments shall equal or exceed the interest due, and subtract the sum of the payments from the amount.*

III. *Proceed in the same manner with the remaining payments until the time of settlement.*

NOTES.—1. This rule is founded upon the decision of Chancellor Kent. The principle is, that neither interest nor payment shall draw interest. It has been adopted by nearly all the States—New Hampshire, Vermont, and Connecticut being the principal exceptions.

2. Although the whole aim of legislative enactments and judicial decisions on the subject of interest, has been to disallow compound interest, yet this very rule really maintains the principle of compound interest in a most objectionable shape, for it makes interest due (not every year, as compound interest generally does) as often as a payment is made: by which it happens that the *more prompt* the debtor is in paying installments, *the greater his loss*. Thus, supposing the note to be for \$5000 at 6 per cent., and that the debtor pays \$25 every month, at the end of the year he still owes \$5000. If he had invested the \$25 every month, he would have had at the end of the year \$308.25 towards the payment, while the interest on the debt would only be \$300, leaving the debt \$4991.75, instead of \$5000.

1. $1000. HARRISBURG, JAN. 1, 1870.

Three years after date I promise to pay Joseph Hughes, or order, for value received, One Thousand Dollars, with interest from date, without defalcation.

WILLIAM WILSON.

Indorsements: July 8, 1870, $200. Oct. 1, 1870, $10. Sept. 25, 1871, $100. March 18, 1872, $400.

What was due Jan. 1, 1873?

OPERATION.

Principal, or face of note,	$1000 00
Interest to 1st payment, 6 mo. 7 da.,	31 17
Amount due July 8, 1870,	1031 17
Amount to be deducted,	200 00
Balance due after 1st payment,	831 17
Interest on balance to second payment, $11.50.	
The payment being less, is not deducted.	
Interest from 1st payment to 3d payment,	60 54
Amount due Sept. 25, 1871,	891 71
Sum of second and third payments to be deducted,	110 00
Balance due after third payment,	781 71
Interest on balance to fourth payment,	22 54
Amount due March 18, 1872,	804 25
Amount to be deducted,	400 00
Balance due after fourth payment,	404 25
Interest from March 18, 1872, to Jan. 1, 1873,	19 07
Balance due on settlement, Jan. 1, 1873,	423 32

2. $5600. PHILADELPHIA, JAN. 11, 1871.

For value received, on demand, I promise to pay James Jones, or order, Five Thousand Six Hundred Dollars, with interest, without defalcation. JOHN SMITH.

Indorsements: May 19, 1871, $500; Sept. 5, 1871, $200; Jan. 1, 1872, $300; April 17, 1872, $150.

What is due Jan. 11, 1873? *Ans.* $5060.54.

3. $2000. MINNEAPOLIS, AUG. 9, 1872.

For value received, sixty days after date, I promise to pay William B. Dana & Co., Two Thousand Dollars.

PHILIP BUTLER.

Indorsements: Dec. 7, 1872, $50; March 11, 1873, $35; July 25, 1873, $150; Oct. 12, 1873, $200; Jan. 1, 1874, $500.

What is due March 25, 1874? *Ans.* $1248.47.

4. $3870. NEW YORK, OCT. 9, 1873.

Thirty days after date, for value received, I promise to

pay F. Ibach, or order, without defalcation, Three Thousand Eight Hundred and Seventy Dollars, with interest at 5%. S. C. DELAP.

Indorsements: Jan. 1, 1874, \$500; June 11, 1874, \$750; Oct. 9, 1874, \$1000; Jan. 1, 1875, \$250; April 10, 1875, \$25; June 9, 1875, \$250.

What is the amount due Dec. 9, 1875? *Ans.* \$1347.13.

5. \$4000. MILLERSVILLE, PA., JULY 11, 1870.

Three months after date, I promise to pay Annie Lyle, or order, for value received, Four Thousand Dollars, without defalcation. JANE E. LEONARD.

Indorsements: Dec. 1, 1870, \$25; March 10, 1871, \$50; July 14, 1871, \$180; Jan. 1, 1872, \$200; April 25, 1872, \$450; Sept. 9, 1872, \$75; Jan. 1, 1873, \$300.

The note was paid Sept. 9, 1873; what was then due?
Ans. \$3357.09.

MERCHANTS' RULE.

645. Business men generally settle notes and interest accounts, payable within a year, by the following

Rule.—I. *Find the amount of the principal till the time of settlement, and also the amount of each payment till the time of settlement.*

II. *Subtract the amount of the payments from the amount of the principal; the remainder will be the balance due.*

NOTES.—1. In some States merchants apply this rule to notes for longer periods by reckoning the interest for 1 year, and subtracting from the amount the amounts of the payments made during the year, and taking this balance for a new principal.

2. As the periods in these notes are all short, the interest should be calculated for the number of days.

1. A note was given for \$5760, Sept. 20, 1869.

Indorsements: Nov. 30, \$200; Feb. 2, 1870, \$600; April 9, 1870, \$350.

What was due Sept. 20, 1870? *Ans.* \$4913.23.

2. A note was given for \$2500, April 1, 1873.

Indorsements: June 11, \$200; July 5, \$100; Sept. 9, \$450.

What is due 6 mo. from date, at 7%? *Ans.* \$1830.96.

3. A note was given for \$1750, May 11, 1870.

Indorsements: July 1, \$100: Aug. 12, \$45; Sept. 30, \$60; Jan. 19 1871, \$250; March 10, \$150.

What was due April 1, 1871, at 8%? *Ans.* \$1255.81.

NOTE.—For Connecticut and Vermont rules, *see page 508.*

TRUE DISCOUNT AND PRESENT WORTH.

647. Discount is an allowance made for the payment of money before it becomes due.

648. The **Present Worth** of a debt payable at a future time without interest, is such a sum as, being on interest for the time, at a certain rate, will amount to the debt.

649. The **True Discount** is the difference between the face of the debt and the present worth.

NOTES.—1. The *true discount* is the *interest* on the present worth for the time between the payment of the debt and the time it becomes due.
2. The present worth corresponds to the principal, discount to interest, and debt to amount; hence the cases may be solved as in Interest.

1. What is the present worth of $375, due 3 yr. 3 mo. hence, without interest, money being worth 6%?

SOLUTION.—The amount of $1 for 3 yr. 3 mo. at 6%, is $1.195; hence the present worth of $1.195 is $1, and the present worth of $375 is as many times $1 as $1.195 is contained times in $375, which is $313.81. Hence the

OPERATION.

$$\$0.06 \times 3\tfrac{1}{4} = \$0.195$$
$$\text{Amount of } \$1 = \$1.195$$
$$\$375 \div 1.195 = \$313.81$$

Rule.—I. *Divide the given sum by the amount of* $1 *for the given rate and time, to find the present worth.*

II. *Subtract the present worth from the given sum, to find the discount.*

NOTE.—When several payments are made without interest, find the present worth of each separately, and take their sum.

2. What is the present worth of $460.50, due 3 yr. 9 mo. 18 da. hence, without interest, at 6%? *Ans.* $375.

3. What is the discount of $401.05 due 5 yr. 7 mo. 11 da. hence, without interest? *Ans.* $101.05.

4. Which is worth most, $320 in 12 months, $310 in 6 months, or $300 cash, money worth 8%? *Ans.* The last.

5. Which is the more profitable, to buy pork at $15 a barrel, at 3 months, or at $16 at 6 months, money being worth 10%? *Ans.* First, $0.60.

6. A merchant buys $2500 worth of goods on 6 months credit, but settles his account by paying cash, a discount of 5% on the face of the bill being taken off. What was the discount, and how much does it exceed true discount, money worth 6%? *Ans.* Discount, $125; excess, $52.18

7. August 9, 1875, I received a note for \$700 at 6%, due Feb. 12, 1876, with interest; what will it be worth Oct. 8, 1875, if discounted at 8%? *Ans.* \$702.004.

8. A merchant having bought a bill of goods, is offered the choice between paying the amount, \$560, in 60 days, or paying cash at a discount of 3%; which will be more profitable, money worth 10%? *Ans.* The latter, \$7.62.

9. What will be gained by borrowing money at 6% to pay a debt of \$5800, due 6 months hence, if 5% is deducted from the face of the bill, for cash? *Ans.* \$124.70.

10. A merchant had two notes to pay, one for \$435.10, due Jan. 1, 1873, the other for \$769.84, due March 11, 1873; how much money did it take to pay both notes Sept. 19, 1872, money being worth 7%? *Ans.* \$1171.26.

11. Mr. Baker bought a house and lot March 25, 1871, for which he was to pay \$2700 on October 1 following, and \$2500 Jan. 1, 1872. If a discount of 8% was allowed for an immediate payment, how much would he gain by borrowing the amount, money being worth 7%, and how much must he borrow? *Ans.* Gain, \$198.11; sum borrowed, \$4784.

BANK DISCOUNT AND BANKING.

650. A **Bank** is an incorporated institution which receives and loans money, or furnishes a paper circulation.

651. A **Bank of Deposit** is one which receives money or its equivalent on deposit, to be drawn at the order of the depositor.

652. A **Bank of Discount** is one that lends money, discounts notes, drafts, etc.

653. A **Bank of Issue** is one that makes and issues notes to circulate as money.

Some banks unite two and some all of these offices. A *Savings Bank* is one that receives small sums on deposit, and pays interest to its depositors.

A bank is generally managed by a board of directors, elected by the stockholders; the principal officers are the *president* and *cashier*.

654. A **Check** is an order on a bank, given by one of its depositors, to pay a certain amount to some person or his

order, or to bearer. Checks drawn "to order" must be indorsed when presented for payment.

655. Bank Discount is the interest on the face of the note from the day of discount to the day of payment.

656. The **Proceeds** or **Avails** of a note is the sum received for it when discounted, and equals the face less the discount.

657. The **Term of Discount** is the number of days from the day of discount to the day of maturity of the note.

When a person wishes to borrow money at a bank, he presents a note, either made or indorsed by himself, payable at a certain time, and receives for it a sum equal to the face *less* the interest for the time the note has to run. This amount is withheld by the bank in consideration of advancing money on the note prior to its maturity.

In Pennsylvania, Delaware, Maryland, Missouri, and the District of Columbia, the *day of discount* and *day of payment* are both reckoned. A 60-day note in Pennsylvania would be discounted for 61 days, in the others named (grace being allowed) for 64 days.

Notes are often discounted by business men, who deduct the interest for a given time, with or without grace, as may be agreed upon. The rate is fixed by agreement, and is usually greater than the legal rate. A check should always be dated on the day it is issued.

658. The difference between bank discount and true discount may be shown as follows:

If I take my note to the bank promising to pay \$106 at the end of 1 year, to get it cashed, by the method of true discount I would receive \$100; but by the method of bank discount, not counting days of grace, I would receive \$106 minus the interest of \$106 for one year, that is, \$106 − \$6.36 = \$99.64.

CASE I.

659. ***Given, the face of the note, the rate, and the time, to find the discount and the proceeds.***

1. What is the present worth, or proceeds, of a note for \$350, due in 18 days, discounted at a bank at 6%?

SOLUTION.—The interest on \$350 for 18 da. is \$1.05, which is the discount. Subtracting this from \$350, we have the proceeds, which equal \$348.95.

OPERATION.

$$\begin{array}{r} \$350 \\ .003 \\ \hline \$1.050 \end{array} \qquad 18 \div 6 = 3$$

\$350 — \$1.05 = \$348.95.

Rule.—I. *Find the interest on the face of the note for the time; the result will be the discount.*

II. *Subtract the discount from the face, to find the present worth.*

NOTE.—The discount of an *interest-bearing* note is computed on the *amount* of the note at its maturity.

2. A note for $375, due in 60 days, was discounted at a bank at 7%; what was the discount? *Ans.* $4.375

3. A note for $750, at 96 days, was discounted at a bank at 7%; required the proceeds. *Ans.* $736.

4. Required the difference between the true discount and the bank discount of a note for $3500, due in 1 yr. 10 mo., at 6%, without grace. *Ans.* $38.15.

5. Sold goods to the amount of $475.60, for which I received a note at 6 mo.; wishing the money, I got it discounted in 30 days. What were the proceeds? *Ans.* $463.71

Find the time when due, the time to run, the discount, and the proceeds of the following notes:

6. 745\frac{85}{100}$. CINCINNATI, NOV. 12, 1887.

Four months after date, for value received, I promise to pay James Curry, or order, Seven Hundred and Forty-five $\frac{85}{100}$ *Dollars.* JOHN VANHORN.

Discounted Dec. 2, 1887, at 6%.

Ans. Due March 12|15, 1888; 104 da. to run; proceeds, $732.92; discount, $12.93.

7. $625. LANCASTER, JUNE 19, 1883.

Six months after date, I promise to pay J. V. Montgomery, or order, for value received, Six Hundred and Twenty-five Dollars, without defalcation. T. R. BAKER.

Discounted July 29, 1883, at 6%.

Ans. Due Dec. 19; 144 da. to run; proceeds, $610; discount, $15.

8. $1075. NEWARK, JULY 11, 1889.

Three months after date, for value received, I promise to pay John Stratton, or order, One Thousand and Seventy-five Dollars, with interest, without defalcation or discount.

G. H. RICHARDS.

Discounted Sept. 1, 1889, at 7%.

Ans. Due Oct. 11, 1889; 40 days to run; proceeds, \$1082.99; discount, \$8.49.

9. \$2000. DETROIT, SEPT. 9, 1887.

Six months after date, for value received, I promise to pay John Winchell, or order, Two Thousand Dollars, with interest at 7%. EDWARD WINTHROP.

Discounted Nov. 11, 1887, at 10%.

Ans. Due March 9|12, 1888; 122 da. to run; proceeds, \$2001.72; discount, \$70.22.

CASE II.

660. ***Given, the rate, the time, and the proceeds or discount, to find the face.***

1. I wish to borrow \$600 from a bank; for what must I give my note at 60 days, discount at 6%?

SOLUTION.—We find the interest of \$1 for 60 days, and subtract it from \$1, which gives the proceeds of \$1. If for every \$1 in the face of the note, the proceeds are \$0.99, to give \$600 proceeds will require as many times one dollar as \$0.99 is contained times in \$600, which are \$606.06+.

OPERATION.

$$\begin{array}{r} 1.00 \\ .01 \\ \hline .99 \end{array}, \text{ proceeds of \$1.}$$

$$\frac{\$600}{.99} = \$606.06+$$

Rule.—*Divide the given proceeds by the proceeds of* \$1 *for the given time and rate; or, divide the discount by the discount of* \$1.

2. What is the face of a note at 90 days, the proceeds of which, when discounted at 6%, are \$443.25? *Ans.* \$450.

3. For what sum must a note be drawn at 30 days, to produce \$2000 when discounted at 5%? *Ans.* \$2008.37.

4. A broker buys a 90-day note for \$11.81¼ less than the face; what was the face, discount 7%? *Ans.* \$675.

5. Required the difference between the present worth and proceeds of \$500 due in 4 yr. 2 mo., at 6% *Ans.* \$25.

6. Mr. Bowman buys goods in Philadelphia to the amount of \$5764.75, and gives in payment his note for 3 months at 4½%; what must be the face of the note? *Ans.* \$5830.34.

7. I give a 90-day note to pay a debt of \$657.50; what must be the face of the note to yield the exact debt if discounted at 2% a month, allowing grace? *Ans.* \$700.96.

8. John Johnson discounted at the Bank of Commerce, a note made by Edward Wilson, having 120 days to run; he obtained $645.75; what was the face of the note, discount being $1\frac{1}{2}\%$ a month? *Ans.* $686.97.

CASE III.

661. ***Given, the face, the rate, and the proceeds or the discount, to find the time.***

1. The proceeds of a note for $500, discounted at 6%, were $492.50; what was the time?

SOLUTION.—We subtract $492.50 from $500, which gives the discount, $7.50. If on $500 the discount is $7.50, on $1 the discount will be as many dollars as $500 is contained times in $7.50, or $0.015. The discount on $1 for 1 day is $\frac{1}{6}$ of a mill, therefore the note was given for as many days as $\frac{1}{6}$ of a mill is contained times in $0.015, or 90 days. Hence the time was 90 days.

OPERATION.

$$\begin{array}{r} \$500.00 \\ 492.50 \\ \hline \$7.50, \text{ discount.} \end{array}$$

$$\frac{7.50}{500} = .015, \text{ discount on } \$1.$$

$$\frac{.015}{.000\frac{1}{6}} = 90 \text{ days.}$$

Rule.—*Divide the discount on* $1 *by the interest on* $1 *for one day.*

NOTE.—If grace is allowed, we subtract 3 days from the quotient.

2. A broker buys a note for $20 discount, the face being $2000; what was the time, discount 6%? *Ans.* 60 da.

3. A merchant sold a consignment of tobacco for $7470, and received a note, which being discounted, yielded $7245.90; what time had the note to run? *Ans.* 6 mo.

4. Mr. Martin, owing $1000, gave a note for $1038.96. which was discounted at $1\frac{1}{4}\%$ a month; how long had it to run if the proceeds discharged the debt? *Ans.* 90 days.

5. A note dated April 1, 1874, was discounted May 10 at 8%; the face was $745.85, and the proceeds $736.73; how long did it run after it was discounted? *Ans.* 55 days.

6. A note dated June 11th, 1874, at 3 months, was discounted at a Baltimore bank at 7%; the face of the note was $600, and the proceeds $591.13; what was the date of discount, grace being allowed? *Ans.* July 1st.

7. An interest-bearing note dated July 1st, 1873, at 6 months, was discounted at 5% ; the face of the note was \$750, and the proceeds \$762.44 ; what was the date of discount? *Ans.* October 30.

8. An interest-bearing note for \$1200 at 10%, dated Cleveland, March 1, 1877, at 3 months, was discounted in New York at 6%, the proceeds being \$1226.53; what was the date of discount, allowing grace? *Ans.* May 10.

CASE IV.

662. ***Given, the face, the time, and the proceeds or the discount, to find the rate.***

1. The proceeds of a note for \$800, at 60 days, are \$790.66⅔ ; what is the rate?

SOLUTION.—We find as in Case III., the discount on \$1 for the given time and required rate to be \$0.01⅙; the discount on \$1 for 60 days at 1% is ⅙ of a cent; hence the required rate will be as many times 1% as ⅙ of a cent is contained times in \$0 01⅙, which is 7%.

OPERATION.

$$\begin{array}{r} \$800.00 \\ 790.66\tfrac{2}{3} \\ \hline 9.33\tfrac{1}{3} \end{array}\text{, discount.}$$

$$\frac{9.33\frac{1}{3}}{800} = .01\tfrac{1}{6}.$$

$$.01\tfrac{1}{6} \div .00\tfrac{1}{6} = 7.$$

Rule.—*Divide the discount on \$1 by the interest on \$1 at 1% for the given time.*

2. A broker buys a note for \$20 discount at 60 days, the face being \$2000 ; what was the rate? *Ans.* 6%.

3. A merchant buys goods to the amount of \$2500, and to pay for them gets his note for 90 days discounted at a bank ; if the face is \$2596.875, what is the rate? *Ans.* 1¼% a mo.

4. A note dated July 18, 1876, at 3 months, was discounted at a Philadelphia bank Aug. 1 ; the face was \$600 and the proceeds \$590.78⅓ ; what was the rate? *Ans.* 7%.

5. A note bearing interest at 6%, dated Aug. 13, 1876, at 6 months, was discounted November 3 ; the face of the note was \$1150, and the proceeds \$1168.475 ; what was the rate of discount? *Ans.* 5%.

CASE V.

663. ***Given, the rate of bank discount to find the corresponding rate of interest, or the rate of interest to find the corresponding rate of bank discount.***

1. If I discount a 60-day note at 2% a month, what rate of interest do I obtain for my money?

SOLUTION.—The discount on \$1 for the given time and rate is \$0.04; and the proceeds of \$1 are \$0.96, which is the amount paid for \$1 of the face. The interest on \$1 for 60 days at 1% is \$0.00$\frac{1}{6}$, and on \$0.96 is .96 times \$0.00$\frac{1}{6}$, or \$0.0016. Dividing \$0.04, the discount at the required rate, by \$0.0016, the interest at 1%, we have 25%, the rate of interest.

OPERATION.

.04, int. on \$1.

$$\begin{array}{r} 1.00 \\ .04 \\ \hline .96 \end{array}$$, proceeds of \$1.

$.96 \times .00\frac{1}{6} = .0016$, int. at 1%.

$.04 \div .0016 = .25$.

2. At what rate must I discount a 30-day note, to obtain 2% interest a month?

SOLUTION.—The amount of \$1 for the given time and rate is \$1.02, which is the face of a note costing \$1. The dis count on \$1 at 1% is \$0.0008$\frac{1}{3}$, and on \$1.02 it is \$0.00085. Dividing \$0.02, the discount at the required rate, by \$0.00085, the discount at 1%, we have $23\frac{9}{17}$, the rate of discount.

OPERATION.

.02, int. on \$1.

$$\begin{array}{r} 1.00 \\ .02 \\ \hline 1.02 \end{array}$$, amt. of \$1.

$1.02 \times .0008\frac{1}{3} = .00085$.

$.02 \div .00085 = 23\frac{9}{17}$.

Rule I.—*Find the discount and the proceeds of* \$1 *for the given time, and divide the discount by the interest of the proceeds at* 1 *per cent. for the same time.*

Rule II.—*Find the interest and the amount of* \$1 *for the given time, and divide the interest by the discount of the amount at* 1 *per cent. for the same time.*

3. A note payable in 60 days is discounted at $1\frac{1}{2}$% a month; what is the rate of interest? *Ans.* $18\frac{54}{97}$.

4. What is the rate of interest on a note for 8 mo. at a discount of 10, 12 or 15%? *Ans.* $10\frac{5}{7}$, $13\frac{1}{23}$, $16\frac{2}{3}$.

5. What rate of interest is paid when a note running 1 year is discounted at 6, 7, 8, 9, or 10%?

Ans. $6\frac{18}{47}$, $7\frac{49}{93}$, $8\frac{16}{23}$, $9\frac{81}{91}$, $11\frac{1}{9}$.

6. At what rate should a 90-day note, with grace, be discounted to produce 10% interest? *Ans.* $9\frac{28551}{38161}$.

7. At what rate should a 6 month note be discounted to produce 1, $1\frac{1}{2}$, 2, or 3 % a month?

Ans. $11\frac{17}{53}$, $16\frac{56}{109}$, $21\frac{3}{7}$, $30\frac{30}{59}$.

SAVINGS BANK ACCOUNTS.

664. Savings Banks are institutions intended to receive on deposit small sums of money, and to return the same with a moderate interest at some future time.

Savings banks differ from other banks in paying interest on all sums of money above a certain amount (generally $1 or $5) deposited with them, and in adding this interest, if not withdrawn, to the principal and paying interest on the amount. They, therefore, pay compound interest.

A savings bank furnishes each depositor with a book, in which are recorded the sums deposited and the sums drawn out. It is customary to add to each depositor's account, at the end of a fixed term, the interest due on his deposits for that term.

Interest is usually declared Jan. 1st and July 1st of each year, and when declared is carried at once to the credit of each depositor and is then entitled to interest. No interest is allowed on the fractional parts of a dollar, nor on any sum withdrawn previous to the first day of January or July for the time it may have been on deposit since the last dividend.

Deposits are generally paid on demand, though the banks reserve a right to require notice. In some banks deposits begin to draw interest at the beginning of each quarter; in others, on deposits made on or before the first of any month, interest is allowed from the first day of that month.

NOTE.—In the following examples interest will be reckoned both from the beginning of the quarter and the first of the month. The rate will be 4%, unless otherwise stated. Notice that 4% per annum is 2% for 6 mo., 1% quarterly.

EXAMPLES FOR PRACTICE.

1. The following deposits were made in a savings bank: Dec. 10, 1895, $100; Feb. 12, 1896, $25; Mar. 20, $15; May 16, $50; Aug. 30, $70; what amount was due Jan. 1, 1897?

SOLUTION.—As the account is to be balanced semi-annually, if the interest is reckoned from the 1st of the quarter, the first deposit, $100, on July 1, is on interest 6 mo., and amounts to $102; the second and third, $25+$15=$40, are on interest 3 mo., and amount to $40.40; the fourth, $50, draws no interest. Hence the balance due July 1 is $192.40, which on Jan. 1 amounts to $196.24; the fifth deposit amounts to $70.70, and the balance due Jan. 1, 1897, is $266.94.

OPERATION.

$102,	amt. of 1st deposit.
40.40	" 2d "
50	" 3d "
$192.40,	bal. due July 1.
$196.24,	amt. of bal.
70.70,	" 4th dep.
$266.94,	bal. due Jan. 1.

If the interest is reckoned from the first of the month, the first deposit is on interest 6 mo. as before, and amounts to $102; the second 4 mo., and amounts to $25.33; the third 3 mo., and

amounts to $15.15; the fourth 1 month, and amounts to $50.17, and the balance, July 1, is $192.65, which during the next term amounts to $196.49; the 4th deposit being on interest 4 mo., amounts to $70.93, and the amount due Jan. 1, 1897, is $267.42.

2. Mr. Arnold made the following deposits in a savings bank: Jan. 5, 1893, $50; April 11, $47.50; May 15, $18.25; Aug. 25, $46.75; Oct. 19, $30; Dec. 18, $19.50; what was the balance due Jan. 1, 1894? *Ans.* $215.28; $216.33.

3. Balance the following account July 1, 1896: Balance on hand, Jan. 1, $205; Deposits, Feb. 1, $20; Mar. 20, $45; April 30, $70. Withdrawals, Mar. 10, $15; April 15, $10; May 5, $25; June 11, $50.

OPERATION.

Date Int. Com.	Bal. 1st of Quar.	Smallest Bal. during Quar.	Int. for Quar.	Int. for 6 Mo.
Jan. 1	205	205	2.05	
April 1	255	240	2.40	
July 1	244.45			

SOLUTION.—If interest commences the first of each quarter, for the first quarter we shall have the interest on $205, which is $2.05. The balance on hand, April 1, is $255, but the smallest balance during the quarter is $240, which is the balance at the close, and therefore draws interest for the quarter amounting to $2.40, making the interest for the term $4.45, which we add to the balance, making $244.45 as the balance to begin the new term.

If interest is reckoned from the first of the month, we shall have the interest on $205 for 6 mo.; the interest on $5 from Feb. 1 for 5 mo.; the int. on $30 from Apr. 1 for 3 mo., which being added, give $4.48, the balance on hand July 1 being $244.48.

4. Jane Osborne had the following account with the Franklin Saving Fund: Deposits, Mar. 11, 1895, $195; July 10, $75; Aug. 25, $45.50; Nov. 1, $62.50; Apr. 10, 1896, $43.50. Withdrawals, May 1, 1895, $25; June 7, $50; Sept. 9, $18.45; Jan. 1, 1896, $100; May 10, $27.50. Balance her account, July 1, 1896. *Ans.* $208.97; $210.10.

5. Required the balance due July 3, 1897, on Mary Simpson's account with the Union Savings Bank. Deposits, Jan. 10, 1896, $61.25; April 7, $72.81; July 9, $52.75: Sept. 20, $48; Feb. 12, 1897, $87.50; May 17, $75. Withdrawals, Feb. 9, 1896, $15.10; May 19, $24.16; Aug. 15, $16; Oct. 20, $15.75; Mar. 12, 1897, $23.81. *Ans.* $309.49; $310.83.

STOCK INVESTMENTS WITH INTEREST.

665. In **Stock Investments** operators take into consideration the interest on the money invested.

In the previous articles on stock investments, no account was taken of *time;* to know accurately, however, the gain or loss on an investment, we should consider the interest on the money used.

In speculating in stocks, a person frequently does not pay for what he buys, but merely deposits with his broker a certain part of the value in money or securities, to secure the latter against loss. This deposit is called a "margin," and varies with the character of the stock. Thus, on reliable stocks, the margin would be about 10%, while on fancy stocks it might be as high as 20%. This margin must be kept up by the speculator; that is, if the stock falls before he is ready to sell, he must increase his deposit. Often no stock is actually bought or sold, but at the end of a certain time the account between the broker and speculator is settled just as if the stock had been bought and sold at current rates.

Note.—As the following examples are worked principally by a combination of methods previously given, it is unnecessary to divide them into cases. Brokerage at $\frac{1}{8}$% is reckoned on all purchases and sales. Money is regarded as worth 6% unless otherwise stated.

1. What is the annual rate of interest of an investment which pays 3% semi-annually, if reinvested at 6%?

Solution.—The interest for the first half-year may be on interest during the second half-year, at 6%; hence, at the end of the year the interest for the first half-year will amount to \$.03×1.03, or \$.0309, which added to \$.03, the interest of the second half-year, gives \$0.0609 as the yearly interest on \$1.

OPERATION.

$\$.03 \times 1.03 = \$0.0309.$

$\$0.03 + \$0.0309 = \$0.0609.$

2. When the Reading Railroad pays $2\frac{1}{2}$% dividend quarterly, what yearly dividend will be equal to this, money being worth 6%? *Ans.* $10\frac{9}{40}$%.

3. If I buy 20 shares Central Transportation Company at $48\frac{1}{2}$ (50) and receive \$30 dividend quarterly, what annual rate of interest do I receive? *Ans.* $12\frac{240}{389}$%.

4. If I buy 12 shares Bank of North America (100) at 240, Jan. div. $17\frac{1}{2}$%, July div. $12\frac{1}{2}$%, what rate of interest do I get for the year reckoning from July? *Ans.* $12\frac{678}{961}$%.

5. Mr. Whitmore bought \$4000 first bonds Union Pacific R. R. (int. 6% gold) at 90; after two years he sold them at $104\frac{1}{8}$; what did he make more than by loaning the money at 6%, gold averaging $112\frac{1}{2}$? *Ans.* \$663.10.

6. Bought 50 shares Harlem R. R. (100) at 133; what

would I gain or lose by buying 15 days later at 130, ex-div-4%, my money lying idle during the time, and what if my money was returning 8%? *Ans.* \$50 loss; \$27.79 loss.

7. I bought 40 shares of Reading R. R. at 52, (50) May 12, and having received dividends at $2\frac{1}{2}$% July 20th and October 20th, I sold them on the 16th of December at $53\frac{1}{4}$; what per cent. more did I obtain than if the money had been invested at 6%? *Ans.* $3\frac{2047}{12510}$%.

8. I ordered my broker to buy for me 600 shares of Lake Shore Railroad (100) at 58, depositing \$7000 as "margin;" 25 days afterward he sold them at 61; what was my gain per cent., interest 6%? *Ans.* $19\frac{257}{336}$%.

9. On April 1st, I bought 25 shares N. Y. Central (100) at 102 (div. 4% 10th of Feb. and Aug.) and 16 shs. N. J. Central (100) at 97 (div. 2% quarterly, 15th of Jan., Apr., July, Oct.); and sold them both on Jan. 2d following, the former at $103\frac{1}{4}$ and the latter at 101; what more did I gain % than if I had loaned the money at 7%? *Ans.* $1\frac{587911}{14804110}$%

10. Buy in June 25 shares Pennsylvania R. R. stock, par 50, and receive in December 5% dividend and in June 3% cash and 5% scrip worth 105; what do I realize after receiving the last dividend, and what is the actual interest on the par value of the stock? *Ans.* \$167.50; $13\frac{3}{5}$%.

11. Buy 250 shares Lehigh Valley R. R. (50) at $58\frac{3}{4}$, received July 15th 5% dividend, and January 15th 3% dividend and the privilege of buying 1 share at par for every 4 shares and fraction of a share; these latter I sell at $57\frac{1}{4}$; what did I receive in a year? *Ans.* $10\frac{1}{3}$%.

12. I bought 500 shs. of Erie (100) at $16\frac{1}{4}$, depositing with my broker \$1625 as "margin" to secure him; 15 days after he sold them for $17\frac{3}{4}$; how much must I receive from the broker beside the deposit, interest at 7%? *Ans.* \$480.68.

13. I bought of Hassler & Co., Bankers, 75 shares Delaware, Lackawanna and Western R. R. stock (50), *b* 60, for 57; in 30 days drew a dividend of 4%, and at the end of 60 days sold the stock for 59; what was my actual gain, money being worth 6%? Ans. \$239.16

14. A speculator bought on the 15th of December 1000 shares of Harlem Railroad at $127\frac{1}{2}$, depositing government bonds worth $12,750 as a margin. He sold the stock on January 31st, for 127, having received a dividend of 5%; how much does he gain, interest at 7%? *Ans.* $2949.03.

EXCHANGE.

666. Exchange is the method of making payments in distant places by means of *Drafts* or *Bills of Exchange*.

667. Exchange is of two kinds, *Domestic* and *Foreign*. Exchange between two places in the same country is called *Domestic* or *Inland Exchange;* that between different countries is called *Foreign Exchange.*

668. A **Draft** or **Bill of Exchange** is a written order for the payment of money. In domestic exchange it is usually called a *Draft.*

669. A **Sight Bill** is one payable "at sight," or on its presentation. A *Time Bill* is one payable at a specified time after sight or after date.

670. The **Drawer** of a bill is the party who signs it. The *Drawee* is the party to whom the bill is addressed.

671. The **Payee** is the party to whom or to whose order the bill is payable. The *Owner* or *Holder* is the party who has possession of the bill, and the person remitting it is called the *Remitter*.

672. The **Indorsement** of a bill is the writing upon the back of it, by which the payee transfers the payment to another.

A *special* indorsement is an order to pay the bill to some particular person, who is then called the *Indorsee,* and he alone can collect the bill. An indorsement *in blank* is the writing of the holder's name upon the back, which makes the bill payable to the bearer.

673. The **Acceptance** of a bill is the promise of the Drawee, when presented, to pay it at maturity. The Drawee *accepts* by writing across the face of the bill, "Accepted," with the date and his signature; the bill is then called an *Acceptance,* and is of the character of a promissory note.

If a bill is protested for non-acceptance, the drawer is under obligations to pay it immediately, although the time specified in it has not expired. Bills of exchange are entitled to "days of grace" according to the custom of the place where the draft is payable, unless a particular day is named. In New York, Pennsylvania, etc., no grace is allowed, and in most States no grace is allowed on sight drafts. If a note is payable on demand, it is legally due when presented, as bank-notes, etc. If a particular time is specified in a note, it is legally due on that day.

When a bill is drawn "acceptance waived," it is not subject to protest until maturity. When an indorser writes over his name, "demand and notice waived," he is liable even if the bill is not protested. If the indorser writes "without recourse" over his indorsement, he is not liable for the payment of the bill.

On ordinary time drafts, it is not necessary to write the date with the "acceptance;" but on sight drafts or those due a number of days specified after acceptance, it is necessary, to fix the time of payment.

In reckoning the time of maturity of a bill payable after date, the day on which it is dated is not included, and in the case of a bill payable after sight, the day of presentment is not included.

674. The **Rate of Exchange** is the rate per cent. which is reckoned upon a draft.

675. The **Course of Exchange** is the current price paid in one place for bills of exchange on another.

The *brokerage* is usually included in the quotation of exchange.

676. The **Par of Exchange** is the established value of the monetary unit of one country in the monetary unit of another; it is either *intrinsic* or *commercial.*

677. The **Intrinsic Par** is the standard of real value, as determined by the weight and purity of the coins of different countries.

678. The **Commercial Par** is the standard of value, as determined by the nominal or market price of the coins of different countries.

679. Exchange is at *par* when a draft sells for its face; at a *premium* when it sells for more than its face; and at a *discount* when it sells for less than its face.

The rate of exchange between two places or countries depends upon the course of trade. If the trade between New York and St. Louis is equal, exchange is at par. If New York owes St. Louis, the demand in New York for drafts on St. Louis is greater than the demand in St. Louis for drafts on New York, hence the drafts are at a *premium* in New York. But if St. Louis owes New York, the demand for drafts is less in New York than in St. Louis; hence drafts in New York on St. Louis are at a discount.

The reason why the banks in New York should charge a premium, is that they must be at the expense of actually sending money to the St. Louis banks, or be charged with interest on their unpaid balance; the reason why the St. Louis banks will sell at a discount is that they are willing to sell for less than the face of a draft in order to get the money owed them in New York immediately.

Exchange is charged from $\frac{1}{8}$ to $\frac{1}{2}$%, and is designed to cover the cost of transporting the funds from one place to the other. Sometimes 1 or 2 days' interest is charged in addition.

A check, draft, or certificate of deposit *on a bank in the place where drafts are selling at a premium,* is often sent to pay a debt in the place where drafts are *selling at a discount,* and such a check or draft will command a premium.

If the course of exchange is unfavorable in drawing, the discount is sometimes avoided by means of a circuitous exchange through several intermediate places between which the course is favorable.

DOMESTIC EXCHANGE.

680. Domestic or **Inland Exchange** is the exchange between two places in the same country.

681. The **Base** of an inland bill is the *face;* the *Rate* is the rate of premium or discount.

682. The **Forms** and **Use** of drafts may be seen by the following examples and explanations:

FIRST NATIONAL BANK OF MOBILE,

$8000. MOBILE, ALA., *July* 16, 1887.

At sight, pay to the order of James Brown, Eight Thousand Dollars. EDWARD PICKENS,

To the MERCHANTS' NATIONAL BANK, *Cashier*

BALTIMORE, MD.

EXPLANATION.—Suppose James Brown, of Mobile, owes John Wilson & Co., of Baltimore, $8000; he goes into a bank in Mobile and gets the above draft. He then writes on the back of the note, "Pay to the order of John Wilson & Co.," signing his name, and forwards it to John Wilson & Co., in Baltimore, who take it to the Merchants' National Bank, and writing the name of their firm on the back, receive the money.

THIRD NATIONAL BANK,

$5600. LOUISVILLE, KY., *Jan.* 11, 1887.

At ten days sight, pay to the order of A. M. Taylor & Co., Five Thousand Six Hundred Dollars, and charge the same to the account of JAMES HARRISON,

To the FIFTH NATIONAL BANK, *Cashier.*

CINCINNATI, O.

EXPLANATION.—Suppose that William Johnson, of Louisville, wishing to pay a debt of $5600 to A. M. Taylor & Co., of Cincinnati, buys the above draft on the Fifth National Bank of Cincinnati. He forwards it to A. M. Taylor & Co., who, having indorsed it, will present it at the bank. The "ten days after sight" means after acceptance. It should be presented to the bank upon which it is drawn as soon as received, when the cashier writes upon it "accepted," with the date of acceptance, and signs his name as cashier. This makes the bank liable for its payment, and is an agreement to pay it after ten days.

NOTE.—If William Johnson has an account with the Fifth National Bank, he may draw on it directly as one bank draws on another. A person sometimes draws on a party who owes him in order to collect the bill.

CASE I.

683. ***To find the cost of a bill of exchange at sight, or on time.***

1. What must I pay in Philadelphia for a draft of $500 on Boston, exchange being $1\frac{1}{4}\%$ premium?

SOLUTION.—At a premium of $1\frac{1}{4}\%$ the cost of exchange of $1 is $1 + $1\frac{1}{4}$ ct. = $1.0125, and the cost of $500 is 500 times $1.0125, which are $506.25. Hence for sight exchange we have the following

OPERATION.

$1.0000	
.0125	rate.
$1.0125	cost of $1.
500	
$506.2500	

Rule.—*Find the cost of* $1 *by adding the rate to* $1, *when at a premium, or subtracting it, when at a discount, and multiply the result by the face of the draft.*

2. What must be paid in Cleveland for a draft of $3000 on Buffalo at 60 days, exchange 2% premium?

SOLUTION.—The draft being on time should be purchased at a discount. The discount of $1, at the rate in Cleveland for 60 days is $.01, which, subtracted from $1, equals $0.99, the cost of $1 of the draft if the exchange was at par; but there is a premium of 2 per cent., hence adding $.02, we find the actual cost of $1 of the draft to be $1.01, and multiplying this by 3000, we have $3030.00, the entire cost. Hence, for time exchange, the following

OPERATION.

$1.00	
.01	discount for 60 da.
.99	cost of $1 at par.
.02	rate of exchange.
1.01	cost of $1 of draft.
3000	
$3030.00	whole cost.

Rule.—*From* $1 *subtract the bank discount of* $1 *for the time and rate, where the draft is purchased; to this result add the rate of exchange when at a premium, and subtract it when at a discount, and multiply the result by the face of the draft.*

3. John Simpson bought in Pittsburgh a draft on Philadelphia for \$3500, exchange $\frac{1}{4}\%$ premium; what did it cost him? *Ans.* **\$3508.75.**

4. A merchant in Cincinnati remitted to New York a draft for \$7500, payable 30 days after sight, at 6%, exchange $2\frac{1}{2}\%$ premium; what did he pay for the draft? *Ans.* **\$7650.**

5. A Philadelphia grain dealer bought a quantity of wheat in Chicago, and remitted in payment a draft for \$1250, at 3 mo., at 6%, exchange at $\frac{3}{4}\%$ discount; what did he pay for the draft? *Ans.* **\$1221.875.**

6. What will be the cost of a sight draft on Philadelphia for \$550 at $\frac{3}{4}\%$ premium, and a 30-day draft for \$2000 at 1% premium? *Ans.* **\$2564.12½.**

7. Mr. Jones, of Philadelphia, sends his check for \$7500 to a firm in Chicago, where drafts on Philadelphia sell at $\frac{7}{8}\%$ premium; what will a Chicago bank pay for it?

Ans. **\$7565.62½**

CASE II.

684. ***Given, the cost of a bill of exchange, to find its face.***

1. I paid \$3030 for a 60-day draft on Buffalo, exchange 2% premium; required the face of the draft.

SOLUTION.—We find by Case I. that a draft for \$1 will cost \$1.01, therefore a draft that costs \$3030 must be for as many dollars as \$1.01 is contained times in \$3030, which are \$3000. From this solution we derive the following

OPERATION.

1.00	
.01	discount for 60 da.
.99	cost of \$1 at par.
.02	rate of exchange.
1.01	cost of \$1.

$$\frac{\$3030}{1.01} = \$3000.$$

Rule.—*Find the cost of a draft of* \$1 *and divide the given cost by it; the quotient will be the face of the draft.*

2. Joseph Hudson owes a debt in St. Louis; to pay it he purchases in Buffalo a 45-day draft, premium $1\frac{1}{2}\%$, for \$5541.25; what was his debt? *Ans.* **\$5500.**

3. A Philadelphia merchant wishes to pay a debt of \$2500 in St. Paul by a sight draft on the City National Bank, Phila.; if exchange on Philadelphia is $\frac{5}{8}\%$ premium *at St. Paul*, what must be the face of the draft?

Ans. \$2484.47

4. If the merchant in the previous problem buy, instead of a sight draft, a draft at 60 days, what will be the cost of the draft? *Ans.* $2459.63.

5. A merchant in Detroit buys goods in Boston, and remits in payment a 4-month draft on Detroit for $760, exchange on Detroit being at a premium of $\frac{3}{4}\%$; what was his bill, reckoning grace? *Ans.* $750.

6. I received a draft for $50, which cost $\frac{7}{8}\%$ to get it cashed; what should have been the face, that I might have realized $50? *Ans.* $50.44.

7. Sold on commission goods to the amount of $2375; after deducting 3% as my commission, I purchase with the proceeds a draft for 60 days, at 2% premium; what was the face of the draft? *Ans.* $2280 94.

CASE III.

685. ***Given, the face and the cost of a draft, to find the rate of exchange.***

1. If I pay $3508.75 in Cincinnati for a draft of $3500 on St. Louis, what is the rate of exchange?

SOLUTION.—If a draft of $3500 cost $3508.75, the premium will be $8.75; and dividing the premium, $8.75, by the base, $3500, we have the rate, $\frac{1}{4}\%$.

OPERATION.

$$\$3508.75 - \$3500 = \$8.75$$

$$\frac{\$8.75}{\$3500} = .0025 = \tfrac{1}{4}\%.$$

Rule.—*Find the premium or discount and divide it by the face, to find the rate.*

2. An agent in Cincinnati remitted to Philadelphia $5500 by a 45-day draft which cost $5541.25; what was the premium? *Ans.* $1\frac{1}{2}\%$.

3. A merchant in Omaha remitted to Baltimore a draft for $7500, payable 30 days after sight at 6%; the draft cost $7646.25; what was the rate of exchange? *Ans.* $2\frac{1}{2}\%$ prem.

4. A Boston merchant remitted to Cleveland a draft for $1250 at 3 mo., at 6%, paying for it $1221.25; what was the rate of exchange with grace? *Ans.* $\frac{3}{4}\%$ discount.

5. Mr. Johnson sold on commission goods to the amount of $2375; having deducted 3% as commission, he remitted a draft at 60 days for $2280.94; what was the rate of exchange? *Ans.* 2% premium

FOREIGN EXCHANGE.

686. Foreign Exchange is the exchange that takes place between different countries.

687. A Set of Exchange consists of three bills of the same tenor and date, each containing a condition that it shall continue payable only while the others are unpaid.

To prevent loss, or delay, each bill of a set is remitted in a different manner, and when one bill of the set has been paid, the others are worthless.

688. Bills of Exchange are usually made payable either 3 days after sight or 60 days after sight. The latter are quoted at a lower rate, on account of the discount.

Bills of exchange are also drawn 75 days after date, allowing 15 days for sight or passage.

689. Quotations of foreign exchange are expressed by equivalents, either by giving the number of cents in the foreign monetary unit (as the pound), or the number of foreign monetary units (as francs) to the dollar.

690. The par of exchange of the English money unit, the pound, is fixed by act of Congress at $4.8665.

Previous to 1834 the par of exchange between the United States and England was at the rate of £9 = $40, or £1 = 4.44\frac{4}{9}$, which is called the *old par of exchange*. A change in the U. S. coinage in 1834 and subsequently, caused the pound sterling to be worth about 9$\frac{2}{5}$ per cent. more than the old par; but exchange, however, was usually given with reference to the old par value; hence, when sterling money was quoted at 9$\frac{2}{5}$ per cent. premium it was really at par. By an Act of Congress, taking effect on the 1st of January, 1874, the par of the English pound sterling was fixed at $4.8665 in American gold coin, which is now the basis of quotation by bankers.

691. The **Money of Account** of any country consists of the denominations of the money of that country in which accounts are kept.

692. The Act of March 3, 1873, provides that "the value of the standard coins . . . of the world shall be estimated annually by the Director of the Mint, and be proclaimed on the first day of January by the Secretary of the Treasury."

693. In accordance with this law, the following table was published by the Secretary of the Treasury, Jan. 1. *1888:*

TABLE.

COUNTRY.	MONETARY UNIT.	STANDARD.	VALUE IN U. S. MONEY.
Austria,	Florin,	Silver,	.34,5
Belgium,	Franc,	G. and S.,	.19,3
Bolivia,	Boliviano,	Silver,	.69,9
Brazil,	Milreis of 1000 reis,	Gold,	.54,6
British America,	Dollar,	Gold,	$1.00
Central America,	Dollar,	Silver,	.91,8
Chili,	Peso,	Gold,	.91,2
Cuba,	Peso,	Gold,	.92,6
Denmark,	Crown,	Gold,	.26,8
Ecuador,	Sucre,	Silver,	.69,9
Egypt,	Pound of 100 piastres,	Gold,	4.94,3
France,	Franc,	G. and S.,	.19,3
Great Britain,	Pound Sterling,	Gold,	4.86,6½
Greece,	Drachma,	G. and S.,	.19,3
German Empire,	Mark,	Gold,	.23,8
Japan,	Yen,	Gold,	.99,7
India,	Rupee of 16 annas,	Silver,	.33,2
Italy,	Lira,	G. and S.,	.19,3
Liberia,	Dollar,	Gold,	1.00
Mexico.	Dollar,	Silver,	.75,9
Netherlands,	Florin,	G. and S.,	.40,2
Norway,	Crown,	Gold,	.26,8
Peru,	Sol,	Silver,	.69,9
Portugal,	Milreis of 1000 reis,	Gold,	1.08
Russia,	Rouble of 100 copecks,	Silver,	.55,9
Spain,	Peseta of 100 centimes,	G. and S.,	.19,3
Sweden,	Crown,	Gold,	.26,8
Switzerland,	Franc,	G. and S.,	.19,3
Tripoli,	Mahbub of 20 piasters,	Silver,	.63
Tunis,	Piaster of 16 caroubs,	Silver,	.11,8
Turkey,	Piaster,	Gold,	.04,4
U.S. of Colombia,	Peso,	Silver,	.69,9
Venezuela,	Bolivar,	Silver,	.19,3

694. Most of the dealings in foreign exchange are with the commercial centres mentioned in the following table, taken from a recent New York paper:

	60 days.	3 days.
Prime bankers' sterling bills on London	4 87½@4 88	4 90 @4 91
Good bankers' do	4 86¾@4 87¼	4 89¾@4 90
Prime com. sterling do.	4 86¼@4 86¾	4 89¼@4 89¾
Paris (francs), ...	5 17½@5 16¼	5 14⅜@5 13¾
Antwerp (francs), ...	5 17½@5 16¼	5 14⅜@5 13¾
Swiss (francs), ...	5 17½@5 16¼	5 14⅜@5 13¾
Amsterdam (guilders), ...	40⅞@ 41	41¼@ 41⅜
Hamburg (reichsmarks), . . .	94⅝@ 94¾	95¼@ 95⅜
Frankfort (reichsmarks), ...	94⅝@ 94¾	95¼@ 95⅜
Bremen (reichsmarks), ...	94⅝@ 94¾	95¼@ 95⅜
Berlin (reichsmarks), ...	94⅝@ 94¾	95¼@ 95⅜

Remittances to and from other places are frequently made in bills on *these leading ones*, especially London.

In the London quotations "prime" bills are those on the best banking houses; "good" are those on houses in good credit, but less in demand than the prime. "Commercial" signifies merchants' drafts, which generally rate below bankers'. In the quotations on Paris, Antwerp, and Switzerland, the franc is the unit, and the quotation gives the number of francs and centimes to the dollar. The exchange on Amsterdam is the number of cents to the guilder; while on Hamburg, Frankfort, Bremen, and Berlin the quotation gives the number of cents in 4 reichsmarks. 4.87¼@4.88 indicates the lowest and highest prices on the day on which the quotations were made.

REMARK.—United States securities are quoted in London on a gold basis instead of a greenback one, of 4 shillings to the dollar, hence they usually appear lower than with us.

695. A **Letter of Credit** is a letter from a banking house in one country to one or more of their correspondents in another, directing them to pay to the person in whose favor the letter is written, any sum required, not exceeding a certain amount specified in the letter.

Letters of credit are much used by travelers, and are preferred to bills of exchange for several reasons: 1st, The traveler need not draw the whole amount mentioned at once, but such a part as he may find convenient; 2d, They are usually addressed to one or more bankers in all the principal cities of Europe, so that money can be obtained at any one of these places; 3d, They can be bought either by depositing the full amount mentioned, and receiving at the time of settlement any balance remaining, or by depositing securities to that amount, and settling the account at the end of the journey.

On presenting a letter of credit, the holder is frequently required to sign a draft at 60 days for the amount drawn, which is forwarded at once to the house issuing the letter, the foreign banker making a profit usually from the course of exchange. The amount is charged by the house issuing the letter to the account of the drawer, with interest from the maturity of draft to the time of settlement, and a commission of 1%.

It is the custom of some houses, however, to require the holder of the letter of credit to merely sign a receipt, which is forwarded to the home bankers, and in this case the latter charge interest from the date of the receipt. The foreign banker pays the amount demanded, less a commission of about ½%, and the expenses of remitting.

When securities are deposited, no commission is charged by the home banker, and interest at the rate of 3% is allowed by some houses on balances.

696. Circular Notes are issued by the Bank of England, which may be bought in London, and are taken for their full value on the Continent. These are made payable to the order of the person who buys them, and there is no commission charged on them when cashed.

697. Circular Notes are also issued by bankers in this

country on some bank in Europe, which are cashed by any of the correspondents of the bank issuing them, without any other expense than a small commission.

The notes are accompanied with a letter from the home bank, countersigned by the bearer, which is to be presented at the same time as each of the circular notes. The latter must be signed in the presence of the correspondent.

1. What must be paid in Philadelphia for a bill of exchange on London for £325, at $4.87½ to the pound sterling?

OPERATION.

$$\begin{array}{r} \$4.87\tfrac{1}{2} \\ 325 \\ \hline \$1584.37\tfrac{1}{2} \end{array}$$

SOLUTION.—If 1 pound costs $4.87½, £325 cost 325 times $4.87½, which is $1584.37½. Hence the following

Rule.—*Find the cost of the unit of the currency in which the bill is given, and multiply the face by it for the cost, or divide the cost by it for the face.*

2. What was the cost in New York of the following draft, exchange at 60 days being $4.88 to the pound, when gold was selling at 113½? *Ans.* $3101.73.

Exchange for £560. NEW YORK, JUNE, 8, 1874.

Sixty days after sight of this first of Exchange (second and third of same tenor and date unpaid), pay to Thomas Elliott, or order, Five Hundred and Sixty Pounds Sterling, value received, with or without further advice, and charge the same to account of FISK, HATCH & CO.

TO MESSRS. BARING BROTHERS, LONDON.

3. What is the face of a draft on Lisbon, bought in New York for $648, if 1 milreis=$1.08? *Ans.* 600 milreis.

4. What is the cost of a draft on Vienna for 375 florins, if 1 florin=45 cents? *Ans.* $168.75.

5. For what can a merchant in St. Petersburg purchase a draft on New York of $2500? *Ans.* 4472.27+ roubles.

6. A merchant wishes a draft on Leghorn for 3000 lire; how much must he pay? *Ans.* $579.

7. By the second quotation in the table, Art. 694, what amount of exchange on Paris, at 60 days sight, will $1500 in gold buy? *Ans.* 7743.75 francs.

8. At the lower quotation of the table, how much ex-

change on Amsterdam, 3 days, will $1025.64 currency buy?

Ans. 2486½ guilders, nearly.

9. How much in gold will a bill on Geneva for 5200 francs cost, at the first quotation (Art. 694), at 60 days sight?

Ans. $1004.83.

10. A merchant in Hamburg wishes to remit 7860 reichsmarks to his correspondent in New York; what will be the face of a draft for 60 days at the highest quotation given in Art. 694? *Ans.* $1859.38.

11. Bought at higher quotations, exchange on Amsterdam, 60 days, for 1500 guilders; on Hamburg at 3 days for 1200 reichsmarks, and on Antwerp, 60 days, for 2000 francs; what was the cost in gold? *Ans.* $1289.66.

12. When U. S. bonds were quoted in London at 108¾, and in Philadelphia at 112¼, exchange $4.89½, gold quoted at 107, how much more was a $1000 U. S. bond worth in London than in Philadelphia? *Ans.* $16.689.

13. Bought in New York, $25,000 U. S. bonds, at 119½, when gold was at 136¼, and sold them in London at 83⅝, the proceeds being remitted to New York by draft; what did I gain or lose, exchange $4.91=£1? *Ans.* $1902.96 loss.

14. Bought in London $50,000 U. S. bonds at 83⅛, when U. S. greenbacks were quoted at at 73¼; and sold them in New York at 122¾, the proceeds being remitted by prime bankers' draft at 60 days to London at the highest quotation (Art. 694); did I make or lose by the transaction, and how much? *Ans.* Gain, £900 9 d—.

15. A gentleman sending his son to Europe, gives him a letter of credit from Drexel & Co., Philadelphia, depositing Government bonds as security; the son draws £75 in London, May 5, the bill of exchange at 60 days sight reaching Philadelphia, May 20; what must be paid to settle the account on August 1, commission of Drexel & Co. being 1%, exchange, $4.875=£1? *Ans.* $370.07.

16. A gentleman before taking a trip to Europe, gets a letter of credit from Knauth, Nachod & Kuhne, Bankers, New York, for £500; he draws for £50 in London, May 1, £100 in Paris, May 15, £100 in Berlin, June 30, £100 in Rome,

August 20, and £50 in London, Oct. 1. He arrives in New York, December 1, and wishes to settle his account with the bankers; the rate of exchange being \$4.90=£1, and premiums on gold being respectively 12%, $12\frac{1}{2}$%, 13%, 15%, and $13\frac{1}{3}$%, commission being charged at 1% and interest at 6%, what is his bill? *Ans.* \$2297.59.

ARBITRATION OF EXCHANGE.

698. Arbitration of Exchange, also called *Circular Exchange,* is the method of making exchange between two places by means of one or more intermediate exchanges.

699. Simple Arbitration is when there is only one intermediate exchange; *Compound Arbitration* is when there are two or more intermediate exchanges.

As rates of exchange constantly vary, it is often more advantageous to make the exchange through several intermediate places than by a direct remittance, and the object of arbitration is to enable a person to ascertain which will be most profitable.

The exchange between distant points is made in broken stages which form the course of exchange. The course of exchange is determined by the course of trade, but goes in the reverse direction.

1. A New York merchant wishes to pay a debt in Hamburg of 1000 marks, remitting through London, exchange between London and Hamburg being £1=20 reichsmarks, and between London and New York, \$4.90=£1; what will be the cost of the draft if the agent in London charge $\frac{1}{4}$% for remitting?

SOLUTION.—If we represent the required number of dollars by x, we have $x = 1000$ marks, 20 marks = £1, £1 remitted equals £1.00¼ paid by the debtor, and £1 = \$4.90. Now, the product of the first set of values will equal the product of the second set; hence the product of the second set divided by the product of all the first set except x, will equal x, from which we have $x = \$245.61\frac{1}{4}$.

OPERATION.

$$
\begin{aligned}
\$x &= 1000 \text{ m.} \\
20 \text{ m.} &= £1. \\
£1 \text{ (net)} &= £1.00\tfrac{1}{4}. \\
£1 &= \$4.90 \\
\hline
x &= \$245.61\tfrac{1}{4}
\end{aligned}
$$

2. A merchant in New Orleans remits \$4560 to New York; exchange on St. Louis is $1\frac{1}{4}$% premium, between St. Louis and Cincinnati, 1% discount, and between Cincinnati and New York $\frac{1}{2}$% discount; what was the value of the remittance in New York if sent through these cities?

SOLUTION.—According to the given rates, $1.01¼ in New Orleans = $1 in St. Louis; and $0.99 in St. Louis = $1 in Cincinnati, and $0.99½ in Cincinnati = $1 in New York; hence we have $x =$ $4572.06.

OPERATION.

x N. Y.	= $4560 N. O.
$1.01¼ N. O.	= $1 St. L.
$0.99 St. L.	= $1 Cin.
$0.99½ Cin.	= $1 N. Y.
	x = $4572.06

Rule.—I. *Represent the sum required by* x, *affix the proper unit of currency, place it equal to the given sum, and arrange the given rates of exchange so that in any two consecutive equations, the same unit of currency shall stand on opposite sides.*

II. *If commission is charged for* DRAWING, *place* 1 *minus the rate on the* LEFT *if the* COST OF EXCHANGE *is required, and on the* RIGHT *if* PROCEEDS *are required; but if commission is charged for* REMITTING, *place* 1 *plus the rate on the* RIGHT *if* COST *is required, and on the* LEFT *if* PROCEEDS *are required.*

III. *Divide the product of the numbers on the right by the product of the numbers on the left, cancelling equal factors; the result will be the required sum.*

3. When exchange between Boston and London is £9 = $44, between London and Paris is £1 = 27 francs, and between Paris and Stockholm is 4 francs = 1 rix dollar; how much must be paid in Boston for a bill on Stockholm for 2400 rix dollars? *Ans.* $1738.27.

4. A firm in New York remitted $5734.50 to their correspondent at St. Petersburg; the direct exchange was $1 = 1 rouble 35 copecks, while through London, Frankfort, and Copenhagen, it was as follows: £1 = $4.90; £1 = 12 florins; 1 rix dollar = 2.75 florins; 1 rix dollar = 1 rouble 30 copecks; which was the better mode of remittance?

Ans. The direct, by 1102 roubles 75 copecks.

5. A Philadelphia merchant has a debt in Hamburg of 10000 reichsmarks He remits through London, Paris, and Amsterdam, at the following rates: £1 = $4.87; £1 = 25 francs; 1 guilder = 2.3 francs; 1 guilder = 1.75 marks; what must he pay in Philadelphia, allowing $\frac{3}{4}$% brokerage in London and $\frac{1}{4}$% in Paris? *Ans.* $2602.

6. A merchant in Chicago wishes to remit $5000 to Savannah to purchase cotton. Exchange on Savannah is $1\frac{1}{2}\%$ premium, but on St. Louis it is 1% premium; from St. Louis to New Orleans $\frac{1}{4}\%$ discount; from New Orleans to Savannah $\frac{1}{2}\%$ discount; what will be the value of the remittance in Savannah by each method, and which is more profitable? *Ans.* Direct, $4926.11; circular, $4987.84.

7. A person living in Chicago is informed that a legacy of 10,000 marks has been left to him in Frankfort; he directs it to be remitted to London by a bill of exchange, and then directs his New York agent to draw on London and remit a draft to Chicago for the amount. If exchange is at 20.5 marks=£1, and £1=$4.87, and $\frac{3}{4}\%$ discount on Chicago, allowing the agent $\frac{1}{4}\%$ for drawing and $\frac{1}{4}\%$ for remitting, what will the legatee receive? *Ans.* $2381.62.

8. A gentleman residing in Florence, wishing to obtain $5000 from property in Baltimore, directs his London agent to draw on Baltimore, and send him the money through Amsterdam and Paris, exchange as follows: £1=$4.91; 1 guilder=18 d.; 1 guilder=2 francs 10 centimes; 1.15 lire= 1 franc. If the agent charges $\frac{1}{4}\%$ both for drawing and remitting, which is best, the circular exchange, or the direct at 17¢ per lira? *Ans.* Circular, by 3214.92 lire.

DUTIES OR CUSTOMS.

700. **Duties,** or **Customs,** are taxes levied by government upon imported goods; they are of two kinds, *ad valorem* and *specific.*

701. An **Ad Valorem** duty is a certain percentage assessed on the cost of the goods in the country from which they were imported.

702. A **Specific Duty** is a certain sum assessed on goods without regard to their cost.

703. A **Tariff** is a schedule showing the rate of duty fixed by law on all kinds of imported merchandise.

704. Certain **Allowances** are made in specific duties called *Tare* and *Breakage.*

705. Tare is an allowance for the weight of the box, cask, or covering, containing the goods.

For some articles certain rates of tare are fixed by law; in other cases the *real* tare only, ascertained under regulations prescribed by the Secretary of the Treasury, is allowed. If the tare is specified in the original invoice, the collector may, if he chooses, with the consent of the consignee, accept it as the correct tare.

706. Breakage is an allowance for the loss of liquors imported in bottles.

The allowance for breakage is 5% on ale, beer, porter, liquors, and sparkling wine in bottles; no allowance is now made on still wines.

707. Gross Weight or **Value** is the weight or value of the goods before any deductions have been made.

708. Net Weight or **Value** is the weight or value of the goods after all allowances have been deducted.

By the present tariff, most duties of the United States are *ad valorem*, but some duties are specific, and some articles are charged both a specific and an *ad valorem* duty. The duty is reckoned on the actual cost at the place of purchase or manufacture, increased by all charges for transportation previous to final shipment.

Seaport towns, where customs are collected, are called *ports of entry*. The offices in which they are collected are called *custom-houses;* and the officer who superintends the collection of duties and other business of the custom-house is called *collector of the port*.

A vessel is *entered* at a port by lodging at the custom-house a *manifest*, or statement of its cargo, and also a list of passengers if it have any, these papers verified by the oath of the master. The *clearance* from its port of departure and papers proving its nationality must also be deposited, before it is permitted to discharge its cargo.

A vessel is *cleared* from a port by lodging at the custom-house a manifest of its outward cargo, verified by oath, and agreeing with the shippers' manifests of parts of cargo. All government charges must be paid also, and everything connected with the discharging of the inward cargo settled, after which a "general clearance" is issued, and the vessel is at liberty to leave the port, having received its papers of nationality again.

The illegal introduction of goods into a country otherwise than through the regular ports of entry is called *smuggling*.

709. All merchandise imported from foreign ports or places, must be consigned in the manifest, invoice, or bill of lading, to some person or firm at the port of importation, by whom it must be duly entered, either for immediate consumption or for warehouse.

Upon the arrival of the vessel in port containing goods consigned to John Smith, he will proceed to make, in duplicate, the following entry *(if intended* for immediate consumption):

Entry for consumption of Mdse. imported by ——, on the ——, —— Master, from ——, on the —— day of ——, 18—.

Marks of Mdse.	Description of Goods.	Quantity.	Rate of Duty.	Rate of Duty.	Rate of Duty.	Rate of Duty.	Rate of Duty.
				40			
FCS 100	One Case Linens Ship ch.	1500 yd.		£100 8-6			
	Com. 2½ pr. ct.			£100- 8-6 2-10-3			
	Duty Linens $501 @ 40 p.c.	$200.40		£102-18-9 or $501			

Port of ——, —— day of ——, 18—.

The entries in this form, stating in full all the particulars (including as above all charges up to place of original shipment), with the invoice and bill of lading, must be presented at the collector's office to the clerk charged with this duty, who will examine the entry by the invoice and bill of lading, estimate the amount of duties, and if correct, will transmit all the papers to the Naval Office, where a like examination will be made, and if found correct, they will be checked. After necessary bonds are taken, oaths administered, packages ordered for examination and duties paid, a permit will be issued for the delivery of goods, in the following form;

Port of —— ——
Custom-house —— 18—

To the Inspector,

Duties thereon having been paid, permission is hereby given to land the following described merchandise, imported in the ——, — Master, from ——, ——, 18—.

FCS
100 One Case Linen.

—— ——, Naval Officer. —— ——, Collector.

Upon the presentation of the permit for delivery to the officer in charge, the merchandise will be delivered, excepting such packages as may be ordered to the appraisers' stores for examination—a subsequent permit being necessary after packages are examined and found correct.

Merchandise not intended for immediate consumption may be deposited in the public stores (or U. S. Bonded Warehouses) "there to be kept with due and reasonable care, at the charge and risk of the importer, owner, consignee, or agent, and subject at all times to their order upon payment of the proper duties and expenses, to be ascertained on entry thereof for warehousing, secured by bond of the owner, consignee, or importer, with surety or sureties, to the satisfaction of the collector, in double amount of duties."

Merchandise deposited in Bonded Warehouses may be withdrawn within one year from date of importation, on payment of duties and charges; after expiration of one year and until the expiration of three years from said date, they can be withdrawn on payment of duty assessed on original entry, and an additional duty of 10% on account of such duties and charges. Three years is the limit allowed for goods remaining in warehouse; after that time they cannot be withdrawn.

710. The following weights and measures are frequently found in *foreign invoices*:

WEIGHTS.

Arroba, Brazil,	= 32.38 lb. U.S.		Pounds, Spain, 100	= 101.44	lb. U.S.
" Buenos Ayres	= 25.36 "	"	" Germany "	= 110.25	" "
Cheke of opium, Smyrna	= 1.66 "	"	Cantaro, Sicily,	= 175	" "
Picul, Manilla	= 139.50 "	"	Kilogramme, France	= 2.2046	" "
" Siam	= 133⅓ "	"	Centner { Denmark & Norway }	= 110.11	" "
Pounds, Austria, 100	= 123½ "	"	Stone, England,	= 14.	" "
" Bremen, "	= 110.12 "	"			

WEIGHTS OF OILS PER GALLON.—Cocoanut, 7½ lb.; linseed, 7¾ lb.; olive, 7.56 lb.; palm, 7.50 lb.

MEASURES.

Velt	= 2 gal.		Palm of marble,	= .512 cu. ft.
Ohm, Antwerp	= 40 "		Aune,	= 1.25 yd.
Eimer Wine, Bavaria,	= 16.944 "		Ell, Berlin,	= .73 yd.

In custom-house business the *long* ton, cwt., and qr. are used. Foreign money is reduced by the table given in Art. 693, unless the invoice is accompanied by a consular certificate stating that a different rate of exchange is ruling at the time the invoice is made out.

711. The **Quantities** considered are: 1. The *Cost* of the goods, or the *Quantity*; 2. The *Duty*; 3. The *Rate*; 4. The *Allowances*.

CASE I.

712. ***Given, the base and the rate, to find the duty.***

Rule I.—*For ad valorem duties, multiply the cost of the goods by the rate of duty.*

Rule II.—*For specific duties, deduct first the allowances, and compute the duty on the remainder.*

NOTE.—In reckoning duties, whole dollars, pounds, gallons, etc., are used as the base, fractions less than ½ being rejected and more than ½ being reckoned as 1. Duties are payable in gold.

1. A receives from London an invoice of 200 dozen bottles of porter, valued at 5 s. a dozen; what is the duty at 50 cts a dozen, breakage 5%? *Ans.* $95.

2. What is the duty on 72 boxes of wax candles, each weighing 1 cwt., duty 8¢ a lb.? *Ans.* $645.12.

3. What is the duty in currency on 60 pieces of English prints, 27 in. wide, each containing 32 yd. @ 6 d., duty 7½¢ per sq. yd., and 15% *ad valorem*, gold at 116⅞? *Ans.* $167.25.

4. What is the duty on 75 hhd. of sugar, each weighing 4

cwt. 3 qr. 8 lb., tare 28 lb. per hhd., duty $2\frac{3}{4}$¢ per lb. and 25% additional; and 25 hhd. of molasses, 90 gallons each, duty 5¢ a gallon, and 25% additional, the price of gold being $112\frac{3}{4}$? *Ans.* $1646.85.

5. Received from Havana 750 boxes of cigars, 100 cigars in each, invoiced at $85 per M., and weighing 12 lb. per M.; what is the duty in currency at $2.50 per lb. and 25% *ad valorem*, gold being $110\frac{5}{8}$? *Ans.* $4252.15.

6. A quantity of wines forwarded to Philadelphia from Cologne via Antwerp was invoiced as follows: Mark C. R. S No. 2, 3 dozen @ 80 reichsmarks; No. 4, 4 dozen @ 70 reichsmarks; No. 6, 3 dozen @ 60 reichsmarks; charges $5; commission $2\frac{1}{2}$%; duty $1.60 per dozen; what was the cost of the invoice? *Ans.* $191.765.

7. Mr. A. B. Carey bought of Matthew Eyre at Rome one quarter cask of Marsala wine, costing 140 lire, commission $2\frac{1}{2}$%; it was shipped free of export duties on board brig Robert Ward for Philadelphia, consigned to order of owner; the wine being estimated at 30 gal., duty 40¢ a gallon, what did it cost on its arrival in Philadelphia? *Ans.* $39.695.

8. A merchant bought in Bordeaux and ordered to be shipped per steamer Mary, in which he returned to the United States, for his account and risk, $\frac{10}{8}$ pipes of brandy marked A, B, C, etc., Nos. 21 to 30, containing $81\frac{1}{4}$ velts @ 15.30 francs; com. $2\frac{1}{2}$%; duty $2 a gallon; what was the whole cost of the brandy? *Ans.* $570.92.

9. Invoice of 5 bales Goat Skins *Tanned*, shipped at Madras per steamer Pacific to Liverpool for transhipment to Philadelphia, on account of and consigned to G. R. Lamar, Esq.

G R L	1|5	5 bales cont'g 2000 skins w'g 2200 lb. @ Rs. 1		
			Export duty	40
			Freight to Liverpool Rs. 100 (not dutiable)	
		Duty,	Com. 5%	——
		Tanned Skins $	@ 10% = $102.50	Rs.

What is the duty on this invoice? Ans. $102.50

NOTE.—In the preceding invoice the skins are worth 1 rupee per lb. the mark being Rs. and the part in italics is the calculation of the duty added by the custom-house officer. The freight to Liverpool must not be added in to find the base for the duty.

10. Messrs. Smith & Son, Philadelphia, bought of A. Fogel & Co., Lyons, Jan. 1, 1875, one case marked S. N., No. 110, containing Nos. 10004-10005 silk, 20 in. wide, 839¾ aunes @ 7.10 francs, and Nos. 10006-10008 silk, 24 in. wide, 263 aunes @ 11 francs; discount was 2% and commission 3%; what was the duty at 60%, and what the cost of the silk per yard? *Ans.* Duty, $1035; price, $2.

11. Invoice of one case dress goods purchased by B. Vogert & Co., and forwarded to McGregor & Co., Havre, for shipment per steamer Atlantic, for account and risk of Charles Daniels, Philadelphia.

CD 61	5 Pcs.	8¢ □ yd 40%	Black Cashmere No. 20, 38in. Net Wt. 24 Kilo.	Mtrs. 232@1.95 or □ yds. 268	
	5 Pcs.	50¢ lb. 40%	" " No. 34, Net Wt. 28 Kilo. or 61½ lb. U. S.	" 173 3.06	
				Discount 5%	
				Case & Packing	10.00
				Com. 3%	
				Fcs.	

What is the duty on the above invoice? *Ans.* $126.99.

NOTE.—In this example it will be noticed there are two duties; on the first lot a specific duty of 8 cents per square yard and an *ad valorem* duty of 40 per cent.; on the second a specific duty of 50 cents per lb. and 40 per cent. *ad valorem*. The pupil may be required to make out invoices for the preceding examples.

CASE II.

713. *Given, the rate and the duty or the whole cost, to find the base.*

Rule I.—*Divide the duty by the rate, to find the base.*

Rule II.—*Divide the whole cost by 1 plus the rate to find the base.*

NOTE.—As the duty is reckoned only on whole dollars, it is evident that the original value cannot be *exactly* obtained from the duty, and hence this case is more theoretical than practical.

1. Paid $278.25 duty on watches from Geneva, at 35%;

what were they invoiced at, and what was the whole cost in store? *Ans.* Fcs. 4119.17; $1073.25.

2. The duty on an invoice of Lyons velvet, at 24%, was $1595.28; what were the goods invoiced at in Lyons, 650 francs having been paid as charges? *Ans.* Fcs. 33790.41.

3. The cost in store of 30 pipes of Port wine, 120 gallons each, is $10829.25; duty, 20%; freight and other charges, $65.25; what was the cost per gallon in Lisbon, 1 milreis being equal to $1.08? *Ans.* 2 milreis, 307 reis.

4. Received from Havre an invoice of 50 baskets of champagne, 1 dozen bottles each, duty being 35%, freight and other charges 175 francs, and the whole cost $515.70; what did it cost per bottle at Havre, what in store, and what must I charge to clear 20%?

Ans. 3 francs at Havre; 86¢ in store; selling price, $1.03.

CASE III.

714. ***Given, the base and the duty, to find the rate.***

Rule.—*Divide the duty by the base, to find the rate.*

1. 375 tons of Swedish railroad iron, invoiced at $60 per ton, cost when the duties were paid, $28,125; what was the rate of duty? *Ans.* 25%.

2. The duty on 300 drums of figs, each containing 28 lb., invoiced at 7½ cents per pound, was $25.20; what was the rate of duty? *Ans.* 4%.

3. A quantity of French merinoes, invoiced at 44,475 francs, cost $9689.08 in store, after paying the duties and $75 for freight; what was the rate of duty? *Ans.* 12%.

4. The duty on 2000 yards of Spitalfields silk was $3019.80, invoice price being 10 s. 3 d. per yard, and freight £9 5 s.; what was the rate of duty and what should I charge to clear 25%? *Ans.* Rate, 60%; selling price, $5.03.

5. I imported from Havana 750 boxes of cigars, 100 cigars in each, invoiced at $85 ℘ M. and weighing 12 lb. ℘ M; the specific duty in gold is $2250 and the *ad valorem* $1593.75; what are the respective rates? *Ans.* $2.50 per lb.; 25%.

SECTION IX.

RATIO AND PROPORTION.

RATIO.

715. **Ratio** is the measure of the relation of two similar quantities; thus, the ratio of 12 to 4 is 3.

716. The **Symbol** of ratio is the colon (:); thus, **12 : 4** signifies the ratio of 12 to 4. Ratio is also expressed by writing the numbers in the form of a fraction; thus, $\frac{12}{4}$.

717. The **Terms** of a ratio are the two numbers compared, called respectively the *antecedent* and the *consequent.*

718. The **Antecedent** is the number compared with the consequent. The *consequent* is the number with which the antecedent is compared.

719. A **Ratio** is found by dividing the antecedent by the consequent; thus, in 12 : 4, the ratio is $\frac{12}{4}$, or 3.

720. A **Simple Ratio** is the ratio of two numbers, as 8 : 4. A *Compound Ratio* is the product of two or more simple ratios; as $(3 : 6) \times (4 : 8)$, or $\frac{3}{6} \times \frac{4}{8}$.

721. A **Compound Ratio** is usually expressed by writing the simple ratios one under another; thus, $\left\{\begin{matrix} 3 : 6 \\ 4 : 8 \end{matrix}\right\}$.

722. The **Reciprocal** of a ratio is a unit divided by the ratio, or the ratio inverted; thus, the reciprocal of $\frac{3}{4}$ is $1 \div \frac{3}{4}$ or $\frac{4}{3}$.

723. **Inverse Ratio** is the quotient of the consequent divided by the antecedent. The ordinary ratio is sometimes called a *direct ratio.*

724. Ratio exists only between similar quantities; there is no ratio between 6 yd. and \$8. A ratio is always an abstract number.

NOTES.—1. The symbol of ratio (:) is supposed to be a modification of the symbol of division.

2 Ratio is usually defined as the *relation* of two numbers. This is indefinite, for the ratio is the *measure* of the relation.

3. A few authors divide the *consequent* by the *antecedent*, calling it the *French Method*. The method and the name are both founded in error; nearly all the French mathematicians, like the German, English, etc., divide the *antecedent* by the *consequent*.

PRINCIPLES.

1. *The ratio equals the quotient of the antecedent divided by the consequent.*

Thus, if the antecedent is represented by a, and the consequent by c, and the ratio by r, we have $a \div c = r$, or $\frac{a}{c} = r$.

2. *The antecedent is equal to the product of the consequent and ratio.*

For, since $\frac{a}{c} = r$, multiplying by c, we have $a = c \times r$.

3. *The consequent is equal to the quotient of the antecedent divided by the ratio.*

For, since $\frac{a}{c} = r$, $a = c \times r$, from which we see $c = \frac{a}{r}$.

4. *Multiplying the antecedent or dividing the consequent multiplies the ratio.*

For, $a : c$ equals the fraction $\frac{a}{c}$, and multiplying the numerator or dividing the denominator multiplies the value of the fraction.

5. *Dividing the antecedent or multiplying the consequent divides the ratio*

For, $a : c$ equals the fraction $\frac{a}{c}$, and dividing the numerator or multiplying the denominator divides the value of the fraction.

6. *Multiplying or dividing both terms of a ratio by any number does not change its value.*

For, $a : c$ equals the fraction $\frac{a}{c}$, and multiplying or dividing both terms of a fraction by the same number does not change its value.

CASE I.

725. ***To find the value of a simple ratio.***

1. What is the ratio of 12 to 18?

SOLUTION.—The ratio of 12 to 18 equals 12 divided by 18, or $\frac{12}{18}$, which reduced to its lowest terms, equals $\frac{2}{3}$.

OPERATION.

$12 : 18 = \frac{12}{18} = \frac{2}{3}$

What is the value of

2. 95 : 19? *Ans.* 5.
3. \$65 : \$13? *Ans.* 5.
4. £117 : £9? *Ans.* 13.
5. $\frac{3}{8} : \frac{5}{6}$? *Ans.* $\frac{9}{20}$.
6. $6\frac{1}{4} : 87\frac{1}{2}$? *Ans.* $\frac{1}{14}$.
7. 3.5 : 6.25? *Ans.* $\frac{14}{25}$.
8. $5.\dot{6} : 7.\dot{4}\dot{5}$? *Ans.* $\frac{187}{246}$.
9. $.0\frac{\dot{1}}{2} : .0\frac{\dot{1}}{3}$? *Ans.* $\frac{3}{2}$.
10. 1 A. : 1 sq. ch.? *Ans.* 10.
11. 24 G. : £28? *Ans.* $\frac{9}{10}$.
12. 5 yd. 2 ft. 7 in. : 1 ft. 6 in.? *Ans.* $11\frac{13}{18}$.
13. 9 mi. 198 rd. : 21 mi. 120 rd. *Ans.* $\frac{9}{20}$.
14. 3 lb. 7 oz. 15 pwt. : 5 lb. $2\frac{1}{2}$ oz. Av. *Ans.* $\frac{32}{55}$.
15. The reciprocal of the ratio of 17 to 51? *Ans.* 3.
16. What is the inverse ratio of 76 to 19? *Ans.* $\frac{1}{4}$.
17. What is the ratio of a yard to a meter? *Ans.* $\frac{3600}{3937}$.
18. What is the ratio of a pound Troy to a pound Avoirdupois? *Ans.* $\frac{144}{175}$.
19. What is the ratio of a liquid gallon to the old beer gallon? *Ans.* $\frac{77}{94}$.
20. What is the ratio of an English ton to a U. S. ton? *Ans.* $\frac{28}{25}$.
21. Which is greater and how much, the ratio of 2.6 : 5.45, or that of $2.\dot{6} : 5.\dot{4}\dot{5}$? *Ans.* The latter, $\frac{58}{49005}$.
22. Required the difference between the direct and inverse ratio of $4\frac{2}{3}$ to $5\frac{1}{4}$. *Ans.* $\frac{17}{72}$.

CASE II.

726. ***Given, the ratio and one of the terms, to find the other term.***

1. The antecedent is 24 and the ratio is 4; required the consequent.

SOLUTION.—The consequent equals the antecedent divided by the ratio; hence the consequent is $24 \div 4$, or 6.

OPERATION.
$24 \div 4 = 6$

2. The ratio is $\frac{2}{3}$ and the antecedent $\frac{5}{6}$; required the consequent. *Ans.* $1\frac{1}{4}$.

3. The consequent is 2.8 and ratio 3.5; required the antecedent. *Ans.* 9.8.

4. The antecedent is 9 yd. 1 ft., and ratio $2.\dot{3}$; required the consequent. *Ans.* 4 yd.

5. The ratio is $2.0\dot{}\frac{1}{2}$ and the antecedent $\dot{5}.\dot{4}$ yd.; what is the consequent? *Ans.* $2\frac{266}{407}$.

6. $2\frac{1}{2}$ times the ratio is $7\frac{1}{2}$, and the antecedent is 72; what is the consequent? *Ans.* 24.

7. The ratio of two numbers, divided by $3.\dot{3}$, equals 3, and the consequent is 42; what is the antecedent? *Ans.* 420.

8. The reciprocal of the ratio of two numbers is $2\frac{1}{4}$, and the antecedent is 16; required the consequent. *Ans.* 36.

9. The inverse ratio of two numbers is $3\frac{2}{3}$, and the consequent is \$3.41; what is the antecedent? *Ans.* \$0.93.

10. The inverse ratio of two numbers is $5\frac{5}{6}$, and the reciprocal of the antecedent is $\frac{7}{18}$; what is the consequent? *Ans* 15.

11. The inverse ratio of two numbers is $3.0\frac{1}{2}$ and the reciprocal of the consequent is $.0\dot{1}\dot{8}$; what is the antecedent? *Ans.* 18.

CASE III.

727. ***To find the value of a compound ratio.***

1. Required the value of the compound ratio $\left\{\begin{matrix} 4:6 \\ 5:8 \end{matrix}\right\}$.

SOLUTION.—This compound ratio equals $(4:6)\times(5:8)$, Art. 721, which equals $\frac{4}{6}\times\frac{5}{8}=\frac{5}{12}$.

2. Required the value of $\left\{\begin{matrix} 7:11 \\ 9:14 \end{matrix}\right\}$. *Ans.* $\frac{9}{22}$.

3. Required the value of $\left\{\begin{matrix} 4\frac{1}{2}:3\frac{1}{4} \\ 3\frac{1}{8}:4\frac{4}{9} \end{matrix}\right\}$. *Ans.* $1\frac{1}{26}$

4. Required the value of $\left\{\begin{matrix} 6.\dot{5}:4.\dot{1} \\ 6\frac{1}{6}:9\frac{5}{6} \end{matrix}\right\}$. *Ans.* 1.

5. Given the compound ratio $\left\{\begin{matrix} a:8 \\ 6:9 \end{matrix}\right\}=\frac{8}{15}$, to find the first antecedent, or a. *Ans.* $6\frac{2}{5}$.

6. Given the compound ratio $\left\{\begin{matrix} 9:12 \\ a:18 \end{matrix}\right\}=\frac{1}{4}$, to find the second antecedent. *Ans.* 6.

7. Given the compound ratio $\left\{\begin{matrix} 16:15 \\ 18:c \end{matrix}\right\}=4$, to find the second consequent. Ans. $4\frac{4}{5}$.

8. The ratio of $7\frac{1}{2} : 13\frac{1}{2}$ equals the compound ratio $\left\{\begin{matrix} 18 : c \\ 20 : 27 \end{matrix}\right\}$; required the first consequent. *Ans.* 24.

9. The ratio of $2.\dot{3} : 6.\dot{5}\dot{1}$ equals the compound ratio $\left\{\begin{matrix} 7 : 5 \\ a : 43 \end{matrix}\right\}$; required the second antecedent. *Ans.* 11.

MISCELLANEOUS EXAMPLES.

Required the value

1. Of 35 pwt. : $\frac{1}{2}$ oz. Av. *Ans.* $3\frac{21}{25}$.
2. Of 1 oz. Av. : 5 ℈. *Ans.* $4\frac{3}{8}$.
3. Of $.\dot{9}28571\dot{4} : .351\dot{3}\dot{5}$. *Ans.* $2\frac{9}{14}$.
4. Of $\dfrac{13\frac{1}{2}}{\frac{9}{10}} : \dfrac{\frac{3}{4} \text{ of } \frac{5}{6}}{\frac{5}{7}}$. *Ans.* $17\frac{1}{7}$.
5. Of $\dfrac{\frac{2}{3}+\frac{3}{4}}{2.\dot{6}\dot{3}-2.\dot{2}\dot{7}} : \dfrac{4.\dot{3}-2\frac{1}{8}}{.\dot{8}\dot{1}-\frac{5}{11}}$. *Ans.* $\frac{17}{24}$.
6. Of $\dfrac{.5\dot{9}\dot{0}-.2\dot{2}\dot{7}}{.2\dot{5}\dot{4}-.2\dot{3}\dot{6}} : \dfrac{.\dot{2}1\dot{6}-.\dot{1}3\dot{5}}{.\dot{2}0\dot{7}-.\dot{0}8\dot{1}}$. *Ans.* $31\frac{1}{5}$.
7. What is the antecedent of $\dfrac{.4\dot{7}\dot{2}-.3\dot{2}\dot{7}}{.5\dot{8}\dot{1}-.5\dot{1}\dot{8}}$, the ratio being 4? *Ans.* $9\frac{1}{7}$.
8. What is the consequent of $\dfrac{.2\dot{7}\dot{2}+.4\dot{2}\dot{7}}{.3\dot{8}\dot{1}+.2\dot{1}\dot{8}}$, the ratio being $1\frac{1}{6}$? *Ans.* 1.
9. The great bell of Moscow weighs 198 tons 2 cwt. 25 lb., and the Great Tom of Christ Church, Oxford, weighs 17000 lb.; required the ratio of the latter to the former. *Ans.* $\frac{680}{15849}$.
10. The ratio of the circumference of a circle to its diameter is 3.141592; find the approximate values for this ratio. *Ans.* 3, $\frac{22}{7}$, $\frac{333}{106}$, $\frac{355}{113}$, or 3, $3\frac{1}{7}$, $3\frac{15}{106}$, etc.
11. In 57551 years the earth makes 36000 conjunctions with Venus; find approximate values for the fraction expressing the ratio of the two numbers. *Ans.* 1, 2, $\frac{3}{2}$, $\frac{8}{5}$, $\frac{227}{142}$, $\frac{235}{147}$, etc.

SIMPLE PROPORTION.

728. A **Proportion** is the expression of equality between equal ratios, the terms of the ratios being indicated.

729. The **Symbol** for proportion is the double colon, (: :); thus, 8 : 4 : : 6 : 3 means the same as 8 : 4=6 : 3.

730. A **Proportion** is *read* in two ways; thus, 8 : 4 : : 6 : 3 is read "the ratio of 8 to 4 equals the ratio of 6 to 3," or "8 is to 4 as 6 is to 3."

731. The **Terms** of a proportion are four numbers used in the comparison. The first and fourth terms are the *Extremes;* the second and third are the *Means.*

732. The **Couplets** are the two ratios compared. The *first couplet* consists of the first and second terms; the *second couplet* consists of the third and fourth terms.

733. A **Mean Proportional** of two numbers is a number which may be made the means of a proportion in which the two numbers are the extremes.

734. **Proportion** may be *Simple* or *Compound.* In *Simple Proportion* both ratios are simple; in *Compound Proportion* one or both of the ratios are compound.

735. A **Simple Proportion** is the expression of the equality of two simple ratios.

736. The **Principles** of proportion are the truths relating to proportion. They enable us to find any one term when the other three are given.

NOTE.—Ratio arises from the *comparison* of two *numbers;* proportion arises from a comparison of *two ratios.* A proportion is therefore a comparison of the results of two previous comparisons.

PRINCIPLES.

1. *In every proportion the product of the means equals the product of the extremes.*

Take any proportion, as 6 : 3 : : 8 : 4. Then we have $\frac{6}{3}=\frac{8}{4}$, and multiplying these equals by 4 and 3, we have $6\times4=8\times3$; that is, the product of the two means, 8 and 3, equals the product of the two extremes, 6 and 4.

2. *Either extreme equals the product of the means divided by the other extreme*

For, from the proportion 6 : 3 : : 8 : 4, we have $6\times4=3\times8$; hence, $6=3\times8\div4$, or $4=3\times8\div6$. Therefore, etc.

3. *Either mean equals the product of the extremes divided by the other mean.*

For, from the proportion 6 : 3 : : 8 : 4, we have $6\times4=3\times8$; hence, $3=6\times4\div8$, or $8=6\times4\div3$. Therefore, etc.

4. *The first term of a proportion equals the second term multiplied by the ratio of the third to the fourth.*

For, from the proportion 8 : 6 : : 12 : 9, we have $\frac{8}{6}=\frac{12}{9}$; hence $8=\frac{12}{9}\times6$, or 12 : 9 multiplied by 6. Therefore, etc.

5. *The fourth term of a proportion equals the third term divided by the ratio of the first to the second.*

For, from the proportion 8 : 6 : : 12 : 9, we have $8\times9=6\times12$, or $9=6\times12\div8$, which equals $12\times\frac{6}{8}$, which equals $12\div\frac{8}{6}$, or $12\div(8:6)$. Therefore, etc.

NOTES.—1. Let the pupils be required to demonstrate these principles by using symbols of any numbers, that is, by letters.

2. French authors usually represent the unknown term by x; the same is done in this work.

3. Principle 1 may be demonstrated by showing that in a proportion we have 2d term$\times$ratio : 2d term : : 4th term$\times$ratio : 4th term; in which we see the factors in the means are the same as the factors in the extremes.

EXAMPLES FOR PRACTICE.

Find the value of x in each of the following proportions:

1. \$13 : \$27 : : x : 9 qt. *Ans.* $4\frac{1}{3}$ qt.
2. $\frac{3}{5}:\frac{7}{8}::x:\frac{5}{9}$. *Ans.* $\frac{8}{21}$.
3. $\frac{9}{10}:\frac{21}{25}::3.5:x$. *Ans.* $3\frac{4}{15}$.
4. $x:2.\dot{0}\frac{1}{2}::.\dot{9}4\dot{5}:.5$. *Ans.* $3\frac{1}{2}$.
5. \$2.50 : \$1.50 : : $16\frac{1}{4}$ yd. : x. *Ans.* $9\frac{3}{4}$ yd.
6. $5\frac{3}{4}$ yd. : $17\frac{1}{4}$ yd. : : £7 : x. *Ans.* £21.
7. 11.34 cwt. : 1 cwt. 62 lb. : : x : \$4.05. *Ans.* \$28.35.
8. x : 36 mi. 298 rd. 1 ft. 6 in. : : 6 h. 30 min : 9 h. 45 min. *Ans.* 24 mi. 198 rd. 4 yd.
9. Form a proportion having 28 : 35 for the first couplet.
10. Form a proportion having \$4.80 and 56 yards for the means.

APPLICATION OF SIMPLE PROPORTION.

737. Simple Proportion is employed for the solution of problems in which three of four quantities are given, so related that the fourth may be determined from them, by *the equality* of the ratios.

738. The required quantity must bear the same relation to a given quantity of the same kind that one of the remaining quantities does to the other. We can then form a proportion containing one unknown quantity, and find the unknown term by the principles of proportion.

NOTE.—Proportion was formerly called the "Rule of Three." Some of the old arithmeticians thought so highly of it that they called it "The Golden Rule of Three."

1. What will 25 pounds of butter cost if 7 pounds cost $2.45?

SOLUTION.—It is evident that the cost of 25 lb. bears the same relation to the cost of 7 lb. that 25 lb. bears to 7 lb. hence we have the proportion, *cost* of 25 lb. is to $2.45 (the *cost* of 7 lb.) as 25 lb. is to 7 lb.; from which, by Prin. 2, we have, *cost* of 25 lb. $= \frac{\$2.45 \times 25}{7} = \$8.75$.

OPERATION.

Cost of 25 lb. : $2.45 : : 25 : 7

Cost of 25 lb. $= \frac{\$2.45 \times 25}{7} = \$8.75$.

SOLUTION 2D.—It is evident that the relation of 7 lb. to 25 lb. is the same as the relation of the *cost* of 7 lb. to the *cost* of 25 lb.; hence we have the proportion, 7 lb. is to 25 lb. as $2.45 (the *cost* of 7 lb.) is to the *cost* of 25 lb., from which we have, by Prin. 2, *cost* of 25 lb. $= \frac{\$2.45 \times 25}{7} = \$8.75$.

OPERATION.

lb. lb. $

7 : 25 : : 2.45 : cost of 25 lb.

Cost of 25 lb. $= \frac{\$2.45 \times 25}{7} = \$8.75$.

Rule.—I. *Write the required quantity for the first term and the similar known quantity for the second term, and place the other two quantities for the third and fourth terms, so that the two ratios will be equal.*

II. *Find the first term by dividing the product of the second and third terms by the fourth.*

NOTES.—1. The rule given will state the method of the 2d solution by merely changing the number of the terms. Require the pupils to apply it to the second method.

2. Teachers may place the unknown term in the first or fourth term, as they prefer. The author prefers the first method, the law of reasoning being to compare the *unknown with the known.*

3. Pupils should be required to place the unknown quantity in different terms, that the subject may be thoroughly understood. In practice, let the unknown term be represented by x.

2. If 5 acres of grass keep 20 cattle a month, how many acres will keep 75 cattle a month? Ans. 18¾ acres.

3. What is the time by rail from Philadelphia to New York, 90 miles, at 4 mi. 240 rd. in 10 min.? *Ans.* 3 h. $9\frac{9}{19}$ min.

4. How much will 57 cwt. 50 lb. of sugar cost at the rate of 22 cwt. 50 lb. for $121.50? *Ans.* $310.50.

5. How high a staff will cast 4 ft. 6 in. of shadow, when a staff 2 ft. 9 in. casts a shadow 1 ft. 6 in.? *Ans.* 8 ft. 3 in.

6. A does a job in $24\frac{4}{5}$ days, working 10 h. a day; in what time will he do it working 8 h. a day? *Ans.* 31 days.

7. If 5 A. 120 P. of land cost $718.75, what will 25 A. 40 P. cost at $25 less an acre? *Ans.* $2525.

8. If 32 lb. $8\frac{1}{2}$ oz. of drugs cost $78.075, how much will 6 lb $8\frac{3}{8}$ oz. cost? *Ans.* $15.70.

9. If £62 8 s. 3 d. are worth $303.73, how many dollars are £409 11 s. 6 d. worth? *Ans.* $1993.19.

10. What cost 11.63125 ℔. of drugs, if $26\frac{1}{2}$ oz. Avoirdupois cost $76.35? *Ans.* $441.20—.

11. Find the mean proportional between 16 and 9; also between $\frac{4}{5}$ and $\frac{16}{45}$. *Ans.* 12; $\frac{8}{15}$.

12. A's fortune is $6000, B's is $2\frac{1}{4}$ times as much, and C's a mean proportional between A's and B's; required C's. *Ans.* $9000.

13. If 10 men harvest a field of wheat in 15 days, how many men can harvest it in 6 days? *Ans.* 25 men.

14. A bankrupt's debts are $4500, and assets $2000; what will be received for a claim of $1800? *Ans.* $800.

15. The two hands of a clock are together at 12; when are they next together? *Ans.* 5 min. $27\frac{3}{11}$ sec. past 1.

16. If a cane 3 ft. 6 in. long, held vertically, casts a shadow 2 ft. 8 in. long, how high is a tree whose shadow is at the same time 45 ft. 4 in. long? *Ans.* 59 ft. 6 in.

17. If a man perform a journey in 27 days of 10 hours each, how many days will it take him of 12 hours each? *Ans.* $22\frac{1}{2}$ days.

18. A man borrows $1800 and keeps it 2 yr. 6 mo.; how long should he loan $1500 to return the favor? *Ans.* 3 yr.

19. A milkman has a false gallon $\frac{1}{2}$ pt. too small; what is the value of the milk he sells for $154.40? *Ans.* $144.75.

20. A regiment of soldiers are 25 in rank when they are 32 in file; how many are there in file when there are 40 in rank? *Ans.* 20 men.

21. If 5 men can do the brickwork of a house in 40 days, how many men must be added to the number to do it in 25 days? *Ans.* 3 men.

22. Robert has 30 seconds start in a foot-race, and runs 18 rods in a minute; how long will it take Richard, who runs 24 rods in a minute, to overtake him? *Ans.* $1\frac{1}{2}$ minutes.

23. The distance around a rectangular lot of land is 70 rods, and the length is to the breadth as 3 to 2; required the length and the breadth. *Ans.* 21 and 14.

24. Two bodies, one weighing 12 lb., the other 8 lb., attract each other inversely as their weights; if the smaller body moves 35 ft., how far will the large one move? *Ans.* $23\frac{1}{3}$ ft.

25. A drover has 119 horses and cows, and the number of horses is to the number of cows as $\frac{2}{3}$ to $\frac{3}{4}$; how many has he of each? *Ans.* 56 horses; 63 cows.

26. A garrison of 1500 men has provisions for 12 months; how long will the same provisions last if the garrison is reinforced by 300 men? *Ans.* 10 months.

27. A garrison of 12,000 men has bread enough to give each soldier 16 oz. a day for 120 days; how long will it last them if they are cut down to 12 oz. a day? *Ans.* 160 days.

28. An oarsman can row a boat 6 miles an hour, and he drives his boat 11 ft. in 3 strokes of his oar; how many strokes are made in a minute? *Ans* 144.

29. Required the quantity of flannel $\frac{7}{8}$ yd. wide, necessary to line the clothes of 500 soldiers, each suit containing $4\frac{1}{2}$ yards of cloth, $1\frac{1}{4}$ yd. wide. *Ans.* $3214\frac{2}{7}$ yd.

30. A grocer has a false balance which gives $14\frac{1}{2}$ oz. to the pound; what does he gain by it in selling sugar for which he receives $258.56? *Ans.* $24.24.

31. A boy bought 200 eggs at the rate of 5 for 2 cents, and then sold 100 at 3 for a cent and the rest at 2 for a cent; did he gain or lose, and how much? *Ans.* Gained $3\frac{1}{3}$ cents.

32. A has flour worth $7.75 a barrel and B has grain worth

1.37\frac{1}{2}$ a bushel; now if in exchange B puts his grain at $1.28 a bushel, what should A charge for his flour? *Ans.* 7.21\frac{5}{11}$.

33. Two cog wheels, one having 42 and the other 30 cogs, run together; in how many revolutions of the smaller will it gain 16 revolutions? *Ans.* 56.

34. A garrison has food to last 6 months, giving to each man 1 lb. 2 oz. a day; how much should be allowed each man daily to make it last 1 yr. 3 mo.? *Ans.* 7$\frac{1}{5}$ oz.

35. Henry and William own a lot of land worth $770, and 2 times Henry's share is to 3 times William's as 8 to 9; required the share of each. *Ans.* H's, $440; W's, $330.

36. A dealer in stock bought 1270 head of cattle, and $\frac{3}{4}$ of the number of horses is to $\frac{4}{5}$ of the number of cows as $\frac{5}{6}$ is to $\frac{7}{8}$; required the number of each. *Ans.* 640; 630.

37. A, B, and C have $2947; A's money is 4 times C's, and B's is a mean proportional between A's and C's; required the amount of each. *Ans.* $1684; $842; $421.

38. The side of a square field, as measured, contained 85 rods 2$\frac{3}{4}$ ft., but it was afterward found that the chain used in measuring contained only 65 ft. instead of 4 rd.; what was the true distance round the field? *Ans.* 335 rd. 8$\frac{1}{8}$ ft.

39. Our chapel clock is set at 12 o'clock Monday noon, and on Tuesday morning at 9 o'clock it had lost 3 minutes; what will be the correct time when it strikes 6 o'clock the next Thursday evening? *Ans.* 6 h. 11 min. 10$\frac{70}{419}$ sec.

40. The fore wheel of a wagon is 8 ft. 6 in. in circumference and the hind wheel is 12 ft. 3 in.; how many times does the fore wheel turn in going a distance in which the hind wheel turns 750 times? *Ans.* 1080$\frac{15}{17}$ times.

41. A garrison of 3500 men has provision sufficient to last 30 days at the rate of 1 lb. 4 oz. a day; how large a reinforcement could be received for the full time if the allowance was reduced to 14 oz. a day? *Ans.* 1500 men.

42. A garrison of 8000 men has "hard tack" sufficient to last 8 weeks, allowing each man 16 oz. a day; but 28000 lb. having been spoiled, what will be each man's daily allowance that the provision may last the eight weeks? *Ans.* 15 oz.

COMPOUND PROPORTION.

739. A Compound Proportion is a proportion in which one or both ratios are compound.

740. Thus, $\left\{\begin{matrix}4:8\\3:9\end{matrix}\right\}::7:42$ and $\left\{\begin{matrix}3:\ 6\\5:10\end{matrix}\right\}::\left\{\begin{matrix}4:\ 8\\6:12\end{matrix}\right\}$ are examples of compound proportion.

PRINCIPLES.

1. *The product of the simple ratios of the first couplet equals the product of the simple ratios of the second couplet.*

For, the value of a compound ratio is the product of the simple ratios, and these compound ratios are equal, since a proportion expresses the equality of ratios. Thus, from the second of the above proportions, we have $\frac{3}{6}\times\frac{5}{10}=\frac{4}{8}\times\frac{6}{12}$.

2. *The product of all the terms in the extremes equals the product of all the terms in the means.*

For, from the nature of proportion, we have from the proportion above, $\frac{3}{6}\times\frac{5}{10}=\frac{4}{8}\times\frac{6}{12}$, and clearing of fractions, we have $3\times5\times8\times12=4\times6\times6\times10$, which by examination, we see is the product of the extremes equal to the product of the means.

3. *Any term in either extreme equals the product of the means divided by the product of the other terms in the extremes.*

For, since from the proportion above we have $3\times5\times8\times12=4\times6\times6\times10$, we will have $3=\frac{4\times6\times6\times10}{5\times8\times12}$, and similarly for any other term in either extreme.

4. *Any term in either mean equals the product of the extremes divided by the product of the other terms in the means.*

For, from the above proportion, we have $3\times5\times8\times12=4\times6\times6\times10$, hence $4=\frac{3\times5\times8\times12}{6\times6\times10}$, and similarly for any other term in the means.

EXAMPLES FOR PRACTICE.

Find the term denoted by x in each of the following:

1. $x:16::\left\{\begin{matrix}7:\ 6\\15:14\end{matrix}\right\}$. *Ans.* 20.

2. $\left\{\begin{matrix}4:\ 8\\5:15\end{matrix}\right\}::\left\{\begin{matrix}7:\ x\\6:12\end{matrix}\right\}$. *Ans.* 21.

3. $\left\{\begin{matrix}7:14\\9:\ 3\end{matrix}\right\}::\left\{\begin{matrix}15:5\\8:x\end{matrix}\right\}$. Ans. 16.

4. $\left\{\begin{matrix} 18 : 12 \\ 17 : x \end{matrix}\right\} : : \left\{\begin{matrix} 19 : 38 \\ 21 : 14 \end{matrix}\right\}.$ *Ans.* **34.**

5. $\left\{\begin{matrix} 9 : 12 \\ 15 : 20 \\ 18 : 27 \end{matrix}\right\} : : \left\{\begin{matrix} x : 34 \\ 21 : 28 \end{matrix}\right\}.$ *Ans.* **17.**

6 $\left\{\begin{matrix} \frac{2}{3} : \frac{6}{5} \\ \frac{7}{8} : \frac{11}{12} \end{matrix}\right\} : : \left\{\begin{matrix} 1\frac{1}{8} : \frac{3}{4} \\ \frac{7}{11} : x \end{matrix}\right\}.$ *Ans.* **$1\frac{4}{5}$.**

APPLICATION OF COMPOUND PROPORTION.

741. Compound Proportion is used in the solution of problems in which the required term depends on a compound ratio.

742. The **Unknown Quantity** in simple proportion depends upon the relation of *one pair* of similar quantities; in compound proportion it depends upon *two or more pairs* of similar quantities.

NOTE.—Problems in compound proportion may be solved by two or more simple proportions, or by analysis.

1. If 6 men earn \$90 in 5 days, how much will 8 men earn in 9 days?

OPERATION.

$$\text{The sum} : \$90 : : \left\{\begin{matrix} 8 : 6 \\ 9 : 5 \end{matrix}\right\}.$$

$$\text{The sum} = \frac{\$90 \times 8 \times 9}{6 \times 5} = \$216, \textit{ Ans.}$$

SOLUTION.—It is evident that the sum that 8 men can earn in a given time is to the sum that 6 men can earn in that time, as 8 to 6, and also the sum that they can earn in 9 days is to the sum that they can earn in 5 days, as 9 is to 5; hence the sum that 8 men can earn in 9 days is to \$90 (the sum that 6 men will earn in 5 days) as 8 : 6 and 9 : 5; hence we have the proportion, *The sum* : \$90 : : $\left\{\begin{matrix} 8 : 6 \\ 9 : 5 \end{matrix}\right\}$; from which, by Prin. 3, we have, *The sum* $= \frac{\$90 \times 8 \times 9}{6 \times 5}$, or \$216.

Rule.—I. *Put the required quantity for the first term and the similar known quantity for the second term, and form ratios with each pair of similar quantities for the second couplet, as if the result depended upon each pair and the second term.*

II. *Find the required term by dividing the product of the means by the product of the fourth terms.*

NOTES.—1. Teachers may put the unknown quantity in the fourth term instead of the first, if they prefer it. The method of solution will be the same in principle, and the rule can be readily changed to correspond with it.

2. Pupils should be required to solve both ways, and to give the rule for both methods.

2. If 47 horses eat 94 bundles of hay in 36 days, how many bundles will 57 horses eat in 48 days? *Ans.* 152.

3. If 25 yd. of muslin $1\frac{1}{8}$ yd. wide cost \$7.25, what cost 27 yd. of the same quality, $1\frac{1}{4}$ yd. wide? *Ans.* \$8.70.

4. If 1053 bricks, 8 in. long, 4 in. wide, are required for a walk 39 ft. long, 6 ft. wide, how many bricks will be required for a walk 144 ft. long and 8 ft. wide? *Ans.* 5184 bricks.

5. If 20 pipes, each delivering 18 gal. a minute, fill a cistern in 3 h. 24 min., how many pipes, each delivering 12 gal. a minute, will fill a cistern twice as large in 4 h. 15 min.? *Ans.* 48 pipes.

6. A farmer has a bin 7 ft. long, 5 ft. wide, and 4 ft. deep, which holds 112 bu. of corn; how deep must he make another which is 20 ft. long and 9 ft. wide, so that it may hold 864 bushels? *Ans.* 6 ft.

7. How many days will it take 15 men to cut 810 cords of wood, working 9 hours a day, if 13 men can cut 364 cords in 14 days, working 12 hours a day? *Ans.* 36 days.

8. Required the cost of 192 loaves of bread, each loaf weighing 7 oz., when flour is worth \$12 a barrel, if 315 loaves, weighing 6 oz. each, cost \$16.20, when flour is \$9 a barrel. *Ans.* \$15.36.

9. If \$7486.50 be paid for a farm of 150 A. 150 P., what will be the cost of 90 A. 75 P., if 6 acres of the latter be worth 5 of the former? *Ans.* \$3739.37½.

10. How many men will be required to dig a trench 450 rods long, 18 ft. wide, and 10 ft. deep, in 18 days, if 45 men can dig a trench 180 rods long, 15 ft. wide, and 9 ft. deep, in 12 days? *Ans.* 100 men.

11. If a cistern 28 ft. long, 14 ft. wide, 11 ft. deep, hold 512 barrels of water, how many barrels of water will a cistern hold that is 21 ft. long, 8 ft. deep, and 11 ft. wide? *Ans.* $219\frac{3}{7}$ bar.

12. If 13 men can cut 364 cords of wood in 14 days by working 12 hours a day, how many hours a day must 15 men work to cut 810 cords in 36 days? Ans. 9 hours.

13. If 24 pipes, each delivering 6 gal. a minute, fill a cistern 8 ft. long, 6 ft. wide, and 5 ft. deep, in $12\frac{36}{77}$ min., how many pipes, each flowing 8 gal. a minute, will fill a cistern 10 ft. long, 7 ft. wide, and 9 ft. deep, in $21\frac{9}{11}$ minutes?

Ans. 27 pipes.

14. What cost 54 planks 35 ft. long, 28 in. wide, and 5 in. thick, if 42 planks 36 ft. long, 25 in. wide, and 7 in. thick, cost $178 when lumber was worth $\frac{2}{7}$ more per foot?

Ans. 138.44\frac{4}{9}$.

15. The first couplet of a compound proportion is made up of the ratios 5 : 15 and 4 : 16, and the first ratio of the second couplet is 4 : 12; what is the second ratio, if the antecedent is 11? *Ans.* 11 : 44.

16. The first couplet of a compound proportion is made up of the ratios 5 : 15 and 4 : 16, and the first ratio of the second couplet is 4 : 12; what is the other ratio, if the antecedents of the second couplet are as 2 to 7? *Ans.* 14 : 56.

17. If 6 compositors in 18 days of 12 hours each, set up 27 sheets of 24 pages each, 45 lines on a page and 48 letters in a line, in how many days, 10 hours long, can 7 compositors set up, in the same type, 35 sheets, 16 pages each, 51 lines to a page, 45 letters in a line? *Ans.* 17 days.

18. The second couplet of a compound proportion consists of the simple ratios 8 : 10 and 14 : 16, and the antecedents of the first couplet are as 9 : 7, and the second consequent of that couplet is 8; required the ratios of the first couplet.

Ans. 18 : 45 and 14 : 8.

19. The first couplet of a compound proportion consists of the simple ratios 7 : 11 and 8 : 14, and the consequents of the second couplet are as 7 : 11, and the first antecedent of that couplet is 9; required the ratios of the second couplet.

Ans. 9 : 21 and 28 : 33.

20. If 27 men in 18 days of 10 hours each dig a ditch 180 rods long, 6 ft. wide, and 3 ft. deep, of 5 degrees of hardness, how many days of 9 hours each will it take 45 men to dig a ditch 300 rods long, 8 ft. wide, and 4 ft. deep, of $7\frac{1}{2}$ degrees of hardness? *Ans.* $53\frac{1}{3}$ days.

PARTITIVE PROPORTION.

743. Partitive Proportion is the process of separating a number into parts which bear certain relations to each other.

744. There are several cases arising from the various relations which may exist between the parts into which a number is divided.

NOTE.—The method of solution is analytical, and no rule is given.

CASE I.

745. ***When one part is a number more or less than another.***

1. Divide 48 into two parts so that the first may be 12 more than the second.

SOLUTION.—The 2d part plus 12 equals the 1st part, which, added to the 2d part, equals 2 times the 2d part, plus 12, which equals 48; if twice the 2d, plus 12 equals 48, twice the 2d part equals 48 minus 12, or 36, and once the 2d part equals ½ of 36, or 18, and the second part plus 12 equals 18 plus 12, or 30.

OPERATION.

$$\begin{aligned} 2 \text{ times } 2d + 12 &= 48 \\ 2 \text{ times } 2d &= 36 \\ 2d &= 18 \\ 1st &= 30 \end{aligned}$$

2. A and B have $20,000, and A has $1500 more than B; what is the fortune of each? *Ans.* A, $10750; B, $9250.

3. A man divided $50,000 among his three sons, giving the first $15,000 more than the second, and the second $5000 less than the third; how much did each receive?
Ans. 1st, $25,000; 2d, $10,000; 3d, $15,000.

4. A and B had the same number of shares of Erie; A sold 60 shares and B bought 54 shares, and they then together had 144 shares; how many had each at first? *Ans.* 75.

5. Four young men, A, B, C, and D, started to Europe with $5000; A had $57 more than B, C had $65 less than D D had $98 more than B; how much money had each?
Ans. A $1260; B, $1203; C, $1236; D, $1301

CASE II.

746. ***When one part is a number of times another or a fractional part of another.***

1. A and B together have \$680, and A has 4 times as much as B; how much has each?

SOLUTION.—Since A's share equals 4 times B's, they together are equal to 5 times B's; hence 5 times B's equals \$680, once B's equals $\frac{1}{5}$ of \$680, or \$136, and A's equals \$136×4, or \$544.

OPERATION.

$$\begin{aligned} B's+4\,B's &= 680 \\ 5\,B's &= 680 \\ B's &= 136 \\ A's &= 544 \end{aligned}$$

2. Two Western farmers together had 9232 sheep, and 3 times what the first has, minus 128, equals what the second has, plus 240; how many sheep has each?

Ans. 1st, 2400; 2d, 6832.

3. Two California miners, C and D, sat down to play with \$840; C lost $\frac{3}{4}$ of his money; D then lost $\frac{1}{4}$ of what he then had, when it was found that C had $\frac{2}{3}$ as much as D; how much had each at first? *Ans.* C, \$800; D, \$40.

4. A Texan farmer owns 5169 cattle; there are 3 times as many horses as cows, plus 569, and 4 times as many cows as sheep, minus 125; how many has he of each?

Ans. 300 sheep; 1075 cows; 3794 horses.

5. Two teachers, Martha and Mary, saved \$1100; if Martha's money be increased by \$20, and Mary's be increased by \$24, Martha's will equal $\frac{5}{6}$ of Mary's; how much had each at first? *Ans.* Martha, \$500; Mary, \$600.

CASE III.

747. ***When a number of times one part equals a number of times another part.***

1. A and B together have \$483, and 3 times A's share equals 4 times B's share; how much has each?

SOLUTION.—Since 3 times A's share equals 4 times B's, once A's share equals $\frac{4}{3}$ of B's, and adding to B's, we have $\frac{7}{3}$ of B's, which equals what both have, or \$483, hence $\frac{1}{3}$ of B's is \$69, B's is \$207, and A's \$276.

OPERATION.

$$\begin{aligned} B's+\tfrac{4}{3}\text{ of }B's &= 483 \\ \tfrac{7}{3}\text{ of }B's &= 483 \\ \tfrac{1}{3}\text{ of }B's &= 69 \\ B's &= 207 \\ A's &= 276 \end{aligned}$$

2. On a Minnesota prairie there are 797 horses and cows, and 3 times the number of cows equals 7 times the number of horses, minus 69; how many are there of each?

Ans. 551 cows; 246 horses.

3. A widow dying, divided her property, worth \$62,000,

among her three sons, A, B, and C; what was the share of each if $\frac{2}{3}$ of A's equals $\frac{4}{5}$ of B's, and $\frac{3}{4}$ of B's equals $\frac{5}{6}$ of C's? *Ans.* A's, \$24,000; B's, \$20,000; C's, \$18,000.

4. Says Mary to Martha, "I see that we together have \$60, but if you give me \$9 and I then give you \$12, three times what you then have will equal twice what I have;" how much had each? *Ans.* Mary, \$39; Martha, \$21.

5. Two men entered business with a united capital of \$25,000; subsequently the first added \$760, and the second took out \$791, and then $\frac{5}{6}$ of the share of the first equaled $\frac{7}{8}$ of the share of the second; what was the original capital of each? *Ans.* 1st, \$12,029; 2d, \$12,971.

CASE IV.

748. ***When the parts are to each other as two or more numbers.***

1. The sum of two numbers is 153; what is each number, if they are to each other as 8 to 9?

SOLUTION.—Since the numbers are to each other as 8 to 9, if we divide 153 into 8 plus 9, or 17 equal parts, 8 of these parts, or 72, equals the first, and 9 of these parts, or 81, equals the second.

OPERATION.

$$8+9=17$$
$$\tfrac{1}{17} \text{ of } 153 = 9$$
$$\tfrac{8}{17} \text{ of } 153 = 72$$
$$\tfrac{9}{17} \text{ of } 153 = 31$$

2. Divide the reciprocal of the fraction $\frac{5}{9}$ into two parts, which shall be to each other as the reciprocals of the fractions $\frac{2}{3}$ and $\frac{4}{5}$. *Ans.* $\frac{54}{55}$ and $\frac{9}{11}$.

3. A, B, and C agree to pay \$1000 toward building an academy which is to be situated $\frac{1}{2}$ of a mile from A, $\frac{2}{3}$ of a mile from B, and $1\frac{1}{2}$ miles from C; how much does each contribute, provided the sums are in proportion to the reciprocals of the distances? *Ans.* A, \$480; B, \$360; C, \$160.

4. A man divided 1015 acres of land among his wife, son, and daughter; the wife's share plus 35 acres was to the son's share as 5 to 6, and the son's share minus 35 acres was to the daughter's share as 5 to 6; required the share of each.
Ans. Wife, 265; son, 360; daughter, 390.

5. A and B constructed 2427 miles of railroad, and 4 times the number of miles A made, plus 48 miles, is to 5 times

the number B made, minus 40 miles, as $\frac{3}{4}$ to $\frac{5}{8}$; how much did each construct? *Ans.* A, 1275 mi.; B, 1152 mi.

6. Emma's fortune plus $\frac{4}{5}$ of Frances's, which equals $\frac{2}{3}$ of Emma's, is \$15,000; and if the sum of Emma's and Frances's be divided in the proportion of $\frac{2}{3}$ to $\frac{4}{5}$, it will respectively give $\frac{2}{3}$ of George's and $\frac{4}{5}$ of Henry's fortune; required the fortune of each.

Ans. E, \$9,000; F, \$7,500; G, \$11,250; H, \$11,250.

CONJOINED PROPORTION.

749. Conjoined Proportion is the process of comparing numbers so related that each consequent is of the same kind as the next antecedent.

750. The method of treatment is analytical, and presents one of the best illustrations of the beautiful process of *Arithmetical Analysis.*

NOTE.—Arbitration of Exchange, which has already been treated, is an application of Conjoined Proportion.

1. How many cents are 20 melons worth, if 6 melons are worth 12 oranges, and 8 oranges are worth 24 cents?

OPERATION.

$\frac{20}{6} \times \frac{12}{8} \times 24 = 120$, *Ans.*

SOLUTION.—If 8 oranges are worth 24 cents, 1 orange is worth $\frac{1}{8}$ of 24 cents, and 12 oranges are worth 12 times $\frac{1}{8}$ of 24 cents, or $\frac{12}{8}$ of 24 cents: if 6 melons are worth $\frac{12}{8} \times 24$ cents, 1 melon is worth $\frac{1}{6}$ of $\frac{12}{8} \times 24$ cents, and 20 melons are worth 20 times $\frac{1}{6} \times \frac{12}{8} \times 24$ cents, which is $\frac{20}{6} \times \frac{12}{8} \times 24$ cents, which, reduced by cancellation and multiplication, equals 120 cents.

OPERATION.

24 cents - 8 oranges.
12 oranges - 6 melons.
20 melons - x

$$x = \frac{24 \times 12 \times 20}{8 \times 6} = 120, \textit{ Ans.}$$

SOLUTION 2D.—We will represent the term we wish to find by x. Now, if we arrange the quantities so that each stands opposite its equivalent, as in the margin, the product of the terms in the first column will equal the product of the terms in the second column; hence the product of the terms in the first column, divided by the product of all the terms in the second column except x, will give the value of x. Hence the following

Rule.—I. *Place the antecedents in one column and the consequents in another, with a hyphen between them.*

II. *Divide the product of the terms in the column containing the odd term by the product of the terms in the other column.*

2. How much will 150 horses cost, if 10 horses are worth 24 cows, 3 cows are worth 18 sheep, 16 sheep are worth 15 pigs, and 20 pigs are worth $100? *Ans.* $10,125.

3. In a cotton factory it was found that 6 men do as much as 8 boys, and 6 boys do as much as 9 girls; how many girls will be required to do as much as 54 men? *Ans.* 108.

4. A can do 3 times as much in a day as B, and B can do 4 times as much as C; in how many days can A do as much as C can do in 12 days? *Ans.* 1 day.

5 A can do 2 times as much as B in a day, B can do 3 times as much as C, C can do 2 times as much as D, and D can do $\frac{1}{3}$ as much as E; in what time can E do as much as A does in 24 days? *Ans.* 96 days.

6. If A can do $\frac{2}{3}$ as much in a day as B, B can do $\frac{3}{4}$ as much as C, and C can do $\frac{4}{5}$ as much as D; in what time can D do as much work as A does in 16 days? *Ans.* $6\frac{2}{5}$ days.

7. If 54 shillings in Vermont equaled 72 shillings in New York, and 56 shillings in New York equaled 35 shillings in Canada, and 40 shillings in Canada equaled 60 shillings in Pennsylvania, how many shillings in Pennsylvania were equal to 50 shillings in Vermont? *Ans.* 62 s. 6 d.

8. If A earns as much in 6 months as B does in 9 months, and B as much in $2\frac{1}{2}$ months as C does in $3\frac{1}{2}$ months, and C as much in 8 months as D in 6 months, in what time could D earn as much as A earns in $\frac{3}{4}$ of a year? *Ans.* $14\frac{7}{40}$ mo.

9. If 10 bushels of wheat are worth $12\frac{1}{2}$ bushels of rye, and $4\frac{3}{4}$ bushels of rye are worth $6\frac{4}{5}$ bushels of corn, and 9 bushels of corn are worth 12 bushels of oats, and $16\frac{2}{3}$ bushels of oats are exchanged for 50 pounds of sugar, how many pounds of sugar could you obtain for 38 bushels of wheat? *Ans.* 272 pounds.

10. If 6 bushels of wheat in Philadelphia are worth 7 bushels at Pittsburgh, and 15 bushels at Pittsburgh are worth 20 in Chicago, and 25 bushels in Chicago are worth 30 in Omaha, and 25 in Omaha are worth 20 in San Francisco, how many bushels in San Francisco will be worth 40 bushels in Philadelphia? *Ans.* $59\frac{11}{15}$ bushels.

MEDIAL PROPORTION.

751. Medial Proportion is the process of combining two or more quantities of different values.

752. The **Mean Value** is the average value of the combination.

NOTE.—The subject has been called *Alligation*, from *alligo*, I bind, a name suggested by the method of linking the figures with a line in solving the problems. Case I., as here presented, was called *Alligation Medial*, and the other cases, *Alligation Alternate*.

CASE I.

753. ***Given, the quantity and the value of each ingredient, to find the mean value.***

1. A grocer mixed 20 lb. of sugar worth 10 cents a pound, 25 lb. at 12 cents, and 30 lb. at 15 cents a pound; what is the mean value of the mixture?

SOLUTION.—20 lb. at 10 cents a pound cost 200 cents, 25 lb. at 12 cents cost 300 cents, 30 lb. at 15 cents cost 450 cents; taking the sum, we find that 75 lb. cost 950 cents; hence 1 lb. cost $\frac{1}{75}$ of 950 cents, which is $12\frac{2}{3}$ cents; hence the mean value of the mixture is $12\frac{2}{3}$ ct.

OPERATION.

lb.		¢		¢
20	@	10	=	200
25	@	12	=	300
30	@	15	=	450
75				950

75)950($12\frac{2}{3}$

Rule.—*Find the sum of the values of the ingredients and divide it by the sum of the ingredients.*

2. A merchant mixed 24 lb. of tea at 60¢ a pound, 35 lb. at 80¢, and 61 lb. at 120¢; what is the mean value of the mixture? *Ans.* $96\frac{1}{3}$¢.

3. A person mixed 24 gal. of brandy at $2.10, 36 gal. at $2.60, and 40 gal. at $3.30, with 20 gal. of water; what was the value of a gallon of the mixture? *Ans.* $2.30.

4. A smith combined 12 oz. of 20 carats fine, 15 oz. 18 carats fine, 24 oz. 22 carats fine, with 9 oz. pure gold; required the fineness of the mixture. *Ans* $20\frac{9}{10}$ carats.

5. On a certain day the thermometer ranged at 64° from 6 o'clock to 9, at 76° from 9 to 12, at 85° from 12 to 3, and at 68° from 3 to 6; what was the average temperature? *Ans.* $73\frac{1}{4}$°.

6. A person mixed 15 gal. of alcohol 80% strong, 12 gal 90% strong, 23 gal. 60% strong, and 20 gal. 70% strong; what is the strength of the mixture? *Ans.* $72\frac{2}{7}$%.

7. Find the specific gravity of a compound of 20 lb. copper, specific gravity $7\frac{3}{4}$; 10 lb. of zinc, specific gravity $6\frac{7}{8}$; and $\frac{1}{2}$ lb. silver, specific gravity $10\frac{1}{2}$. *Ans.* $7\frac{31}{61}$.

CASE II.

754. ***Given, the mean value and the value of each ingredient, to find the proportional quantity of each.***

1. What relative quantities of sugar, worth 6, 7, 12, and 15¢ a pound, must be taken to form a mixture worth 10¢ a pound?

OPERATION. *Ans.*

$$10\left\{\begin{array}{c}6\\7\\12\\15\end{array}\right.\left|\begin{array}{c}\frac{1}{4}\\ \\ \\ \frac{1}{5}\end{array}\right|\begin{array}{c} \\ \frac{1}{3}\\ \frac{1}{2}\\ \\ \end{array}\left|\begin{array}{c}5\\ \\ \\ 4\end{array}\right|\begin{array}{c} \\ 2\\ 3\\ \\ \end{array}\left|\begin{array}{c}5\\2\\3\\4\end{array}\right|$$

SOLUTION.—If we take 1 lb. at 6 cents for the mixture worth 10¢, we gain on it 4¢, and to gain 1 cent we would take $\frac{1}{4}$ of a pound. If we take 1 lb. at 15¢, we will lose 5¢, and to lose 1 cent, what we have just gained, we would take $\frac{1}{5}$ lb.; hence we take $\frac{1}{4}$ lb. at 6¢ as often as $\frac{1}{5}$ lb. at 15¢, or in whole numbers, 20 times $\frac{1}{4}$, which is 5 of the first, as often as 20 times $\frac{1}{5}$, which is 4 of the fourth. In a similar manner, we find that we must take 2 lb. at 7¢, as often as 3 lb. at 12¢; hence the quantities may be mixed in the proportion of 5, 2, 3, and 4.

Rule.—I. *Write the several prices or qualities in a column, and the mean price or quality of the mixture at the left.*

II. *Select two quantities, the one less and the other greater than the average, write the reciprocals of the difference between each quantity and the average opposite the quantity, and reduce these to integers by multiplying by the least common denominator, and proceed in the same manner until all the prices have been used.*

III. *Add two or more proportional numbers if they stand opposite a given quantity; the results will be the proportional numbers required.*

NOTES.—1. When there are three quantities, compare *the one* which is greater or less than the average with *both* the others, and take the sum of the two numbers opposite *this one*.

2. A common factor may be inserted in any couplet or omitted from it without changing the proportional parts; *it is thus seen that there may be any number of answers in the same proportion.*

2. A merchant has three kinds of muslin worth 16, 18, and 25 cts. a yard; how many yards must he sell from each that the price may average 20 cts? *Ans.* 5 yd.; 5 yd.; 6 yd.

3. A silversmith combines gold of $16\frac{1}{2}$ carats, $17\frac{1}{8}$ carats, and $22\frac{1}{2}$ carats, with pure gold to make a mixture of 20 carats; in what proportion should he combine them?

Ans. 8; 15; 16; 7.

4. What relative quantities of alcohol 78%, 82%, 89%, 92%, and 98%, must be taken to form a mixture which shall be 86% strong? *Ans.* 3; 6; 4; 2; 2.

5. A goldsmith wishes to mix gold $\frac{3}{4}$ pure, $\frac{5}{6}$ pure, and $\frac{9}{10}$ pure, to make a mixture $\frac{7}{8}$ pure; required the proportion of each quantity. *Ans.* 3 lb.; 3 lb.; 20 lb.

6. A man has a sum of money consisting of 1-cent, 3-cent, 5-cent, 25-cent, and 50-cent pieces, which he wishes to exchange for 10-cent pieces; what is the relative number of pieces of each exchanged? *Ans.* 40; 15; 8; 7; 10.

7. In what proportion must I mix gold and silver whose specific gravities are $19\frac{1}{4}$ and $10\frac{1}{2}$ respectively, to make a compound whose specific gravity shall be $15\frac{3}{4}$?

CASE III.

755. ***Given, the mean value, the value of each ingredient, and the relative amounts of two or more, to find the quantity of each.***

1. A man bought sheep at \$8, cows at \$35, oxen at \$60, and horses at \$150; the sheep were to the cows as 3 to 2, and the price averaged \$50; required the number of each.

OPERATION.

						Ans.
50	8	$\frac{1}{42}$		5		30
	35		$\frac{1}{15}$		20	20
	60	$\frac{1}{10}$		21		126
	150		$\frac{1}{100}$		3	3

SOLUTION.—We find by Case II., that the sheep and oxen were as 5 to 21, and the cows and the horses as 20 to 3; and since the sheep are to be to the cows as 3 to 2, we must have 6 times 5, or 30 sheep; and since there must be 21 oxen to 5 sheep there must also be 6 times 21, or 126 oxen.

Rule.—I. *Find the proportional quantities by Case II.*

II. *Multiply each of the proportional parts by numbers which will produce the required proportions.*

2. I have some 3-cent, 5-cent, 25-cent, and 50-cent pieces, which I wish to exchange for their value in 10-cent pieces; how many of each will it take, if the 3-cent and 5-cent pieces are in the proportion of 5 to 6? *Ans.* 40; 48; 16; 7.

3. A farmer bought pigs at $\$4\frac{1}{4}$, sheep at $\$5\frac{1}{2}$, and calves at $\$7\frac{1}{3}$; he sold them at an average of $6; how many were there of each, if the number of sheep and pigs were in the proportion of 2 to 3? *Ans.* 48; 32; 75.

4. Having gold $\frac{3}{4}$ pure, $\frac{5}{6}$ pure, and $\frac{7}{8}$ pure, I wish to make two mixtures of them $\frac{4}{5}$ pure, the first to contain equal quantities of the first and second kinds, and the second equal quantities of the second and third; what are the required proportions?

$$Ans. \begin{cases} \text{1st, } 9, 9, 2. \\ \text{2d, } 13, 6, 6. \end{cases}$$

5. A merchant having teas worth $\$\frac{3}{4}$, $\$\frac{5}{6}$, $\$1\frac{1}{4}$, and $\$1\frac{1}{3}$ respectively, wishes to make six mixtures at $1 a pound, the first containing equal quantities of the 1st and 2d, the second equal quantities of the 1st and 3d, the third equal quantities of the 1st and 4th, the fourth equal quantities of the 2d and 3d, the fifth equal quantities of the 2d and 4th, and the sixth equal quantities of the 3d and 4th.

$$Ans. \begin{cases} \text{1st, } 12, 12, 8, 9;\ \text{2d, } 4, 6, 4, 3;\ \text{3d, } 6, 12, 6, 6; \\ \text{4th, } 12, 12, 12, 6;\ \text{5th, } 4, 3, 2, 3;\ \text{6th, } 8, 9, 6, 6. \end{cases}$$

CASE IV.

756. ***Given, the mean value, the value of each ingredient, and the quantity of one or more, to find the other quantities.***

1. A farmer bought 30 sheep at $12 each; how many must he buy at $5 and $6 each, so that they may average $9 each?

OPERATION.

$$9\begin{cases} 5 \\ 6 \\ 12 \end{cases}\begin{array}{c|c|c|c} & 3 & & 3 \\ & & 1 & 1 \\ & 4 & 1 & 5 \end{array} \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} \times 6 = \begin{cases} 18 \\ 6 \\ 30 \end{cases}$$

SOLUTION.—We find by Case II., that the number at $5 and $12 are as 3 to 4, and at $6 and $12 are as 1 to 1; hence, as often as he buys 3 at $5 and 1 at $6, he will buy 4+1, or 5, at $12; but he bought 30, which is 6 times 5, at $12; hence he must buy 6 times 3, or 18, at $5, and 6 times 1, or 6, at $6.

Rule.—I. *Find the proportional quantities by Case II.*

II. *Divide the given quantity by the proportional quantity limited, and multiply each of the other proportional quantities by the quotient.*

2. A drover sold some hogs at $10, some sheep at $6, and 80 cows at $45 a head; the average price was $20; how many hogs and sheep were there? *Ans.* 25 hogs; 125 sheep.

3. How many railroad shares, at 45%, 50%, and 65%, must I buy so that with my 130 shares at 70%, the average price may be 58%? *Ans.* 120; 70; 80.

4. A merchant mixed 76 lb. of tea worth $1.25, and 34 lb. worth $1.12½, with that worth $0.95 and $0.80; how much did he take of each, if the average price was $1?

Ans. 85; 95.

5. I mixed 2 gal. 2 qt. of water with 19 gal. 3 qt. of acid; the mixture has 20% more acid than desired; how much water will reduce it to the required strength?

Ans. 4 gal. 1⅗ qt.

6. A jeweler has 2 pwt. 8 gr. of gold 16½ carats fine, and 2 pwt. 16 gr. 18¾ carats fine; how much pure gold must be added to make a mixture of 20 carats fine? *Ans.* 2 pwt. 21 gr.

CASE V.

757. ***Given, the mean value, the value of each ingredient, and the entire quantity, to find the quantity of each ingredient.***

1. A person has a sum of money in 3-cent, 5-cent, 25-cent, and 50-cent pieces, which he wishes to exchange for 10-cent pieces; how many will it take of each, if there are 306 in all?

OPERATION.

$$10\left\{\begin{array}{r|r|r|r} & a & b & \\ 3 & 40 & & 40 \\ 5 & & 3 & 3 \\ 25 & & 1 & 1 \\ 50 & 7 & & 7 \\ & & & \overline{51} \end{array}\right. \times 6 = \left\{\begin{array}{r} 240 \\ 18 \\ 6 \\ 42 \\ \overline{306} \end{array}\right.$$

$$306 \div 51 = 6$$

SOLUTION.—We find by Case II., that we must have 40 three-cent pieces as often as 7 fifty-cent pieces, and also 3 five-cent pieces as often as one 25-cent piece. Taking the sum of these we have 51 in all; but we wished 306, which is 6 times 51, hence we must take 6 times as many of each, which gives respectively 240, 18, 6, and 42.

Rule.—I. *Find the proportional quantities by Case II*

II. *Divide the required quantity by the sum of the proportional quantities, and multiply each proportional quantity by the quotient.*

NOTES.—1. When the sum of the proportional parts is not an exact divisor of the quantity, each couplet must be multiplied by such numbers as will make the sum of the proportional parts a divisor of the entire quantity.

2. If in the above problem, we had 110 pieces, we would multiply Col. *b* by 2 and add the result to Col. *a*, obtaining 55, and then multiply by 2. If we had 212 pieces, we would take 2 times Col. *a* plus 3 times Col. *b*, and multiply by 2.

2. A man mixed teas worth 40¢, 52¢, 65¢, and 72¢ a pound, making a mixture of 126 lb. worth 60¢ a pound; how much did it take of each kind? *Ans.* 18, 30, 48, 30.

3. A man bought 140 head of poultry for \$56; hens at 20¢, ducks at 35¢, geese at 50¢, and turkeys at 75¢; how many did he purchase of each kind? *Ans.* 70, 20, 10, 40.

4. A broker bought 220 shares of stock (\$50) at an average advance of 10%; some at a discount of 6%, some at a discount of 2%, some at an advance of 16%, and some at an advance of 20%; how many shares did he buy of each kind?

Ans. 50, 30, 60, 80.

5. A man has \$155 in 10-cent pieces which he wishes to exchange for 3-cent, 5-cent, 25-cent, and 50-cent pieces respectively; how many of each kind will it take?

Ans. 750, 400, 350, 50.

6. A banker bought 100 shares of stock (\$50) at an average of 10% below par, and sold it at an average of 10% above par; some at a discount of 20%, some at a discount of 15%, some at par, and some at a premium of 15%; required the number of each kind. *Ans.* 4, 8, 8, 80.

PROBLEMS IN INDETERMINATE ANALYSIS.

758. There is a class of problems in Indeterminate Analysis which can be readily solved by the process of the last case.

NOTE.—Several of these problems have more than one answer. Pupils should be required to ascertain several results, even when but one is given.

1. A person purchased 100 animals for \$100; sheep at \$3½ apiece, calves at \$1⅓, and pigs at \$½; how many animals did he buy of each kind?

OPERATION.

				a	b	c	d	e
6	21	$\frac{1}{15}$	0	1	0	5	0	5
	8	0	$\frac{1}{2}$	0	3	0	42	42
	3	$\frac{1}{8}$	$\frac{1}{3}$	5	2	25	28	53
				6	5	30	70	100

SOLUTION.—We find by Case II., that we must buy 1 sheep for every 5 pigs, and 3 calves for 2 pigs. We now wish to combine these columns in such a way as to obtain 100 animals. By inspection we see that we may take 5 times column **a**, giving column **c**, and 14 times column **b**, giving column **d**; the sum of which gives column **e**. Hence there were 5 sheep, 42 calves, and 53 pigs.

2. A farmer bought 100 animals for \$100; geese at \$$\frac{1}{2}$ each, pigs at \$3, and calves at \$10; how many animals were there of each kind? *Ans.* 94, 1, 5.

3. A lady bought 10 books of three different kinds for \$30; the first kind cost \$4$\frac{1}{4}$ each, the second \$2$\frac{1}{2}$, and the third \$2; required the number of each. *Ans.* 4, 2, 4.

4. A man bought 20 birds for 20 pence, consisting of pigeons at 4 pence, grouse at $\frac{1}{2}$ penny, and larks at $\frac{1}{4}$ penny each; how many were there of each kind? *Ans.* 3, 15, 2.

5. A woman bought 12 loaves for 12 pence; some were two-penny, others penny, and the rest farthing loaves; what number was there of each sort? *Ans.* 3, 5, 4.

6. A farmer buys oxen, sheep, and ducks, 100 in all, for £100; required the number of each, if the oxen cost £5, the sheep £1, and the ducks 1 shilling. *Ans.* 19, 1, 80.

7. A person wishes to purchase 20 animals for £20, sheep at 31 shillings, pigs at 11 s., and rabbits at 1 s. each; how many of each kind will he buy?

Ans.				
	Sheep,	10,	11,	12.
	Pigs,	8,	5,	2.
	Rabbits,	2,	4,	6.

8. A person buys 100 head of cattle for £100; viz., oxen at £10, cows at £5, calves £2, and sheep at 10 s. each; how many were there of each?

Ans.										
	Oxen,	1,	1,	1,	1,	1,	1,	1,	1,	4,	4.
	Cows,	1,	2,	3,	4,	5,	6,	7,	8,	1,	2.
	Calves,	24,	21,	18,	15,	12,	9,	6,	3,	5,	2.
	Sheep,	74,	76,	78,	80,	82,	84,	86,	88,	90,	92.

NOTE.—The 4th and 8th problems are from Euler's Algebra; the 5th from Simpson's Algebra; the 6th from Todhunter's Algebra; and the 7th from Key to Ray's Algebra.

PARTNERSHIP.

759. Partnership is the association of two or more persons for the transaction of business.

760. Partners are the persons associated in business, and are of three kinds, *General, Limited,* and *Special.*

761. The **Capital** of a firm is the money or property invested by the partners. The *Liabilities* are its debts.

762. The **Resources** or **Assets** of a firm are its property of any kind, together with the amounts due it. The excess of resources over liabilities is called the *Net Capital.*

763. Partnership is divided into *Simple* and *Compound Partnership* for convenience of treatment.

General Partners risk their whole property in the business; *Limited and Special Partners* risk only the amount of capital they agree to contribute. Partners whose names do not appear are sometimes called *Silent Partners.*

SIMPLE PARTNERSHIP.

764. In **Simple Partnership** the shares of the partners are employed for equal periods of time.

1. A, B, and C went into partnership; A put in \$600; B put in \$800; and C put in \$1000; they gained \$480; what was each one's share of the gain?

SOLUTION.—The entire capital is \$2400. Since A put in \$600, he furnished $\frac{600}{2400}$, or $\frac{1}{4}$ of the capital, and hence should have $\frac{1}{4}$ of \$480, or \$120; B furnished $\frac{800}{2400}$, or $\frac{1}{3}$ of the capital, and hence should have $\frac{1}{3}$ of \$480, or \$160; and C should have $\frac{5}{12}$ of \$480, or \$200.

OPERATION.

\$600 $\frac{600}{2400} = \frac{1}{4}$, A's share.
800 $\frac{800}{2400} = \frac{1}{3}$, B's share.
1000 $\frac{1000}{2400} = \frac{5}{12}$, C's share.
\$2400 = Stock.

$\frac{1}{4}$ of \$480 = \$120, A's share.
$\frac{1}{3}$ of \$480 = \$160, B's share.
$\frac{5}{12}$ of \$480 = \$200, C's share.

Rule.—*Divide the gain or loss among the partners in proportion to their shares of the stock.*

NOTE.—The division may be made by analysis, or by Simple Proportion.

2. Three persons enter into partnership with \$12,000, of which A owns $\frac{1}{2}$, B $\frac{1}{3}$, and C the remainder; they gain \$900; what sum belongs to each?

Ans. A, \$450; B, \$300; C, \$150.

3. A man wills \$3000 to his daughter, \$3500 to his son, and \$4000 to his wife; but upon settling his estate his fortune was found to be only \$8400; how should the property be divided? *Ans.* D., \$2400; S., \$2800; W., \$3200.

4. A, B, and C entered into a partnership with a capital of \$8080; A's gain was \$1640, B's \$1500, and C's \$900; required each person's stock.

Ans. A's, \$3280; B's, \$3000; C's, \$1800.

5. Four men agree to share 120 gallons of wine, A taking $\frac{1}{3}$, B $\frac{1}{4}$, C $\frac{1}{5}$, and D $\frac{1}{6}$; but upon drawing off these parts there is still a remainder; how should the wine be divided?

Ans. A, $42\frac{2}{19}$ gal.; B, $31\frac{11}{19}$ gal.; C, $25\frac{5}{19}$ gal.; D, $21\frac{1}{19}$ gal.

6. A, B, and C in partnership gained \$2520; A's stock was \$4800, B's \$7200, and C's gain \$840; required C's stock, and A's and B's gain.

Ans. A, \$672; B, \$1008; C's stock, \$6000.

7. A, B, and C have a capital of \$54,000; at the time of closing up business, A has \$40,000, B \$32,000, C \$24,000, what was each one's original stock?

Ans. A, \$22,500; B, \$18,000; C, \$13,500.

8. Three men purchase a tract of land for \$15,000, of which A pays \$6000; they sell at such a price that B gains \$750.20 and C \$937.75; how much do B and C pay, and what is A's gain?

Ans. B, \$4000; C, \$5000; A's gain, \$1125.30.

9. A, B, and C form a company for manufacturing coal oil; A puts in \$14,000, B \$16,000, C \$8000; C receives a salary of \$2000 for personal attention to the business, while the expenses during the year are \$3500, and their receipts are \$9,500; what does each partner receive?

Ans. A, $\$1473.68\frac{8}{19}$; B, $\$1684.21\frac{1}{19}$; C, $\$2842.10\frac{10}{19}$.

10. A, B, and C speculate in flour; A contributes 500 barrels @ \$9.50, B 700 barrels @ \$10, and C 800 barrels @ \$8.50; they lose \$1.25 a barrel, and pay for expenses \$75; what is the loss of each?

Ans. A, $\$659.36\frac{244}{371}$; B, $\$971.69\frac{301}{371}$; C, $\$943.93\frac{197}{371}$.

11. A, B, and C formed a partnership; A put in \$5000.

B $7500, and C $9000; they gained 50 per cent., but receive the whole amount of their gain in notes, which they discount at 8 per cent.; what was each man's gain?

Ans. A, $2300; B, $3450; C, $4140.

COMPOUND PARTNERSHIP.

765. In **Compound Partnership** the capitals of the partners are employed for different periods of time.

CASE I.

766. ***When the profits and losses are divided in proportion to capital and time.***

1. Two persons enter into partnership and gain $452; A puts in $900 for 7 months, and B $1000 for 5 months; what was each one's share of the gain?

SOLUTION.—$900 for 7 mo. is equivalent to $6300 for 1 mo., and $1000 for 5 mo. is equivalent to $5000 for 1 mo.; hence their entire capital is equivalent to $11,300 for 1 mo. The rest of the solution may be given as in Simple Partnership.

OPERATION.

$\$900 \times 7 = \$6300$, A's for 1 mo.
$\$1000 \times 5 = \$5000$, B's for 1 mo.
$\$11,300$, whole for 1 mo.
$\frac{6300}{11300} = \frac{63}{113}$ = A's share of capital.
$\frac{5000}{11300} = \frac{50}{113}$ = B's share of capital.
$\$452 \times \frac{63}{113} = \$252$, A's gain.
$\$452 \times \frac{50}{113} = \$200$, B's gain.

Rule.—*Multiply each partner's capital by the time it was employed, and divide the gain or loss in proportion to these products.*

2. A commenced business with $8000 stock; 3 mo. after he took in B with a capital of $4000, and 4 mo. after he took in C with a capital of $1200; at the end of the year the firm had gained $4600; required the share of each.

Ans. A's, $3200; B's, $1200; C's, $200.

3. A and B are in partnership; A's capital was to B's as 6 to 8; at the end of 6 mo. A withdraws $\frac{1}{3}$ of his and B $\frac{1}{4}$ of his, and during the year they lose $1416; what is each man's share of the loss? *Ans.* A's, $590; B's, $826.

4. A's capital was in trade 4 mo., B's 5 mo., and C's 12 mo.; A's gain was $600, B's $500, and C's $900, and the whole capital was $20,865; how much did each own?

Ans. A, $9630; B, $6420; C, $4815.

5. A's capital is $800 and B's $1000; at the end of 6 mo how much more must A put in, so that at the end of the year he may be entitled to one-half the gain? *Ans.* $400.

6. A, B, and C engage in business; A invests his capital for 5 months and claims $\frac{4}{9}$ of the profits; B's capital is invested 6 months, and C puts in $5000 for 4 months and claims $\frac{2}{9}$ of the profits; what were A's and B's investments?

Ans. A, $8000; B, $5000.

7. A, B, and C engage in manufacturing straw goods; A puts in $4800 for 5 months, B a certain sum for 7 months, and C $8400 for a certain time. On settling accounts, A received $5400 for his share, B $9400, and C $10,080. What were B's stock and C's time? *Ans.* $8000, and 8 months.

8. A, B, and C formed a partnership with a capital of $12,000, of which A puts in $4000, B $5000, C $3000; at the end of 3 months A withdrew $1000 and 4 months later withdrew $500; at the end of 5 months B withdrew $1500 and at the end of 8 months withdrew $500; C withdrew $500 at the end of 4 months, and at the end of 9 months added $1500; at the end of the year they had gained $1800; what was each man's share?

Ans. A, 545.22\frac{198}{241}$; B, 709.54\frac{86}{241}$; C, 545.22\frac{198}{241}$.

9. A and B form a partnership, each putting in $5000; at the end of 3 months A draws out $1500 and B $500, and each draws the same sum at the end of 6 months; at the end of 9 months A draws out $2000 and B $1000; at the end of the year they dissolve partnership with a remaining capital of $2100; how must they divide it?

Ans. B takes $2100 and has a claim on A for $350.

CASE II.

767. ***When the proportion of profits or losses is fixed, and interest is allowed for the difference between each partner's proportion of capital and the amount he actually contributes.***

1. A and B form a partnership; A contributes $2500 and is to have $\frac{3}{4}$ of the profits; B contributes $1500 and is to have $\frac{1}{4}$ of the profits; each partner is to receive or pay in-

terest at the rate of 6 per cent. per annum for any excess or deficit in his proportionate share of capital. At the end of a year the profits are $800; what is the share of each?

SOLUTION.—Total capital is $4000. A should contribute $\frac{3}{4}$, or $3000, and must pay 1 year's interest, or $30 on his deficit. B should contribute $\frac{1}{4}$, or $1000, and is entitled to 1 year's interest, or $30, on his excess. A gained $\frac{3}{4}$ of $800 = $600, less $30 interest, or $570. B gained $\frac{1}{4}$ of $800 = $200, plus $30 interest, or $230.

OPERATION.

$2500	$\frac{3}{4}$ of $4000 =	$3000
1500	A contributed	2500
$4000	Deficit	$500
		.06
	Int. for 1 yr.	$30.00
	B contributed	$1500
	$\frac{1}{4}$ of $4000 =	1000
	Excess	$500
	Int. for 1 yr.	$30.00

$\frac{3}{4}$ of 800 = 600, and −30 = $570, A's gain.
$\frac{1}{4}$ of 800 = 200, and +30 = $230, B's gain.

Rule.—I. *Find the interest on the excess or deficit of each partner's share of capital. If there are additions and withdrawals, subtract the interest on the former from the gross profits, and add the interest on the latter.*

II. *Divide the profits thus obtained in the required proportions, adding or subtracting the interest due to or by each partner respectively, and the result will be the net gain of each. For the present value of each share, add to each partner's original stock the net gain and the additions, and subtract the withdrawals.*

NOTE.—It will readily be seen that the interest on the excesses and deficits of the origiual shares will exactly balance each other, and therefore will not change the profits. For, if one partner puts in $100 more than his share of a certain sum, the other partners must have $100 less than theirs.

2. A and B form a partnership. A contributes $7000, and is to have $\frac{2}{3}$ of the profits; B contributes $3000, and is to have $\frac{1}{3}$ of the profits; each partner is to receive or pay interest at 6 per cent. per annum for any excess or deficit in his share of capital. At the end of the first year the profits are $1800. Required worth of each share.

Ans. A's, $8220; B's, $3580.

3. The second year A adds $2000, averaging May 1; B adds $1000, averaging July 1. Profits $1500. Required worth of each share. *Ans.* A, $11,247.87; B, $5,052.13

4. Third year, neither partner adds any capital, but A

withdraws $1800, averaging Sept. 1, and B withdraws $800, averaging May 1. Profits, $2000. Required worth of each share. *Ans.* A, $10,813.41; B, $4,886.59.

5. Fourth year; A sells $\frac{1}{2}$ of his share to C; A's proportion of profits to be $\frac{1}{3}$, B's $\frac{1}{3}$, C's $\frac{1}{3}$. A adds capital $1000, March 1; B $1500, averaging May 1; C $1800, averaging Sept. 1. A withdraws, July 1, $1500; B, Sept. 1, $1000; and C, Nov. 1, $500. Profits $5000. Required worth of each share. *Ans.* A, $6563.44; B, $7047.12; C, $8389.44.

6. Fifth year; the firm is changed to a limited partnership, taking in D as a special partner with a capital of $8000, for the use of which he is to receive 20% of the profits of the concern annually. The remaining profits are to be divided as before, among A, B, and C, who remain as general partners. A withdraws for living expenses $1000, averaging May 1; B $1200, averaging Nov. 1; and C $900, averaging July 1. Profits $5500, of which D has already drawn his share. Required the worth of each share.

Ans. A, $6970.25; B, $7310.95; C, $9018.80; D, $8000.

7. At the close of the sixth year the firm finds it has lost $50,000, and the partnership is dissolved. Nothing has been withdrawn by either partner. Required the share of loss due from each partner after exhausting the assets of the firm.

Ans. A, $7077.54; B, $6716.39; C, $4906.07; D, nothing.

NOTE.—The special partner is liable only for the amount of his capital. The general partners are liable for losses in the same proportion as profits are shared.

BANKRUPTCY.

768. Bankruptcy is the legal acknowledgment, by a person or firm, of inability to satisfy pecuniary obligations. Such persons are said to be *bankrupt* or *insolvent*.

769. An **Assignee** is appointed in cases of bankruptcy to take charge of the assets of the bankrupt, turn them into cash, and having deducted the necessary expenses, to divide the net proceeds among the creditors in proportion to their claims.

After having thus given up his property, the bankrupt is freed *from his liabilities*, and is at liberty to commence business again.

1. A bankrupt owes A \$3000, B \$6000, C \$4500, and D \$1500, and the net proceeds of his assets are only \$9000; how much can he pay on the dollar, and how much will each receive?

SOLUTION.—Adding the liabilities, we find the amount to be \$15,000. If on \$15000 the creditors receive \$9000, on \$1 they will receive $\frac{1}{15000}$ of \$9000, or \$0.60. If they receive 60 cents on a dollar, A, whose claim is \$3000, will receive 3000 times 60 cents, or \$1800; B will receive \$3600, C will receive \$2700, and D \$900. Hence the

OPERATION.

$$\$3000+\$6000+\$4500+\$1500=\$15000$$
$$\$9000\div 15000=\$0.60$$
$$3000\times \$0.60=\$1800$$
$$6000\times \$0.60=\$3600$$
$$4500\times \$0.60=\$2700$$
$$1500\times \$0.60=\$\ 900$$
$$\overline{\$9000}$$

Rule.—I. *Divide the net proceeds of the estate by the amount of the liabilities, to find the amount paid on a dollar.*

II. *Multiply the amount paid on a dollar by the amount of each man's claim.*

2. Colburn, Robinson & Co. failed for \$100,000; their assets amounted to \$29,000; what would be the shares of A and B, if their claims amounted to \$57,000, and A's is 28% more than B's? *Ans.* A's, \$9280; B's, \$7250.

3. Rowe, Wilson & Co. failed, owing John Henderson \$10,000, Amos Bristow \$17,000, and sundry other persons \$19,750. Henderson agrees to act as assignee for 3% of the assets, and the other expenses amount to \$845. If the assets amount to \$37,500, what are Henderson's and Bristow's shares? *Ans.* Henderson, \$8.725; Bristow, \$12,920.

4. Osgood, Lee & Co. failed for \$75,750; the assignee sold their real estate for \$25,000, and the remainder of their stock of goods for \$3500, and collected debts owing them amounting to \$14,000, and expended \$1,648 in settling up the business; what will James Conger receive, whose claim is \$33,475? *Ans.* \$18,053.08.

5. The above-mentioned firm in seven years were so successful as to be enabled to pay off their debts in full, with interest at 6%; what would Samuel Forsyth now receive, if his original claim was \$27,925? *Ans* \$18,268.35.

EQUATION OF PAYMENTS.

770. Equation of Payments is the process of finding the mean or equitable time for paying several sums, due at different times.

771. The **Term of Credit** is the time allowed for the payment of a debt.

772. The **Average Term of Credit** is the time to elapse before several debts due at different times may in equity be paid together.

773. The **Equated Time** is the date at which several debts due at different times may be paid in one sum.

774. The **Focal Date** is the date from which we begin the reckoning in averaging an account.

CASE I.

775. ***To find the average term of credit, when the terms of credit begin at the same time.***

1. In settling my accounts on the 1st of January, I find I owe Mr. Peck \$300 to be paid in 4 months, \$400 to be paid in 5 months, and \$800 to be paid in 6 months; what is the average term of credit?

OPERATION.

$$\begin{array}{ll} 300\times 4 = 1200 & \\ 400\times 5 = 2000 & \\ \underline{800\times 6 = 4800} & \\ 1500 &)8000(5\tfrac{1}{3} \text{ mo.} \end{array}$$

SOLUTION.—A credit on \$300 for 4 months is regarded as equivalent to a credit on \$1 for 1200 months, and a credit on \$400 for 5 months is equivalent to a credit on \$1 for 2000 months, and a credit on \$800 for 6 months to a credit on \$1 for 4800 months; adding, we have the sum equivalent to a credit on \$1 for 8000 months; if \$1 has a credit for 8000 months, \$1500 would have a credit of $\frac{1}{1500}$ of 8000 months, which is $5\frac{1}{3}$ months. Hence the

Rule.—*Multiply each payment by its term of credit, and divide the sum of the products by the sum of the payments; the quotient will be the average term of credit.*

NOTES.—1. If there are cents in any of the payments, they may be rejected when less than 50, and reckoned as \$1 when more than 50. The fraction of a day in the answer is also rejected when less than $\frac{1}{2}$, and reckoned as 1 day if more than $\frac{1}{2}$.

2. It is objected to this rule that the interest on a certain sum not paid till after it is due, is more than the discount on the same sum paid an equal length of time before it is due. As practically, however, we generally reckon bank discount, which is the same as interest, the rule seems not really to lie open to this objection.

3. The time may also be found by dividing the sum of the interests on the payments, using any rate, by the interest on the sum of the payments for 1 month or 1 day, according to the unit of time used in the calculation. This method is preferred by some accountants.

2. Mr. Smith owes $1200, of which $\frac{1}{8}$ is due in 4 mo., $\frac{1}{2}$ in 6 mo., and the remainder in 9 mo.; required the average term of credit. *Ans.* 5 mo. 25 da.

3. A country merchant bought goods in Philadelphia to the amount of $4500, of which $\frac{1}{8}$ was to be paid down, $\frac{1}{4}$ in 3 mo., and the remainder in 6 mo.; required the average term of credit. *Ans.* $3\frac{1}{4}$ mo.

4. A jobber owes an importer $1000 due in 2 mo., $1500 in 4 mo., $900 in 6 mo., and $3000 in a year; at what time should he pay the whole debt? *Ans.* 7 mo. 22 da.

5. B owes a certain sum, $\frac{1}{2}$ payable in 3 mo., $\frac{1}{4}$ in 4 mo., $\frac{1}{5}$ in 6 mo., and the remainder in a year; required the average term of credit. *Ans.* 4 mo. 9 da.

REMARK.—The result will be the same whatever the sum owed, hence we may assume $1 as the capital, and proceed as before.

6. On the 1st of July A owes B $700 due in 3 mo., $560 due in 7 mo., $450 due in 9 mo., and $825 due in 11 mo.; what will be the average time and the equated time?

Ans. 7 mo. 17 da.; due Feb. 18.

7. Jan. 1st, 1872, I owe $560; Feb. 8th, $470.70; March 10th, $561.50; and April 11th, $749.75; what is the equated time, reckoning from Jan. 1st, for the payment of the whole amount? *Ans.* Feb. 27.

8 A man bought a house and lot for $5000 on the 1st of March, agreeing to pay $1250 down, $1250 on the 18th of May, $1250 on the 3d of July, and $1250 on the 9th of October. On further consideration, he decides to make but one payment; when will it be due? *Ans.* June 15.

9. I owe $500 in 3 mo., $600 in 4 mo., and $400 in 9 mo., but procure an extension of time to 1 year, and my creditor offers to either take my note with interest at 6% for the whole amount from the equated time, or a note with interest from date for the true present worth of all the payments; which will be the most profitable for me?

Ans. The latter, by $1.06

CASE II.

776. ***To find the equated time when the credits begin at different dates.***

1. Bought of Newlin & Fernley the following bill of goods:

Jan. 15, 1871,	a bill	amounting	to	$500	on 2	mo. credit.
Feb. 1,	"	"	"	350	" 3	"
Feb. 19,	"	"	"	400	" 2	"
Mar. 17,	"	"	"	380	" 4	"

If I give my note for the amount, when will it become due?

OPERATION.

Mar. 15,	500×	0 =	0000
May 1,	350×	47 =	16450
Apl. 19,	400×	35 =	14000
July 17,	380×	124 =	47120
	1630		)77570(48

March 15+48 da. = May 2.

SOLUTION.—From the time the first item is due till the time the second is due is 47 da., and till the time the third is due is 35 da., and till the time the fourth is due is 124 da.; hence, reckoning from the time the first is due, the second has a credit of 47 days, the third of 35 days, the 4th of 124 days, and the first of no days. We then average as in Case I., and find the term of credit to be 48 da. from March 15, the time at which the first debt is due; hence the equated time of payment is May 2.

Rule.—I. *Select the date at which the first debt becomes due, and multiply each debt by its term of credit reckoned from the date selected.*

II. *Divide the sum of the products by the sum of the debts, and the quotient will be the average term of credit, estimated from the date selected.*

NOTE.—When the *earliest* date is not the *first* of the month, it is often more convenient to take the *first* of the month as the standard date.

2. Mr. Fletcher bought goods at different dates as stated in the following bill:

June 20,	to the	amount	of	$250	on 3 mo. credit.
July 1,	"	"	"	$300	" 4 mo. credit.
Aug. 15,	"	"	"	$280	" 3 mo. credit.
Sept. 9,	"	"	"	$750	" 2 mo. credit.

What is the average term of credit, and also the equated time for the payment of the bill? *Ans.* 42 days; Nov. 1.

3. I purchased of a merchant at different times the following bills of goods:

March 11,	to the amount of			$359.84	on	2 mo.	credit.
April 30,	"	"	"	$475.15	"	3 mo.	"
June 15,	"	"	"	$278.50	"	4 mo.	"
Aug. 9,	"	"	"	$564.75	"	3 mo.	"
Sept. 14,	"	"	"	$356.25	"	4 mo.	"

What is the equated time for the payment of the whole? *Ans.* Sept. 22.

4. I sold goods to Mr. Peters as follows:

May 1,	a bill of	$565.62	at 60 days	credit.
May 31,	" "	$342.56	at 30 days	"
July 5,	" "	$794.67	at 90 days	"
Aug. 30,	" "	$834.18	at 30 days	"
Sept. 10,	" "	$250.25	at 90 days	"

If he gives me his note for the amount, when, in equity, should it commence to bear interest? *Ans.* Sept. 7.

CASE III.

777. ***When a debt due at some future time has received partial payments, to find when the remainder should be paid.***

1. A merchant bought goods to the amount of $3500 on a credit of 6 months; 4 months before it was due he paid $1000, and 2 months before it was due, $1500; how long after the expiration of the 6 months may the balance remain unpaid?

SOLUTION.—A credit on $1000 for 4 mo. is equivalent to a credit on $1 for 4000 mo., a credit on $1500 for 2 mo. is equivalent to a credit on $1 for 3000 mo.; hence $1000, the sum which remains unpaid, should have a credit of $\frac{1}{1000}$ of 7000 mo., which is 7 mo. Hence the

OPERATION.

$$\begin{array}{ll} 1000\times4 = & 4000 \\ 1500\times2 = & 3000 \\ \hline 2500 & 7000 \end{array}$$

$3500 - 2500 = 1000$

$7000 \div 1000 = 7$ mo.

Rule.—*Multiply each payment by the time it was paid before it was due, and divide the sum of the products by the sum remaining unpaid.*

2. I borrowed of Mr. B. $600 for 3 mo., $500 for 6 mo., and $300 for 9 mo.; at the end of 5 mo. I paid him $1000; how long after the equated time should I keep the remainder? Ans. 27 da.

3. *Mr. Glass* borrowed $250 for 30 days and $540 for 50

days; at the end of 24 days he paid $300, and in 40 days he paid $200; how long after the equated time should the balance be paid? *Ans.* 23 da.

4. Mr. Jones owes me the following notes: one for $400, due July 5; one for $250, due Sept. 1; and one for $850, due Oct. 20; I wish to exchange them for 2 notes of $750 each, one to fall due on Aug. 1; when should the other fall due? *Ans.* Oct. 26.

5. I purchase a farm for $15,000, $\frac{1}{3}$ to be paid down, and the remainder in two equal payments at 6 and 9 months; I pay $\frac{1}{5}$ in cash and the remainder in three equal payments at equal intervals; what are the intervals? *Ans.* $3\frac{1}{8}$ months.

6. I owe two notes, one for $600, due June 11, and the other for $800, due Oct. 9, and wish to discharge the debt by two equal payments made at an interval of 50 days; when must the payments be made? *Ans.* July 25 and Sept. 13.

7. I exchanged four notes, $200 due in 15 days, $300 due in 24 days, $400 due in 35 days, and $800 due in 60 days, for $500 cash, and 3 notes for $250, $350, and $600, due at equal intervals; what were the intervals? *Ans.* 26 days.

8. I exchanged the six following notes for five, each for the same amount and payable at equal intervals: $700 due in 25 days; $1000 in 31 days; $1200 in 42 days; $1500 in 54 days; $1700 in 60 days; $2000 in 75 days; what are the amounts and intervals? *Ans.* 18 days and $1620.

AVERAGING ACCOUNTS.

778. Averaging an Account is the process of finding the mean or equitable time for the payment of the balance of the account.

1. In the following account it is required to find the balance and when it is due.

DR. JAMES HENDERSON. CR.

1873				1873			
March 9	To merchandise,	300	00	March 20	By cash,	247	25
May 12	" "	473	60	April 11	" draft at 30 da.	400	00
June 19	" "	564	20	July 10	" cash,	259	55

OPERATION 1.

Due.	Time.	Items.	Products.	Due.	Time.	Items.	Products.
March 9	00	300	0000	March 20	11	247	2717
May 12	64	474	30336	May 14	66	400	26400
June 19	102	564	57528	July 10	123	260	31980
		1338	87864			907	61097
		907	61097				
		431	26767	26767÷431 = 62 da.; May 10.			

SOLUTION 1.—Select the date of the item first due as the focal date, and find the time the others are due after it, allowing 3 days grace to the draft; then multiplying each item by the corresponding time, and taking the sums of the products, we find that if paid on the 9th of March the *Dr.* items must suffer a discount of $87,864 for 1 day, and the *Cr.* items must suffer a discount of $61,097 for 1 day. Subtracting the two sums, we find that the *Dr.* side must suffer a discount of $26,767 more for 1 day than the *Cr.* side, and on $431, the balance of the items, it should suffer a discount of as many days as 431 is contained times in 26,767, which is 62 days. Hence the balance is due 62 days from March 9, or May 10.

OPERATION 2.

Due.	Time.	Items.	Products.	Due.	Time.	Items.	Products.
March 9	123	300	36900	March 20	112	247	27664
May 12	59	474	27966	May 14	57	400	22800
June 19	21	564	11844	July 10	00	260	00000
		1338	76710			907	50464
		907	50464				
		431	26246	26246÷431 = 61 da.; May 10.			

SOLUTION 2.—Select the item last due as the focal date, and find the time the others are due before it; then multiplying as before and taking the sums of the products, we find that on the 10th of July, the *Dr.* items must bear an interest of $76,710 for 1 day, and the *Cr.* items must bear an interest of $50,464 for 1 day. Subtracting, we find that the *Dr.* side must bear an interest of $26,246 more for 1 day than the *Cr.* side, and on $431, the balance of the items, it should bear an interest of as many days as 431 is contained times in 26,246, which is 61 nearly. Hence, the balance is due 61 days before July 10, or May 10.

Rule.—I. *Find when each item is due, take the earliest or the latest date as the focal date, find the difference between the focal date and the remaining dates, and multiply each item by the corresponding difference.*

II. *Balance the columns of products, and also the columns of items, and divide the former* balance by the latter ; the

ACCOUNT SALES.

782. An **Account Sales** is a written statement, rendered by an agent or consignee to the consignor, of the sales of goods consigned, the charges, and the net proceeds.

783. **Guaranty** is a charge made for securing the owner against the risk of non-payment, when goods are sold on credit.

Expenses incurred in receiving the goods and all charges paid in cash are considered due the consignee when paid, but commission and after charges are due at the average maturity of the sales.

An account-sales is averaged to find when the net proceeds become due, in order that the consignor may draw a bill of exchange to fall due at the equated time. Except that the date of maturity of the commission and guaranty must be found by first averaging the sales, the account is averaged like an account current, the charges being the debits and the sales the credits.

1. Account sales of 200 hhd. molasses received from New Orleans per ship Crescent City, on % of Lafourcade Brothers.

1873					
May	11	Sold 50 hhd. molasses, 6000 gal. @ 40¢ on 30 days,		2400	00
June	1	" 20 " " 2400 " @ 37¢ cash,		888	00
"	20	" 60 " " 7200 " @ 38¢ "		2736	00
July	7	" 70 " " 8400 " @ 42¢ on 60 days,		3528	00
		200		9552	00
		CHARGES.			
May	1	To Freight and Drayage,	256.72		
"	1	" Cooperage,	10.00		
"	1	" Insurance and adv.,	11.40		
July	7	" Storage from May 1,	74.54		
"	14	" Commission on $9552 @ $2\frac{1}{4}$%,	214.92		
"	14	" Guaranty on $5928 @ $2\frac{1}{2}$%,	148.20	715	78

What are the net proceeds of the above account, and when is it due? *Ans.* Net proceeds, $8836.22; due July 17.

2. A commission merchant in Philadelphia received Oct. 1, 1873, a consignment of 2000 bushels of white wheat, paying as charges, freight $75.42, drayage $25, and other expenses $10.50. He sold Oct. 10, 500 bu. @ $1; Oct. 31, 750 bu. @ 1.12\frac{1}{2}$ for 60 days; Dec. 1, 250 bu. @ $1.10; and Jan. 12, 500 bu. @ 1.25. The commission was $2\frac{1}{2}$%, guaranty $2\frac{1}{2}$%, and storage $256. Required the net proceeds and equated time of the account. *Ans.* $1799.64; Dec. 12.

SECTION X.

INVOLUTION AND EVOLUTION.

INVOLUTION.

784. Involution is the process of finding any power of a number.

785. A **Power** of a number is the product arising from using the number several times as a factor. The number itself is called the *first power.*

786. The **Second Power** of a number is the product obtained by using the number twice as a factor. Thus, 16 is the second power of 4, since $4\times4=16$.

787. The **Third Power** of a number is the product obtained by using the number three times as a factor. Thus, 64 is the third power of 4, since $4\times4\times4=64$.

788. The **Fourth Power** of a number is the product obtained by using the number four times as a factor; the *Fifth Power*, five times as a factor.

789. The **Degree** of a power is indicated by a small figure, called an *exponent*, placed at the right and a little above the number. Thus, 5^2 represents the 2d power of 5, 6^3, the third power of 6, etc.

790. The **Exponent** indicates how many times the number is used as a factor. Thus, 8^3 denotes that 8 is used as a factor three times.

The second power of a number is called its *square*, because the area of a square equals the product of its two equal sides. The third power of a number is called its *cube*, because the product of the three equal sides of a cube gives its contents.

NOTES.—1. Exponents were first introduced by *Descartes*, born in 1596. The earliest writers on algebra denoted the power by an abbreviation of its name. *Harriott*, born 1560, repeated the quantity, writing *aaaa* for a^4.

2. The symbol of evolution, $\surd$, was introduced by *Stifelius*, a German mathematician of the 15th century. It is a modification of the letter *r*, the initial of *radix*, or root. Formerly, the letter *r* was written before the quantity whose root was to be extracted.—See Philosophy of Arithmetic.

PRINCIPLES.

1. *A power of a number is obtained by using the number as a factor as many times as there are units in the degree.*

2. *The product of any two powers of a number equals a power of the number denoted by the sum of the exponents.*

For, if we multiply the cube of a number by the 4th power of the number, we will evidently have the number used seven times as a factor, or the 7th power of the number; thus, $5^3 \times 5^4 = (5 \times 5 \times 5) \times (5 \times 5 \times 5 \times 5) = 5^7$; and the same may be shown in any other case.

3. *A power of a number raised to any power equals a power of the number denoted by the product of the exponents.*

For, if we *square* the *cube* of a number, we will evidently use the number as a factor *two* times *three* times, or *six* times; thus, $(5^3)^2 = 5^3 \times 5^3$, which, by Prin. 2, equals 5^6, and the same may be shown in any other case.

NOTE.—By means of this principle we can abbreviate the operation of involution; thus we can raise a number to the sixth power by squaring its cube, or to the 12th power by squaring its sixth power, or cubing its 4th power, etc.

EXAMPLES FOR PRACTICE.

Find the value of the following:

1. 45^2. *Ans.* 2025.
2. 46^3. *Ans.* 97336.
3. 216^2. *Ans.* 46656.
4. 105^3. *Ans.* 1157625.
5. 14^4. *Ans.* 38416.
6. 16^5. *Ans.* 1048576.
7. $(\frac{14}{15})^2$. *Ans.* $\frac{196}{225}$.
8. $(13\frac{2}{3})^3$. *Ans.* $2552\frac{17}{27}$.
9. $(5.6)^6$. *Ans.* 30840.979456.
10. $(2.5)^2$. *Ans.* $6.\dot{5}3086419\dot{7}$.
11. $(2^4)^8$. *Ans.* 4294967296.
12. $(\frac{4}{5})^3 \times (\frac{4}{5})^2$. *Ans.* $\frac{1024}{3125}$.
13. $25^4 \times 25^3$. *Ans.* 25^7.
14. $(9\frac{1}{2})^4 \times (9\frac{1}{2})^2$. *Ans.* $(9\frac{1}{2})^6$.
15. $(4.\dot{6})^5 \times (4.\dot{6})^2$. *Ans.* $(4.\dot{6})^7$.
16. $(\frac{4}{5})^2 \times (\frac{5}{8})^3$. *Ans.* $\frac{5}{32}$.
17. $(\frac{2}{3})^3 \times (\frac{3}{4})^4$. *Ans.* $\frac{3}{32}$.
18. $27^4 \div 3^{12}$. *Ans.* 1.
19. $16^8 \div 2^{32}$. *Ans.* 1.
20. $3^2 \times 7^3 \div (7^2 \times 3.)$ *Ans.* 21
21. $18^3 \times 12^4 \div (12^2 \times 18^2)$. *Ans.* 2592.
22. $3^5 \times 6^4 \times 8^3 \div (8 \times 3^4 \times 6^3)$. *Ans.* 1152.
23. What power is the product of the cube by the 4th power? *Ans.* 7th power.
24. What power is the product of the first power, the second power, and the third power? *Ans.* 6th power.
25. What power is the square of the cube of the fourth power? *Ans.* 24th power.

SQUARING NUMBERS.

791. There are **Two Methods** of squaring numbers, called the *Analytic* or *Algebraic*, and the *Synthetic* or *Geometrical* methods.

792. The object of these methods is to find the law of forming the square, and thus prepare for corresponding methods of explaining Evolution.

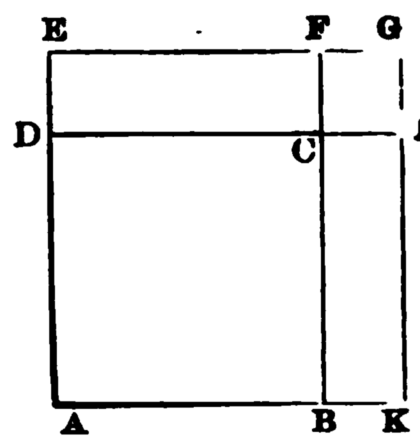
1. Square 45 analytically and synthetically.

Analytical Solution.—Forty-five equals 40 plus 5, or 4 tens plus 5 units. Multiplying in the analytic form, beginning with units, we have 5 times 5 equal 5^2; 5 times 40 equals 40×5; 40 times 5 equals 40×5; 40 times 40 equals 40^2; adding, we have 40^2+2 times $40\times5+5^2$; hence the square of 45 equals *the square of the tens, plus twice the product of the tens by the units, plus the square of the units*, which we find to be 2025.

OPERATION.

$$\begin{array}{rcr}
45 & = & 40+5 \\
45 & = & 40+5 \\
\hline
225 & = & 40\times5+5^2 \\
180 & = & 40^2+40\times5\quad \\
\hline
2025 & = & 40^2+2(40\times5)+5^2
\end{array}$$

Synthetic Solution.—Let the line AB represent a length of 40 units, and BK 5 units. Upon AB construct a square; its area will be $40^2=1600$ square units. On the two sides BC and DC, construct rectangles each 40 units long and 5 units wide; the area of each will be 40×5, and the area of both will be $2\,(40\times5)$, or 400 square units. Now add the little square on CH; its area will be $5^2=25$ square units; and the sum of the different areas, $1600+400+25=2025$, is the area of a square whose side is 45.



Note.—When there are three figures, after completing the second square as above, we must make additions to it as we did to the first square. When there are four figures there are three additions, etc.

Square the following numbers by both methods:

2. 35.	*Ans.* 1225.	8. 234.	*Ans.* 54756.
3. 46.	*Ans.* 2116.	9. 345.	*Ans.* 119025.
4. 57.	*Ans.* 3249.	10. 527.	*Ans.* 277729.
5. 63.	*Ans.* 3969.	11. 1872.	*Ans.* 3504384
6. 75.	*Ans.* 5625.	12. 2345.	*Ans.* 5499025
7. 123.	*Ans.* 15129.	13. 3064.	*Ans.* 9388096

793. The following principles derived from the above solutions are important, and should be committed to memory.

PRINCIPLES.

1. *The square of a number of two figures equals the* TENS^2+2 *times* TENS$\times$UNITS+UNITS^2.

2. *The square of a number of three figures equals* HUNDREDS^2+2 *times* HUNDREDS$\times$TENS+TENS^2+2(HUNDREDS+TENS)$\times$UNITS+UNITS^2.

794. These principles may also be expressed in symbols. Let u represent units figure, t tens, h hundreds, and T thousands, and two letters written together denote multiplication; then we have

$$(t+u)^2=t^2+2tu+u^2.$$
$$(h+t+u)^2=h^2+2ht+t^2+2(h+t)u+u^2.$$
$$(T+h+t+u)^2=T^2+2Th+h^2+2(T+h)t+t^2+2(T+h+t)u+u^2.$$

CUBING NUMBERS.

795. There are **Two Methods** of cubing numbers, called the *Analytical* or *Algebraic*, and the *Synthetic* or *Geometrical* methods.

796. The object of these methods is to find the law of forming the cube, and thus prepare for corresponding methods of explaining evolution.

1. Cube 45 analytically.

OPERATION.

$$\begin{array}{rl}
2025 = & 40^2+2\times40\times5+5^2 \\
45 = & 40+5 \\
\hline
10125 = & 40^2\times5+2\times40\times5^2+5^3 \\
8100 = & 40^3+2\times40^2\times5+40\times5^2 \\
\hline
91125 = & 40^3+3\times40^2\times5+3\times40\times5^2+5^3
\end{array}$$

ANALYTICAL SOLUTION.—Squaring 45 by the method already given, we have $40^2+2(40\times5)+5^2$; we then multiply this by 40+5. 5 times 5^2 equals 5^3; 5 times $2\times40\times5$ equals $2\times40\times5^2$; 5 times 40^2 equals $40^2\times5$; 40 times 5^2 equals 40×5^2; 40 times $2\times40\times5$ equals $2\times40^2\times5$; 40 times 40^2 equals 40^3. Taking the sum of these products, we have 5^3; next, 40×5^2 plus $2\times40\times5^2$ equals $3\times40\times5^2$; next, $2\times40^2\times5$ plus $40^2\times5$ equals $3\times40^2\times5$; and next we have 40^3; hence $45^3=40^3+3\times40^2\times5+3\times40\times5^2+5^3$. Therefore the cube of 45 equals *the cube of the tens, plus 3 times the square of the tens into the units, plus 3 times the tens into the square of the units, plus the cube of the units.*

2. Find the cube of 45 by means of the cubical blocks.

Fig. 1.

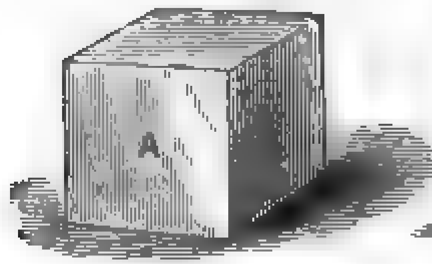


Fig. 3.

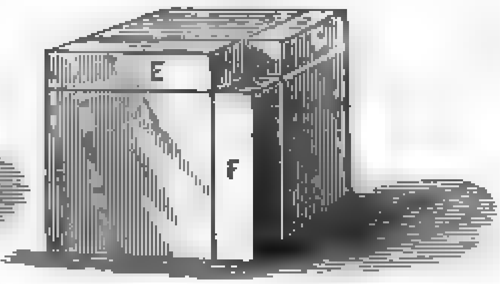


Fig. 2.

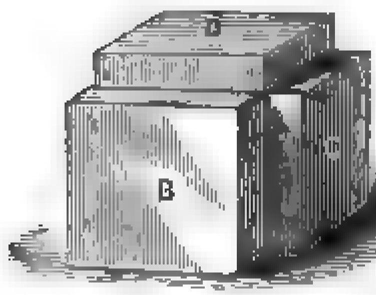
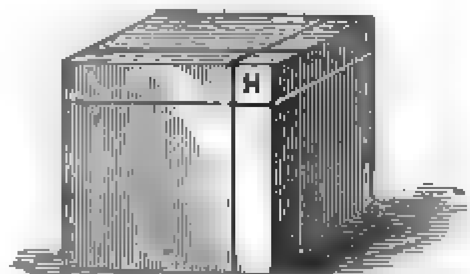


Fig. 4.



OPERATION.

$$\begin{aligned} 40^3 &= 64000 \\ 40^2\times5\times3 &= 24000 \\ 40\times5^2\times3 &= 3000 \\ 5^3 &= 125 \\ \text{Hence } 45^3 &= \overline{91125} \end{aligned}$$

GEOMETRICAL SOLUTION.—Let A, Fig. 1, represent a cube whose sides are 40 units, its contents will be $40^3 = 64000$. To increase its dimensions by 5 units we must add, 1st, the three rectangular slabs, B, C, D, Fig. 2; 2d, the three corner pieces, E, F, G, Fig. 3; 3d, the little cube H, Fig. 4. The three slabs, B, C, D, are 40 units long and wide and 5 units thick; hence their contents are $40^2\times5\times3 = 24000$; the contents of the corner pieces, E, F, G, Fig. 3, whose length is 40 and breadth and thickness 5, equal $40\times5^2\times3 = 3000$; and the contents of the little cube H, Fig. 4, equal $5^3 = 125$; hence the contents of the cube represented by Fig. 4 are $64000+24000+3000+125 = 91125$.

NOTE.—When there are three figures in the number, complete the second cube as above, and then make additions and complete the third in the same manner; or let the first cube represent the cube already found, and then proceed as at first.

EXAMPLES FOR PRACTICE.

Cube the following numbers by both methods:

3. 36.	*Ans.* 46656.	**8.** 245.	*Ans.* 14706125
4. 48.	*Ans.* 110592.	**9.** 306.	*Ans.* 28652616.
5. 72.	*Ans.* 373248.	**10.** 258.	*Ans.* 17173512.
6. 85.	*Ans.* 614125.	**11.** 4036.	*Ans.* 65743598656.
7. 123.	*Ans.* 1860867.	**12.** 5678	*Ans.* 183056925752

797. The following principles are important, and should be committed to memory.

PRINCIPLES.

1. *The cube of a number consisting of two figures equals* TENS^3+3 *times* TENS$^2\times$UNITS+3 *times* TENS$\times$UNITS^2+ UNITS^3.

2. *The cube of a number consisting of three figures equals* HUNDREDS^3+3 *times* HUNDREDS$^2\times$TENS+3 *times* HUNDREDS $\times$TENS^2+TENS^3+3 *times* (HUNDREDS+TENS)$^2\times$UNITS+3 *times* (HUNDREDS+TENS)$\times$UNITS^2+UNITS^3.

798. These principles may also be expressed in symbols as follows:

$$(t+u)^3=t^3+3t^2u+3tu^2+u^3$$

$$(h+t+u)^3=h^3+3h^2t+3ht^2+t^3+3(h+t)^2u+3(h+t)u^2+u^3.$$

EVOLUTION.

799. Evolution is the process of finding a root of a number.

800. A **Root** of a number is *one* of its *equal* factors. Roots are of different degrees; as, *second, third,* etc.

801. The **Square Root,** or *second root,* of a number is *one* of its *two equal* factors. Thus, 8 is the square root of 64, since $8\times8=64$.

802. The **Cube Root,** or *third root,* of a number is *one* of its *three equal* factors. Thus, 4 is the cube root of 64, since $4\times4\times4=64$.

803. The **Fourth Root** is *one* of the *four equal* factors; the *fifth root* is *one* of the *five equal* factors, etc.

804. The **Symbol of Evolution** is $\sqrt{}$; thus, $\sqrt[2]{64}$ or $\sqrt{64}$, denotes the square root of 64; $\sqrt[3]{64}$ denotes the cube root of 64.

805. The **Index** of the root is a small figure placed in the angle of the symbol. The *index* indicates the degree of the root.

Roots are also indicated by the denominator of a fractional exponent; thus $9^{\frac{1}{2}}$ denotes $\sqrt{9}$; $27^{\frac{1}{3}}$ denotes $\sqrt[3]{27}$, etc.

806. The following principles of involution are given to enable us to determine the number of figures in the root.

PRINCIPLES.

1. *The square of a number contains twice as many figures as the number itself, or twice as many, less one.*

DEM.—The square of 1 is 1, and the square of 9 is 81, hence the square of a number consisting of *one* figure is a number consisting of *one* or *two* figures. The square of 10, the smallest number of two figures, is 100, the square of 99, the largest number of two figures, is 9801, hence the square of a number consisting of *two* figures is a number consisting of *three* or *four* figures, that is, *twice two*, or *twice two, less one*, etc. The same may be shown for the square of a number consisting of any number of figures.

$1^2 = 1$
$9^2 = 81$
$10^2 = 100$
$99^2 = 9801$

2. *The cube of a number contains three times as many figures as the number itself, or three times as many, less one or two.*

DEM.—The cube of 1 is 1, and the cube of 9 is 729, hence the cube of any number consisting of one figure is a number consisting of *one, two,* or *three* figures. The cube of 10 is 1000, a number of four figures, the cube of 99 is 970299, a number of six figures, hence the cube of a number consisting of two figures contains *four, five,* or *six* figures, that is, *three times* two, or *three times* two, *less one* or *two*. The same may be shown for the cube of a number consisting of any number of figures.

$1^3 = 1$
$9^3 = 729$
$10^3 = 1000$
$99^3 = 970299$

EVOLUTION BY FACTORING.

807. When the number is a perfect power and the factors are easily found, the root of a number can be readily obtained by the following

Rule.—*Resolve the number into its prime factors, and for the square root form a product by taking* ONE *of every* TWO *equal factors; for the cube root,* ONE *of every* THREE *equal factors; etc.*

1. Find the square root of 2025.

SOLUTION.—We first resolve the number into its prime factors. Since the square root of a number is one of its two equal factors, we take *one* of every *two* equal factors, and have $3 \times 3 \times 5$, which equals 45. Hence the square root of 2025 is 45.

OPERATION.

3)2025
*3)675
3)225
*3)75
5)25
*5

NOTE.—It will be well for the pupil to mark the factor taken with a star, as in the margin.

Solve the following problems:

2.	$\sqrt{144}$.	*Ans.* 12.	7.	$\sqrt[3]{4096}$.	*Ans.* 16.
3.	$\sqrt{4096}$.	*Ans.* 64.	8.	$\sqrt[3]{19683}$.	*Ans.* 27.
4.	$\sqrt{9216}$.	*Ans.* 96.	9.	$\sqrt[4]{5308416}$.	*Ans.* 48.
5.	$\sqrt{6561}$.	*Ans.* 81.	10.	$\sqrt[5]{7962624}$.	*Ans.* 24.
6	$\sqrt{11664}$.	*Ans.* 108.	11.	$\sqrt[7]{170859375}$.	*Ans.* 15.

SQUARE ROOT.

808. There are **Two Methods** of explaining the general process of extracting the square root, called the ***Analytic*** or *Algebraic Method*, and the *Geometrical Method.*

809. The **Analytic Method** of square root is so called because it analyzes the number into its elements and derives the process of evolution from the law of involution.

810. The **Geometrical Method** is so called because it makes use of a geometrical figure to explain the process of extracting the root.

1. Extract the square root of 2025.

ANALYTIC SOL. — Since the *square of a number contains twice as many figures as the number itself, or twice as many less one,* the square root of 2025 will consist of two figures, and hence consist of tens and units, and 2025 consists of *tens*^{2}+ 2× *tens*× *units* +*units*^{2}.

OPERATION.

$$\begin{array}{lrl} t^2+2tu+u^2 & =2025(40 & \\ t^2= \quad 40^2 & =1600 & 5 \\ \hline 2tu+u^2 & = \quad 425 & 45 \\ 2t=40\times 2=80 & & \\ (2t+u)u=(80+5)\times 5= & 425 & \end{array}$$

The greatest number of tens whose square is contained in 2025 is 4 tens; squaring the tens and subtracting, we have 425, which equals 2× *tens*× *units*+*units*^{2}. Now since 2× *tens*× *units* is much greater than *units*^{2}, 425 must consist principally of twice the tens into the units; hence if we divide by 2× *tens* we can ascertain the units. Twice the *tens* equal 40×2=80; dividing, we find the units to be 5; now finding 2× *tens*× *units*+*units*^{2}, or, what is the same, 2× *tens*+*units*, both multiplied by *units*, which equals (80+5)×5=425, and subtracting, nothing remains. Hence the square root of 2025 is 4 tens and 5 units, or 45.

GEOMETRICAL SOL.—Let Fig. 1 represent a square which contains 2025 square units, then our object is to find the number of linear units in the edge. Since the square of a number consists of *twice as many places as the number itself, or twice as many less one,* the square root of 2025 will consist of two places, and hence will consist of tens and units.

OPERATION.

$$\begin{array}{rrl} & 2025(40 & \\ 40^2= & 1600 & 5 \\ 40\times 2=80 & 425 & 45 \\ (80+5)\times 5= & 425 & \end{array}$$

The greatest number of tens whose square is contained in 2025 is 4 tens. Let A, Fig. 1, represent a square whose sides are 40 units, its area will be 40^2, or 1600 square units. Subtracting 1600 from 2025, we find remaining a surface containing 425 square units. By inspection we find this surface to consist principally of the two rectangles B and C, Fig. 2, each of which is 40 units long, and since they nearly complete the square, their area is nearly 425 units; hence if we divide 425 by their length, we will find their width. The length of both is $40 \times 2 = 80$; dividing 425 by 80, we find their width to be 5 units. Adding the length of the little corner square D, Fig. 3, whose sides are 5 units, we find the entire length of the surface remaining after the removal of the square A, is $80+5=85$ units, and multiplying this by the width, we find the whole area of the remainder to be $85 \times 5 = 425$ square units. Subtracting 425 square units from the square units left after subtracting 1600 square units, nothing remains, therefore the side of the square whose area is 2025 square units is 45 units; hence the square root of 2025 is 45.

Fig. 1.

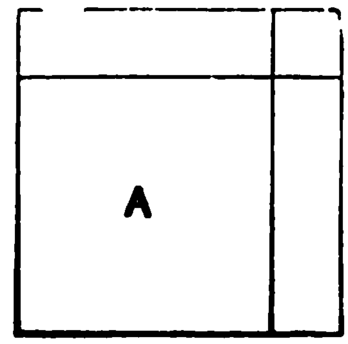


Fig. 2.

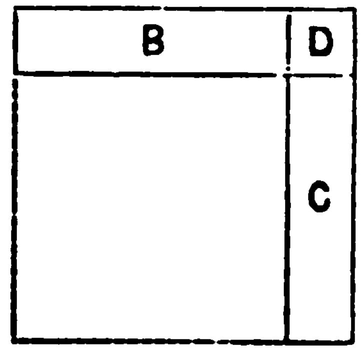
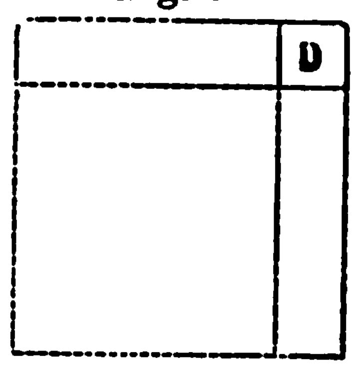


Fig. 3.



NOTES.—1. When there are three figures in the root, by the analytic method we use the formula for three terms; by the geometrical method, after removing the first rectangles and small square, we have two rectangles and a small square remaining, which we remove as before.

2. In practice, we determine the number of figures in the root by pointing off the number into periods of two figures each, beginning at the right; we also abbreviate the work by omitting ciphers and condensing the other parts, preserving only the *trial* and *true* divisors. For illustration see solution in the margin.

3. This can also be explained by building up the square instead of separating it into its parts, for which see *Manual*.

OPERATION.

```
     10·49·76(324
3     9
62   149
644  124
      2576
      2576
```

Rule.—I. *Begin at units, and separate the number into periods of two figures each.*

II. *Find the greatest number whose square is contained in the left hand period, place it at the right as a quotient, subtract its square from the left hand period, and annex the next period to the remainder for a dividend.*

III. *Double the root found and place it at the left for a* TRIAL DIVISOR; *divide the dividend, excluding the right hand term, by this divisor; the quotient will be the second term of the root.*

IV. *Annex the second term of the root to the trial divi-*

sor for the TRUE DIVISOR, *multiply the result by the second term of the root, subtract the product from the dividend, and bring down the next period for the next dividend.*

V. *Double the root now found for a second* TRIAL DIVISOR, *find the third term of the root as before, and thus proceed until all the periods have been used.*

NOTES.—1. If the product of a true divisor by a term of the root exceeds the dividend, the term must be diminished by a unit.

2. When a cipher occurs in the root, annex a cipher to the trial divisor, bring down the next period, and proceed as before.

3. The square root of a common fraction is evidently the square root of each term. When these terms are not perfect squares, reduce the fraction to a decimal, and extract the root. When a number is not a perfect square, annex periods of ciphers and carry the root on to decimals.

4. By squaring 1, .1, .01, etc., we see that the *square of a decimal contains twice as many decimal places as the decimal,* hence to extract the square root of a decimal, we *point off the decimals into periods of two figures each, counting from the decimal point,* and proceed as in whole numbers.

$1^2 = 1$
$.1^2 = .01$
$.01^2 = .0001$

Extract the square root of

2. 1369.	*Ans.* 37.	6. 277729.	*Ans.* 527.
3. 3136.	*Ans.* 56.	7. 1827904.	*Ans.* 1352.
4. 98596.	*Ans.* 314.	8. 7387524.	*Ans.* 2718.
5. 65536.	*Ans.* 256.	9. 9339136.	*Ans.* 3056.

Find the square root of

10. $\frac{256}{289}$.	*Ans.* $\frac{16}{17}$.	20. .00009216.	*Ans.* .0096.
11. $\frac{1225}{5476}$.	*Ans.* $\frac{35}{74}$.	21. 4907 0025.	*Ans.* 70.05.
12. $4\frac{576}{1225}$.	*Ans.* $2\frac{4}{35}$.	22. 89526.025681.	*Ans.* 299.209.
13. $35\frac{134}{361}$.	*Ans.* $5\frac{18}{19}$.	23. $.\dot{1}0\dot{0}$.	*Ans.* .3163859+.
14. .3364.	*Ans.* .58.	24. 642521104.	*Ans.* 25348.
15. .0841.	*Ans.* .29.	25. 185383635844.	*Ans.* 430562.
16. .001225.	*Ans.* .035.	26. 4122544464025.	*Ans.* 2030405.
17. .099856.	*Ans.* .316.	27. 77531660905535929.	*Ans.* 278445077.
18. 061009.	*Ans.* .247.		
19. $36\frac{4}{9}$.	*Ans.* 6.036923+.		

CONTRACTIONS IN SQUARE ROOT.

311. When the square root is to be extracted to many places of decimals, the work may be shortened by the following method:

Rule.—*Find, as usual, more than one-half the terms of*

the root, and then divide the last remainder by the last divisor, using the contracted method, as in Art. 278.

1. **Extract the square root of 10.**

OPERATION.

	10(3.16227766+.
	9
61	100
	61
626	3900
	3756
6322	14400
	12644
63242	175600
	126484
632447	4911600
	4427129
6324547	48447100
	44271829
63245546	417527100
	379473276
632455526	3805382400
	3794733156

CONTRACTED METHOD.

	10(3.16227766+.
	9
61	100
	61
626	3900
	3756
6322	14400
	12644
63242	175600
	126484
	49116
	44269
	4847
	4427
	420
	379
	41
	38

Find the value of the following:

2.	$\sqrt{2}$.	*Ans.* 1.414213+.	7.	$\sqrt{8}$.	*Ans.* 2.828427+.
3.	$\sqrt{3}$.	*Ans.* 1.732050+.	8.	$\sqrt{.9}$.	*Ans.* .948683+.
4.	$\sqrt{5}$.	*Ans.* 2.236067+.	9.	$\sqrt{11}$.	*Ans.* 3.316624+.
5.	$\sqrt{6}$.	*Ans.* 2.449489+.	10.	$\sqrt{12}$.	*Ans.* 3.464101+.
6.	$\sqrt{7}$.	*Ans.* 2.645751+.	11.	$\sqrt{13}$.	*Ans.* 3.605551+.

APPLICATIONS OF SQUARE ROOT.

312. The **Applications of Square Root** to problems involving geometrical figures are extensive.

313. The **Side** of a square is equal to the square root of its area.

1. A man owns a square lot containing 25 hectares; how many meters does its side measure?

SOLUTION.—The 25 hectares equal 250000 sq. meters; extracting the square root, we have 500 meters.

2. I own a square lot containing 7 acres; what is the length of one of its sides? *Ans.* 33.466+ rods.

3. A man owns a rectangular lot containing 20 acres, whose length is twice its breadth; what is the distance around it? *Ans.* 240 rods.

4. What will it cost to enclose a rectangular lot contain ing 12 hectares, whose length is 3 times its breadth, at the rate of 25 cents a meter? *Ans.* $400.

5. A cabinet maker has a board 26 ft. 3 in. long and 2 ft. 11 in. wide; what is the largest square table he can make out of it, no allowance being made for sawing?

Ans. 8 ft. 9 in.

6. If it cost $600 to inclose a farm 96 rods long and 54 rods wide, how much less will it cost to enclose a square farm of equal area with the same kind of fence?

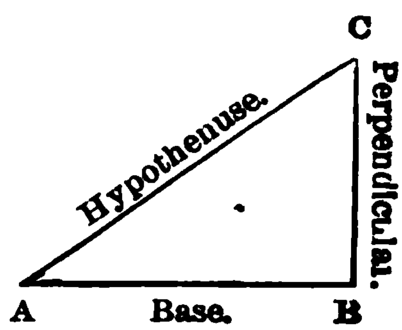
Ans. $24.

7. A general drew up his army of 38000 in three grand divisions in the form of three equal squares, and found he had 354 over in the first, 414 in the second, and lacked 400 in the third; what was the number of men in the side of each square? *Ans.* 112 men.

RIGHT-ANGLED TRIANGLES.

814. A **Right-angled Triangle** is a triangle which has one right angle.

815. The **Base** of a triangle is the side on which it stands; as AB.



816. The **Perpendicular** is the side which forms the right angle with the base; as BC.

817. The **Hypothenuse** is the side opposite the right angle; as AC.

818. The **Principles** of right-angled triangles are as follows:

PRINCIPLES.

1. *The square of the hypothenuse equals the sum of the squares of the other two sides.*

2. *Hence, the square of either side equals the square of the hypothenuse diminished by the square of the other side.*

NOTE.—The smallest integers which can express the relation of the three sides of a right-angled triangle are 3, 4, and 5. We may have an infinite number of right-angled triangles with their sides in this relation. Other integral relations of sides are as follows: 5, 12, 13; 8, 15, 17; 20, 21, 29. These are obtained by substituting in the formula $(2rs)^2+(s^2-r^2)^2=(s^2+r^2)^2$, in which r is less than s.

1. The hypothenuse of a right-angled triangle is 230, and perpendicular 138; required the base.

SOLUTION.—The base $=\sqrt{230^2-138^2}=184$, *Ans.*

2. A rectangular lot containing 103.68 ares is twice as long as wide; required the distance between its opposite corners. *Ans.* $72\sqrt{5}$ meters.

3. A ladder leaning against a house reaches 72 feet, its foot being 30 feet from the house; what is the length of the ladder? *Ans.* 78 ft.

4. Two rafters, each 35 feet long, meet at the ridge of a roof 15 feet above the attic floor; what is the width of the house? *Ans.* 63.2454+ft.

5. Two ships sail from the same port, one going due north, 8 miles an hour, and the other due east, 6 miles an hour; how far are they apart in three days? *Ans.* 720 miles.

6. A ladder 78 feet long stands close against a building; how far must it be drawn out at the foot, that the top may be lowered 6 feet? *Ans.* 30 ft.

7. A tree was broken 51 ft. from the top, and fell so that the end struck 24 feet from the foot; required the length of the tree. *Ans.* 96 feet.

8. A ladder 60 feet long, standing with its foot in the street, will reach on one side to a window 23 ft. high, and on the other to a window 37 ft. high; what is the width of the street? *Ans.* 102.65 ft.

9. A light-house was built upon a rock; if the distance from a point of observation to that point of the rock on a level with the eye is 620 meters, to the top of the rock is 846 meters, and to the top of the light-house 900 meters, what is the height of the light-house? *Ans.* 76.78 meters.

10. Required the distance between the lower corner and the upper opposite corner of a room 60 ft. long, 32 ft. wide, and 51 ft. high. Ans. 85 ft.

SIMILAR FIGURES.

819. Similar Figures are those which have the same form. Thus, circles are similar figures; also squares, etc.

820. The **Principles** of similar figures, derived from geometry, are as follows:

PRINCIPLES.

1. *The areas of all similar figures are to each other as the squares of their like dimensions.*

2. *Hence, the like dimensions of similar figures are to each other as the square roots of their areas.*

EXAMPLES FOR PRACTICE.

1. The area of a rectangle is 648 sq. yd., and one side is 27 yd.; required the area of a similar rectangle whose corresponding side is 36 yd.

OPERATION.

Area of 2d : 648 : : 36^2 : 27^2

Area of 2d $= \dfrac{648 \times 36^2}{27^2} = 1152$, *Ans.*

SOLUTION.—Since the rectangles are similar, their areas are as the squares of their corresponding sides; hence we have the proportion in the margin. Cancelling and multiplying, we have 1152 sq. yd.

2. The area of a circle whose diameter is 10 meters is 78.54 square meters; what is the diameter of a circle whose area is 1963.5 square meters? *Ans.* 50 meters.

3. A farmer has a field 40 rods long and 32 rods wide; required the dimensions of a similar field containing $4\frac{1}{2}$ acres. *Ans.* 30 rd.; 24 rd.

4. A man has two circular gardens; the one is 6.5 meters in diameter, the other 2.6 decameters; the second is how many times the size of the first? *Ans.* 16 times.

5. If a horse tied to a post by a rope 1 ch. $78\frac{1}{4}$ li. can graze upon an acre, what length of rope would allow it to graze upon $11\frac{1}{9}$ acres? *Ans.* 5 ch. $94\frac{1}{6}$ li.

6. The altitudes of two similar triangles are 18 ft. and 5.4 ft.; what is the relation of their areas? *Ans.* $11\frac{1}{9}$.

7. The area of a rectangular building lot is 720 sq. rd.; its sides are as 4 to 5; required the sides. *Ans.* 24; 30.

8. The sides of a rectangular field are as 3 to 4, and its area is 30 acres; required its dimensions. *Ans.* 60 rd., 80 rd.

9. If a pipe $\frac{3}{4}$ of an inch in diameter fill a cistern in 3 hours, what is the diameter of a pipe which will fill it in 1 hour? *Ans.* 1.299 in.

10. If a pipe whose diameter is $1\frac{1}{2}$ in. fill a cistern in 5 hours, in what time will a pipe whose diameter is $3\frac{1}{2}$ inches fill it? *Ans.* $55\frac{5}{49}$ min.

11. If a pipe of 6 inches bore is 4 hours in running off a quantity of water, in what time will three pipes, each 4 inches bore, discharge double the quantity? *Ans.* 6 hours.

12. Four men bought a grindstone 40 inches in diameter; how much of the diameter must each grind off so as to share it equally, no allowance being made for the hole?

Ans. 1st, 5.359+; 2d, 6.357; 3d, 8.284; 4th, 20 inches.

CUBE ROOT.

821. We give **Three Methods** of extracting the cube root; the *Common Method*, a *New Method*, and *Horner's Method.*

822. There are **Two Methods** of explaining the methods of extracting the Cube Root, called the *Analytic* or *Algebraic Method*, and the *Geometrical Method.*

823. The **Analytic Method** of cube root is so called because it analyzes the number into its elements, and derives the process from the law of involution.

824. The **Geometrical Method** of cube root is so called because it makes use of a cube to explain the process.

COMMON METHOD.

1. Extract the cube root of 91125.

OPERATION.

$$
\begin{array}{lrl}
 & & \quad tu \\
t^3+3t^2u+3tu^2+u^3= & & 91125(40 \\
t^3= & 40^3=64000 & \quad 5 \\
\hline
\quad 3t^2u+3tu^2+u^3= & & \vert 27125 \quad 45 \\
\quad 3t^2= & 3\times 40^2=4800 & \vert \\
\quad 3tu= & 3\times 40\times 5=\ \ 600 & \vert \\
\quad u^2= & 5^2=\ \ \ 25 & \vert \\
(3t^2+3tu+u^2)u= & 5\times \overline{5425} & \vert \underline{27125}
\end{array}
$$

ANALYTIC SOLUTION.—Since the cube of a number consists of *three times as many places as the number itself*, or of *three times as many less one or two*, the cube root of 91125 consists of two places, or of tens and units, and the number itself consists of $tens^3+3\times tens^2\times units+3\times tens\times units^2+units^3$.

The greatest number of tens whose cube is contained in 91125 is 4 tens. Cubing the tens and subtracting, we have 27125, which equals $3 \times tens^2 \times units + 3 \times tens \times units^2 + units^3$. Now, since $3 \times tens^2 \times units$ is much greater than $3 \times tens \times units^2 + units^3$, 27125 must consist principally of 3 times $tens^2 \times units$; hence if we divide by 3 times $tens^2$, we can ascertain the *units*. 3 times $tens^2$ equals $3 \times 40^2 = 4800$; dividing by 4800, we find the *units* to be 5. We then find 3 times $tens \times units$ equal to $3 \times 40 \times 5 = 600$, and $units^2 = 5^2 = 25$, and adding these and multiplying by *units*, we have $(3 \times tens^2 + 3 \times tens \times units + units^2) \times units$, which equals $5425 \times 5 = 27125$; subtracting, nothing remains, hence the cube root of 91125 is 45.

Fig. 1.

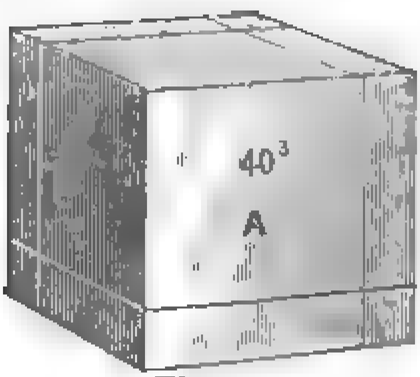


Fig. 3.

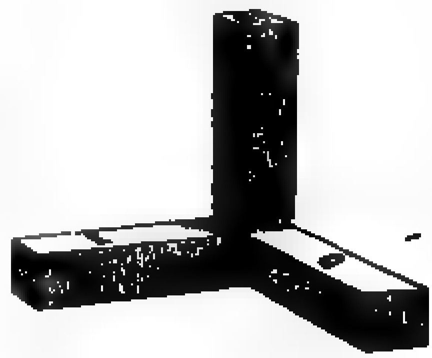


Fig. 2.

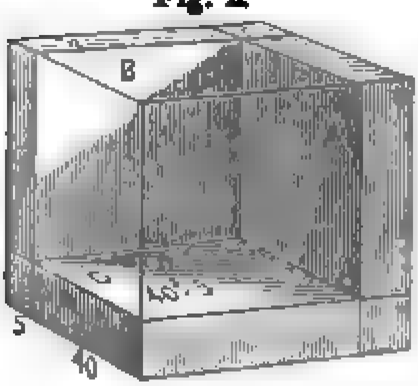
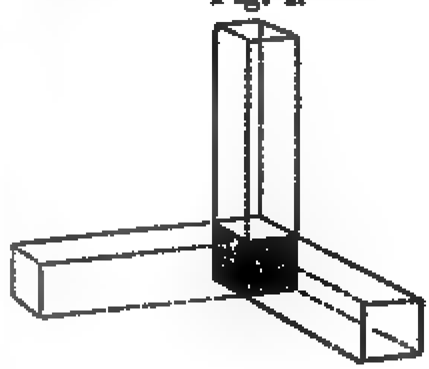


Fig. 4.



Geometrical Solution.—Let Fig. 1 represent the cube which contains 91125 cubic units, then our object is to find the number of linear units in its edge. The number of terms in the root, found as before, is two. The greatest number of tens whose cube is contained in the given number is 4 tens. Let A, Fig. 1, represent a cube whose sides are 40, its contents will be $40^3 = 64000$. Subtracting 64000 from 91125, we find a remainder of 27125 cubic units, which, by removing the cube A from Fig. 1, leaves a solid represented by Fig. 2.

OPERATION.

$$
\begin{array}{rr|l}
 & 91125 & (40 \\
40^3 = & 64000 & 5 \\
\cline{2-3}
 & 27125 & 45 \\
3 \times 40^2 = 4800 & & \\
3 \times 40 \times 5 = 600 & & \\
5^2 = 25 & & \\
\cline{1-1}
5425 & 27125 & \\
\end{array}
$$

Inspecting this solid, we perceive that the greater part of it consists of the three rectangular slabs, B, C, and D, each of which is 40 units in length and breadth; hence if we divide 27125 by the sum of the areas of one face of each regarded as a base, we can ascertain their thickness,

The area of a face of one slab is $40^2 = 1600$, and of the three, $3 \times 1600 = 4800$, and dividing 27125 by 4800 we have a quotient of 5, hence the thickness of the slab is 5 units.

Removing the rectangular slabs, there remain three other rectangular solids, E, F, G, as shown in Fig. 3, each of which is 40 units long and 5 units thick, hence the surface of a face of each is $40 \times 5 = 200$ square units, and of the three, $3 \times 40 \times 5 = 600$ square miles.

Finally, removing E, F, and G, there remains only the little corner cube H, Fig. 4, whose sides are 5 units, and the surface of one of its faces, $5^2 = 25$ square units. We now take the sum of the surfaces of the solids remaining after the removal of the cube A, and multiply this by the common thickness, which is 5, and we have their solid contents equal to $(4800 + 600 + 25) \times 5 = 27125$ cubic units, which, subtracted from the number of cubic units remaining after the removal of A, leaves no remainder. Hence the cube which contains 91125 cubic units is $40 + 5$, or 45 units on a side.

NOTE.—This can also be explained by building up the cube instead of separating it into its parts, for which see *Manual.*

825. When there are three figures in the root, the solution by the analytic method is as follows:

SHOWN BY LETTERS.

	htu	
	14706125(245	
$h^3 = 200^3 =$	8000000	
$3h^2 = 3 \times 200^2 =$	120000	6706125
$3ht = 3 \times 200 \times 40 =$	24000	
$t^2 = 40^2 =$	1600	
	145600	5824000
$3(h+t)^2 = 3 \times 240^2 =$	172800	882125
$3(h+t)u = 3 \times 240 \times 5 =$	3600	
$u^2 = 5^2 =$	25	
	176425	882125

OPERATION AS IN PRACTICE.

		14·706·125(245
$2^3 =$	8	
$2^2 \times 300 =$	1200	6706
$2 \times 4 \times 30 =$	240	
$4^2 =$	16	
	1456	5824
		882125
$24^2 \times 300 =$	172800	
$24 \times 5 \times 30 =$	3600	
$5^2 =$	25	
	176425	882125

NOTES.—1. By the geometric method, when there are more than two figures we remove the first cube, rectangular slabs and solids, and small cube, and we have remaining three slabs, three solids, and a small cube, as before.

2. The method employed in actual practice is derived from the other by omitting ciphers, using parts of the number instead of the whole number each time we obtain a figure of the root, etc. It will also be seen that by separating the number into *periods of 3 figures each*, we have the *number of places in the root*, the *part of the number* used in obtaining each figure of the root, etc.

Rule.—I. *Begin at units and separate the number into periods of three figures each.*

II. *Find the greatest number whose cube is contained in the left hand period, write it for the first term of the root, subtract its cube from the left hand period, and annex the next period to this remainder for a dividend.*

III. *Multiply the square of the first term of the root by 300 for a* TRIAL DIVISOR; *divide the dividend by it, and the result will be the second term of the root.*

IV. *To the trial divisor add 30 times the product of the second term of the root by the first term, and also the square of the second term; their sum will be the* TRUE DIVISOR.

V. *Multiply the true divisor by the second term of the root, subtract the product from the dividend, and annex the next period for another dividend. Square the root now found, multiply by 300, and find the third figure as before, and thus continue until all the periods have been used.*

NOTES.—1. If the product of the true divisor by the term of the root exceeds the dividend, the root must be diminished by a unit.

2. When a dividend will not contain a trial divisor, place a cipher in the root and two ciphers at the right of the trial divisor, bring down the next period, and proceed as before.

3. To find the cube root of a common fraction, extract the cube root of both terms. When these are not perfect cubes, reduce to a decimal and then extract the root.

4. By cubing 1, .1, .01, etc., we see that the *cube of a decimal contains three times as many decimal places as the decimal;* hence, to extract the cube root of a decimal, we *point off the decimal in periods of three figures each, counting from the decimal point.*

$$1^3 = 1$$
$$.1^3 = .001$$
$$.01^3 = .000001$$

Find the cube root of

1. 42875.	*Ans.* 35.	16. $343\frac{216}{512}$.	*Ans.* 7.002+.
2. 166375.	*Ans.* 55.	17. $(3^3+4^3+5^3)$.	*Ans.* 6.
3. 185193.	*Ans.* 57.	18. $(8^3+48^3+64^3)$.	*Ans.* 72.
4. 262144.	*Ans.* 64.	19. $(24^3+32^3+40^3)$	*Ans.* 48.
5. 438976.	*Ans.* 76.	20. 8998912.	*Ans.* 208.
6. 614125.	*Ans.* 85	21. 629422793.	*Ans.* 857.
7. 941192.	*Ans.* 98.	22. 1879080904.	*Ans.* 1234.
8. 14886936.	*Ans.* 246.	23. 16348384872.	*Ans.* 2538.
9. 48228544.	*Ans.* 364	24. 8427392875.	*Ans.* 2035.
10. 105154048.	*Ans.* 472.	25. 46967731712.	*Ans.* 3608.
11. $2.\dot{3}7\dot{0}$.	*Ans.* $1.\dot{3}$.	26. $17040.7277\dot{0}\dot{3}$.	*Ans.* 25.73
12. 1.953125.	*Ans.* $1\frac{1}{4}$.	27. 16503.467336.	*Ans.* 25.46.
13. $1.58\dot{9}6\dot{2}$.	*Ans.* $1\frac{1}{6}$.	28. 46928.689543.	*Ans.* 36.07
14. 129554216.	*Ans.* 506.	29. 8625.214936512.	*Ans.* 20.508.
15. $101\frac{17}{27}$.	*Ans.* $4\frac{2}{3}$.	30. 842182563625.	*Ans.* 20345.

SHORT METHOD OF CUBE ROOT.

826. A Short Method of extracting the Cube Root is presented in the following modification of the ordinary method previously explained. The abbreviation consists in obtaining the successive trial divisors by a law which enables us to use our previous work.

1. Extract the cube root of 14706125.

OPERATION.

		14·706·125(245
		8
12	t. d.	6706
24		
16		
1456	T. D.	
16		5824
1728	t. d.	882125
360		
25		
176425		882125

SOLUTION.—We find, as before, the number of figures in the root, and the first term of the root, cube, subtract, and bring down the first period.

We then find, as before, the trial divisor, 12, by taking three times the square of the first term. Dividing, we find the second term of the root to be 4. We then, as before, take three times the product of the first and second terms, and the square of the second term, and add these to the trial divisor as a *correction* to obtain the *true divisor*, 1456. We then multiply 1456 by 4, and subtract and bring down the next period.

We then, to find the next *trial divisor*, take the *square* of the *last term*, which is 16, and add it to the previous *true divisor* and the *two corrections*, which were added to the previous trial divisor, and we have 1728 as the next trial divisor.

Then, to find the *true divisor*, we add three times the product of the last term of the root into the previous part of the root, and also the square of the last term, and have 176425 for the *true divisor*. Multiplying by 5 we have 882125.

The method is indicated in the following formula:

1. TRUE DIVISOR=TRIAL DIVISOR+PRODUCT+SQUARE.
2. TRIAL DIVISOR=SQUARE+TRUE DIVISOR+CORRECTIONS.

The method is readily explained either by the blocks or by the algebraic formula.

Find the cube root of

1. 37393731584. *Ans.* 3344.
2. 45156047481. *Ans.* 3561.
3. 271091266048. *Ans.* 6472.
4. 45740596939947. *Ans.* 35763.
5. 66814683180552. *Ans.* 40578.
6. 12906401876038605752. Ans. 2345678.

HORNER'S METHOD.

827. Horner's Method is derived from the general method of solving cubic and higher equations invented by Mr. Horner, of Bath, England.

1. Extract the cube root of 14706125.

OPERATION.

1st col.	2d col.	
2	4	14·706·125(245
4	8	8
64	12 t. d.	6706
68	256	
725	1456 T. D.	5824
	272	882125
	1728 t. d.	
	3625	
	176425 T. D.	882125

SHOWN BY LETTERS.

1st col.	2d col.
h	h^2
$2h$	$2h^2$
$3h+t$	$3h^2$ t. d.
$3h+2t$	$3ht+t^2$
$3h+3t+u$	$3h^2+3ht+t^2$ T. D.
	$3ht+2t^2$
	$3h^2+6ht+3t^2$
	or, $3(h+t)^2$
	$3hu+3tu+u^2$
	$3(h+t)^2+3(h+t)u+u^2$

Solution.—We write the first term of the root 2, in the 1st col., its square, 4, in 2d col., and its cube, 8, under 1st period, subtract, and bring down the next period. We then add the first term of the root, 2, to 2, the first term in 1st col., multiply the sum 4 by the root, and place it under the 4 in 2d col., take the sum, and the result 12 is our 1st *trial divisor;* before using it, however, we add 2, the 1st term of the root, to the number 4 in 1st col., giving 6.

We then find the 2d term of the root to be 4, annex it to the 6 in the 1st col., multiply the result 64 by 4, the 2d term of the root, place it under the trial divisor, removing it two places to the right, add, and we have the *true divisor*. We then multiply, subtract, and bring down the next period.

We then add the 2d term of the root, 4, to the last number in the 1st col., making 68, multiply the result by the last term of the root, 4, write the result under the true divisor, add, and the sum is the next *trial divisor*, before using which we add the last term of the root, 4, to the last number in 1st col., making 72. We then find the next term of the root, annex it to the last number in 1st col., 72, multiply the result by the last term of the root, write the result removed two places to the right, under the trial divisor, add, and the sum is the *true divisor*, etc.

Rule.—I. *Begin at units and separate the number into periods of three figures each, and find the greatest number whose cube is contained in the left hand period.*

II. *Write the first term of the root at the left for the first term of the* 1st col., *and its square for the first term*

of the 2D COL., *and its cube under the left hand period; subtract and annex to the remainder the next period for the* FIRST DIVIDEND.

III. *Add the first term of the root to the first term of* 1ST COL., *for its second term; multiply the second term by the root found, and add the product to the first term of* 2D COL., *for a* TRIAL DIVISOR, *before using which add the root to the last term in* 1ST COL.

IV. *Find the second term of the root by dividing the* DIVIDEND *by the* TRIAL DIVISOR *with two ciphers annexed; annex this second term of the root to the last term in* 1ST COL., *multiply the result by the second term of the root, and add the product advanced two places to the right to the trial divisor, and the result will be the* TRUE DIVISOR.

V. *Multiply the true divisor by the last term of the root found, subtract the result from the dividend, annex the next period to the remainder for the next dividend, and proceed in like manner until all the periods have been used.*

NOTE.—Require the pupils to apply this method to the problems giver under the preceding rule.

CONTRACTIONS IN CUBE ROOT.

828. The work of obtaining approximate roots may be shortened by the following method:

1. Extract the cube root of 2 to four decimal places.

SOLUTION.—We perform the division, as in the New Method, till we have found three of the required figures of the root. The trial divisor at this point will be 46875, and the corresponding remainder 46875. Now if we were continuing the operation regularly, we would annex three ciphers to the remainder and two to the divisor, hence when we begin to drop the last figures as in Art. 278, we must drop two from the divisor for one from the

OPERATION.

```
                              2   (1.2599
                              1
 3        t. d.               1000
 6
  4
 364                           728
  4          T. D.            272000
 432      t. d.
 180                          225125
  25         T. D.             46875
 45025                          4219
  25                             468
 46875                            42
                                   4
```

remainder. Then dividing 4687 by 468 the quotient figure must be 9, and multiplying (adding in what would be carried from the product of 9 by 75), we have 4219, and a remainder of 468. Then 4 is contained in 46, 9 times (adding in 9 times 68), and the division is finished, as no more terms remain in the divisor.

Rule.—*Extract the cube root, as usual, until one more than half the terms required in the root have been found; then with the trial divisor and last remainder proceed, as in contracted division of decimals, to find the other terms of the root, dropping two figures instead of one from the divisor at each step, and one from each remainder.*

2. $\sqrt[3]{3}$. *Ans.* 1.4422+.
3. $\sqrt[3]{4}$. *Ans.* 1.5873+.
4. $\sqrt[3]{5}$. *Ans.* 1.7099+.
5. $\sqrt[3]{6}$. *Ans.* 1.8171+.
6. $\sqrt[3]{7}$. *Ans.* 1.9129+.
7. $\sqrt[3]{9}$. *Ans.* 2.08008+.
8. $\sqrt[3]{10}$. *Ans.* 2.15443+.
9. $\sqrt[3]{11}$. *Ans.* 2.22398+.
10. $\sqrt[3]{24}$. *Ans.* 2.8844992.
11. $\sqrt[3]{25}$. *Ans.* 2.924018—.
12. $\sqrt[3]{.0079}$. *Ans.* .1991632+.
13. $\sqrt[3]{.25}$. *Ans.* .62996+.
14. $\sqrt[3]{2\sqrt{8}\times4\sqrt{8}}$. *Ans.* 4.
15. $\sqrt[3]{3\sqrt{64}\times9\sqrt{64}}$. *Ans.* 12.

Find the value of the following expressions:

16. $1.728^{\frac{1}{3}}+91.125^{\frac{7}{3}}-3$. *Ans.* 37365.1453125.

17. $(15625^{\frac{2}{3}}-\sqrt[3]{46656}\times\left(\frac{6561}{5184}\right)^{\frac{1}{2}}$. *Ans.* $662\frac{5}{8}$.

18. $\sqrt[3]{512^4}\div\sqrt[3]{3375}-7\times\sqrt[3]{.729}$. *Ans.* $266\frac{23}{30}$.

19. $2283\div(\sqrt[3]{12.\dot{7}0\dot{3}}+\sqrt[3]{101.\dot{6}2\dot{9}}$. *Ans.* $326\frac{1}{7}$.

20. $(185193^{\frac{2}{3}}-216^{\frac{4}{3}}-2809^{\frac{1}{2}})\div10^2$. *Ans.* 19.

APPLICATIONS OF CUBE ROOT.

829. The **Applications** of cube root to problems involving geometrical volumes, such as cubes, parallelopipedons, spheres, etc., are extensive.

830. The **Edge** of a cube is equal to the cube root of its contents.

EXAMPLES FOR PRACTICE.

1. Required the dimensions of a cubical cistern which contains 3375 cubic feet. *Ans.* 15 ft.

2. Required the entire surface of a cubical block which contains 4096 cubic meters. *Ans.* 1536 sq. meters.

3. Required the edge of a cube equivalent to a solid 40 ft. 8 in. long, 20 ft. 6 in. wide, and 12 ft. 10 in. high. *Ans.* 22.034 ft.

4. A miller wishes to make a cubical bin which shall contain 100 bu. of grain; what must be its depth? *Ans.* 4.992 ft.

5. How many square feet of boards will it take to line the four sides of a cubical cistern which contains 300 barrels of water? *Ans.* 467.42 sq. ft.

6. What would it cost to plaster the bottom and sides of a cubical reservoir which contains 200 barrels of water, at 5 cents a square foot? *Ans.* $22.29.

7. A farmer wishes to have a bin made whose width shall equal its depth, and length equal 3 times its width, and which shall contain 150 hectoliters of grain; required its dimensions. *Ans.* Length, 16.83+ ft.; width and depth, 5.61 ft.

8. There is a granary whose capacity is 5000 bushels; its length is twice its breadth, and breadth twice its height; required its dimensions. *Ans.* 36.784 ft.; 18.392 ft.; 9.196 ft.

9. A farmer wishes to build a granary containing 1920 cu. ft., whose dimensions are in the proportion of 5, 6, and 8; what are the dimensions? *Ans.* 10 ft.; 12 ft.; 16 ft.

10. In digging Mr. Fisk's cellar, the length being 6 times, and the width twice the depth, 324 loads of earth were removed; what are the dimensions? *Ans.* 54 ft.; 18 ft.; 9 ft.

11. I have two cubical boxes, one of which will exactly

hold a bushel of wheat, and the other a gallon of water; what is the inner edge of each?

Ans. 1st, 12.91 in.; 2d, 6.13 in.

12. A brewer has a vat which contains 6 barrels of beer (36 gal.), and its length and height are each equal to twice its breadth; required its dimensions.

Ans. L. and H., 4.13 ft.; B., 2.065 ft.

13. If a hollow sphere 4 feet in diameter and $3\frac{1}{4}$ inches thick weigh 18 tons, what would be the dimensions of a similar sphere that would weigh 1152 tons?

Ans. 16 ft.; 13 inches.

14. Estimating the area of the Mississippi Valley at 1,400,000 sq. miles, and the average annual rain in the whole valley at 169,128,960,000,000 cu. ft., what will be the average annual depth of rain water? *Ans.* 52 in.

SIMILAR VOLUMES.

831. Similar Volumes are such as have the same shape, but differ in size; as, cubes, spheres, etc.

832. A **Dimension** of a volume is a length, breadth, height, diameter, radius, circumference, etc.

833. The **Principles** of similar volumes are derived from geometry.

PRINCIPLES.

1. *Similar volumes are to each other as the cubes of their like dimensions.*

2. *Like dimensions of similar volumes are to each other as the cube roots of those volumes.*

1. If a globe 4 inches in diameter weigh 16 lb., what will a globe 6 inches in diameter weigh?

SOLUTION.—By Prin. 1, we have the weight of the second ball, which we represent by x, is to 16 lb., as 6^3 is to 4^3; whence $x = 16 \times (\frac{6}{4})^3$, or $16 \times (\frac{3}{2})^3$, which equals 54.

OPERATION.

$x : 16 :: 6^3 : 4^3.$

$x = 16 \times (\frac{3}{2})^3 = 54$, *Ans.*

2. If a cubical box 6 ft. long hold 173.58 bu., what will a cubical box 8 ft. long hold? *Ans.* 411.448+ bu.

3. How many globes $2\frac{1}{2}$ inches in diameter are equal to one 10 inches in diameter? *Ans.* 64.

4. There are two spheres whose diameters are respectively $\frac{5}{8}$ in. and $3\frac{3}{4}$ in.; required the relation of their contents. *Ans.* 2d is 216 times the first

5. If a tree 1 foot in diameter yields 2 cords of wood, how much wood is there in a similar tree 3 ft. 6 in. in diameter? *Ans* $85\frac{3}{4}$ cords.

6. There are two balls whose diameters are respectively 4 and 5 inches; required the diameter of a ball whose contents are equal to the contents of both. *Ans.* 5.74—.

7. If a globe of gold 1 in. in diameter is worth $100, what is the diameter of a globe of silver worth $2700, if gold is worth $15\frac{5}{8}$ times as much as silver? *Ans.* $7\frac{1}{2}$ inches.

8. There are three balls whose diameters are 1, $1\frac{1}{3}$, and $1\frac{2}{3}$ inches in diameter; required the diameter of a ball whose volume equals that of the three. *Ans.* 2 inches.

9. There are three balls whose diameters are 3, 4, and 5 inches respectively; required the diameter of a ball which contains as much as the three. *Ans.* 6 inches.

10. Four ladies own a ball of thread 8 inches in diameter; how much of the diameter must each wind off so as to share the thread equally? *Ans.* 1st, .732+in.; 2d, .919 in.; 3d, 1.31 in.; 4th, 5.039 in

EXTRACTION OF ANY ROOT.

334. Horner's Method, invented by Mr. Horner, of England, is the best general method of extracting roots.

Any root whose index contains only the factors 2 or 3, can be extracted by means of the square and cube root.

Rule.—I. *Divide the number into periods of as many figures each as there are units in the index of the root, and at the left of the given number arrange the same number of columns, writing 1 at the head of the left hand column and ciphers at the head of the others.*

II. *Find the required root of the first period, write it in the root, multiply the number in the 1st col. by this first term*

of the root, and add to the 2d col., multiply this sum by the root, and add it to the 3d col., and thus continue, writing the last product under the first period; subtract and bring down the next period for a DIVIDEND.

III. *Repeat this process, stopping one column sooner at the right each time until the sum falls in the 2d col. Then divide the* DIVIDEND *by the number in the last column, which is the* TRIAL DIVISOR; *the result is the second figure of the root.*

IV. *Use the second figure of the root precisely as the first, remembering to place the products* ONE *place to the right in the 2d col.,* TWO *in the 3d col., etc.; continue this operation until the root is completed or carried as far as desired.*

NOTES.—1. Only a part of the dividend is used for finding a root figure, according to the principle of place value. The partial dividend thus used always terminates with the first figure of the period annexed.

2. If any dividend does not contain the trial divisor, place a cipher in the root, and bring down the next period; annex one cipher to the last term of the 2d column, two ciphers to the last term of the 3d, three to the 4th, and then proceed according to the rule.

1. Extract the fourth root of 5636405776.

1	0	0	0	56·3640·5776(274
	2	4	8	16
	2	8	24	403640
	4	12	(1) 32 t. d.	
	2	12	21063	
	6	(1) 24	53063 T. D.	371441
	2	609	25669	321995776
	(1) 8	3009	(2) 78732 t. d.	
	7	658	1766944	
	87	3667	80498944 T. D.	321995776
	7	707		
	94	(2) 4374		
	7	4336		
	101	441736		
	7			
	(2) 108			
	4			
	1084			

2. $\sqrt[4]{2}$.	*Ans.* 1.1892+.	5. $\sqrt[4]{100}$	*Ans.* 3.16227+
3. $\sqrt[4]{3}$.	*Ans.* 1.2457+.	6. $\sqrt[6]{6}$.	*Ans.* 1.34801—.
4. $\sqrt[5]{5}$.	*Ans.* 1.37974—.	7. $\sqrt[11]{11}$.	*Ans.* 1.2436—.

SECTION XI.

ARITHMETICAL AND GEOMETRICAL SERIES.

835. A **Series** is a succession of numbers, each derived from the preceding by some fixed law.

836. The **Law of a Series** is the constant relation existing between two or more terms of the series.

837. The **Terms** of a series are the numbers which compose it. The *Extremes* are the first and last terms; the *Means* are the terms between the extremes.

838. An **Ascending Series** is one in which the terms increase from left to right; a *Descending Series* is one in which the terms decrease from left to right.

839. There are many different kinds of series; the only two suitable for arithmetic are Arithmetical and Geometrical Series. These series are usually called *Progressions.*

ARITHMETICAL PROGRESSION.

840. An **Arithmetical Progression** is a series of numbers which vary by a common difference; as,

3, 5, 7, 9, 11, 13, 15.

841. The **Common Difference** is the difference between any two consecutive terms; thus, in the above series the common difference is 2.

842. The **Quantities** considered are five, any three of which being given, the others may be found.

QUANTITIES CONSIDERED.

	Symbols.		*Symbols*
1. The first term,	a.	3. The common difference,	d.
2. The last term,	l.	4. The number of terms,	n.

5. The sum of all the terms, S.

SOLUTION.—By Case I. we have $39 = 7 +$ (No. of terms$-1) \times 3\frac{1}{5}$; hence (No. of terms$-1) \times 3\frac{1}{5} = 39 - 7$; and No. of terms$-1 = \frac{39-7}{3\frac{1}{5}}$, or No. of terms $= \frac{39-7}{3\frac{1}{5}} + 1 = 11$.

OPERATION.

$$39 = 7 + (n-1) \times 3\frac{1}{5}$$
$$(n-1) \times 3\frac{1}{5} = 39 - 7$$
$$n = \frac{39-7}{3\frac{1}{5}} + 1 = 11$$

SOLUTION 2D.—Since $l = a + (n-1)d$, $39 = 7 + (n-1) \times 3\frac{1}{5}$; hence $(n-1) \times 3\frac{1}{5} = 39 - 7$, or 32, and $n = 32 \div 3\frac{1}{5} + 1$, or 11.

NOTE.—Require the pupils to derive the formula and solve the problems by substituting the values of the terms in the formula. See Art. 843.

Rule.—*To find the number of terms, divide the difference between the extremes by the common difference, and add* 1.

2. How many days will it take a student to walk 51 miles a day, if he goes $3\frac{1}{2}$ miles the first day, 6 miles the second day, etc. *Ans.* 20 days.

3. How many pigs must a man buy, giving \$2.25 for the first, \$2.37$\frac{1}{2}$ for the second, etc., that the last may cost \$4.75? *Ans.* 21.

CASE V.

847. ***To insert a given number of arithmetical means between two given numbers.***

1. Insert 3 arithmetical means between the numbers 4 and 12.

SOLUTION.—Since there are 3 means, there are 3+2, or 5 terms in the whole series; hence by Case III., the common difference equals $\frac{12-4}{4}$, or 2; hence the means are 6, 8, and 10.

OPERATION.

$$d = \frac{12-4}{4} = 2$$
$$\therefore 4, 6, 8, 10, 12.$$

Rule.—*Take the given numbers as the extremes, and the number of means plus 2 as the number of terms; find the common difference by Case III., add this to the smaller number for the 1st mean, and so complete the series.*

2. Insert 6 arithmetical means between 3 and 24.

Ans. 6, 9, 12, 15, 18, 21.

3. If 2 means be found between the successive terms of the series, 1, 7, 13, 19, what will the new series be?

Ans. 1, 3, 5, 7, 9, 11, 13, etc.

4. Form an arithmetical series by writing 3 means between the successive terms of the series 3, 15, 27.

Ans. 3, 6, 9, 12, 15, 18, etc.

5. A man bought teas at prices increasing in arithmetical progression, the cheapest costing 25 cents, and the dearest $1.10 a pound; what were the prices of the four intermediate kinds? *Ans.* 42¢, 59¢, 76¢, and 93¢.

CASE VI.

848. ***Given, the first term, the last term, and the number of terms, to find the sum of the series.***

1. The first term is 3, the last term 19, and the number of terms 5; required the sum of the series.

SOLUTION.—To derive the rule, we find by Case III., the common difference to be 4. Writing the series in its natural, and then in an inverted order, we take the sum of the two series, and we have *twice the sum*, equal to 22 taken 5 times, that is, $(3+19)\times 5$; hence, the sum equals $\frac{1}{2}$ of $(3+19)\times 5$, or 55. Now, $3+19$ is the *sum of the extremes*, and 5 is the *number of terms*; hence we have the following

OPERATION TO DERIVE THE RULE.

$$\begin{aligned} \text{Sum} &= 3+7+11+15+19 \\ \text{Sum} &= 19+15+11+7+3 \\ 2\times\text{Sum} &= 22+22+22+22+22 \\ 2\times\text{Sum} &= 22\times 5=(3+19)\times 5 \\ \text{Sum} &= \frac{3+19}{2}\times 5=55 \end{aligned}$$

Rule.—*To find the sum of an arithmetical series, multiply half the sum of the extremes by the number of terms.*

NOTE.—This is expressed in the following formula: $S=\frac{a+l}{2}\times n$. The problem may be solved by substituting the values of the terms in this formula.

2. How many strokes does an ordinary clock strike in 24 hours? *Ans.* 156.

3. The last term of a series is 18.75, the common difference .25, and the number of terms 18; required the sum of the series. *Ans.* 299.25.

4. The clocks in Venice strike from 1 to 24; how many strokes does such a clock strike in a day? *Ans.* 300.

5. I discharge a mortgage in 15 payments; my last payment was $850, and each payment was $50 greater than the preceding; what was the mortgage? *Ans.* $7500.

6. A stone falling from an altitude will descend $16\frac{1}{12}$ feet in 1 second, 3 times as far the next second, 5 times as far the next second, etc., how far will it fall in half a minute?

Ans. 2 mi. 237 rd. 4½ ft.

7. 150 apples are placed in a row $2\frac{1}{2}$ yards apart, the first being 3 yards from a basket; how far will a boy travel, starting from the basket, to gather them singly into the basket? *Ans.* 32 mi. 455 yd.

8. I wish to set out 75 fruit trees 4 yards apart around a circular field which will exactly contain them in its circumference; how far shall I have walked when the last one is planted, if I plant the first one at the starting point, and always go on the circumference, returning to the starting point every time? *Ans.* 12 mi. 196 rd. 2 yd.

9. Suppose, in the last example, I had returned to the starting point every time, but had taken the shortest distance on the circumference of the circle; how far would I have walked? *Ans.* 6 mi. 125 rd. 1 ft. 6 in.

CASE VII.

849. ***Given, the sum and any two of these three—the first term, the last term, or the number of terms—to find the one not given.***

1. The sum of an arithmetical series is 63, the first term 3, and the last term 18; what is the number of terms?

SOLUTION.—By Case VI., we have $63 = \frac{3+18}{2} \times$ (No. of terms); hence, No. of terms $= 63 \div \frac{3+18}{2}$, or No. of terms $= 6$.

OPERATION.

$$63 = \frac{3+18}{2} \times n$$

$$n = 63 \div \frac{3+18}{2}$$

$$n = 63 \times \tfrac{2}{21} = 6$$

SOLUTION 2D.—By Case VI., we have $S = \frac{a+l}{2} \times n$; multiplying by 2, we have $2S = (a+l) \times n$; dividing by $a+l$, we have $n = \frac{2S}{a+l}$; substituting the values of a and l, we have $n = 63 \times 2 \div (3+18)$, which equals $126 \div 21$, or 6.

Rule.—*To find the number of terms, divide twice the sum of the terms by the sum of the extremes.*

NOTE.—The other cases are solved in a similar manner. Let the pupils derive and state the rules.

2. How long will it take to pay a debt of \$3500, the payments being made yearly in a decreasing series, if the first and last are respectively \$575 and \$125? *Ans.* 10 years.

3. If I travel 660 miles in 15 days, going 65 miles the last day, increasing regularly each day, how far did I go the first day? *Ans.* 23 miles.

4. The sum of the terms is 4935, the first term 197, and number of terms 21; what is the last term? *Ans.* 273.

5. I owe a debt of $7200; I wish to cancel it in 16 payments, increasing regularly at each payment, the first being $300; required the last payment. *Ans.* $600.

850. Since there are five quantities in Arithmetical Series, any three of which being given the other two may be found, there are twenty distinct cases.

851. The rules for the eight simple cases are expressed in the following formulas:

1. $l=a+(n-1)d.$	1. $S=\frac{a+l}{2}\times n.$
2. $a=l-(n-1)d.$	2. $n=\frac{2S}{a+l}.$
3. $d=\frac{l-a}{n-1}.$	3. $a=\frac{2S}{n}-l.$
4. $n=\frac{l-a}{d}+1.$	4. $l=\frac{2S}{n}-a.$

GEOMETRICAL PROGRESSION.

852. A **Geometrical Progression** is a series of numbers which vary by a common multiplier; as, 2, 6, 18, 54, etc.

853. The **Rate** or *Ratio* is the common multiplier; thus, in the above series, the rate is 3.

854. In an **Ascending** series, the rate is greater than a unit; in a *Descending* series, the rate is less than a unit.

855. The **Quantities** considered are five, any three of which being given, the others may be found.

QUANTITIES CONSIDERED.

	Symbols.		*Symbols*
1. The first term,	a.	3. The number of terms,	n.
2. The last term,	l.	4. The rate,	r.
5. The sum of the terms,	S.		

7. 150 apples are placed in a row $2\frac{1}{2}$ yards apart, the first being 3 yards from a basket; how far will a boy travel, starting from the basket, to gather them singly into the basket? *Ans.* **32 mi. 455 yd.**

8. I wish to set out 75 fruit trees 4 yards apart around a circular field which will exactly contain them in its circumference; how far shall I have walked when the last one is planted, if I plant the first one at the starting point, and always go on the circumference, returning to the starting point every time? *Ans.* **12 mi. 196 rd. 2 yd.**

9. Suppose, in the last example, I had returned to the starting point every time, but had taken the shortest distance on the circumference of the circle; how far would I have walked? *Ans.* **6 mi. 125 rd. 1 ft. 6 in.**

CASE VII.

849. ***Given, the sum and any two of these three—the first term, the last term, or the number of terms—to find the one not given.***

1. The sum of an arithmetical series is 63, the first term 3, and the last term 18; what is the number of terms?

SOLUTION.—By Case VI., we have $63 = \frac{3+18}{2} \times$ (No. of terms); hence, No. of terms $= 63 \div \frac{3+18}{2}$, or No. of terms $= 6$.

OPERATION.

$$63 = \frac{3+18}{2} \times n$$

$$n = 63 \div \frac{3+18}{2}$$

$$n = 63 \times \frac{2}{21} = 6$$

SOLUTION 2D.—By Case VI., we have $S = \frac{a+l}{2} \times n$; multiplying by 2, we have $2S = (a+l) \times n$; dividing by $a+l$, we have $n = \frac{2S}{a+l}$; substituting the values of a and l, we have $n = 63 \times 2 \div (3+18)$, which equals $126 \div 21$, or 6.

Rule.—*To find the number of terms, divide twice the sum of the terms by the sum of the extremes.*

NOTE.—The other cases are solved in a similar manner. Let the pupils derive and state the rules.

2. How long will it take to pay a debt of \$3500, the payments being made yearly in a decreasing series, if the first and last are respectively \$575 and \$125? *Ans.* **10 years.**

3. If I travel 660 miles in 15 days, going 65 miles the last day, increasing regularly each day, how far did I go the first day? *Ans.* 23 miles.

4. The sum of the terms is 4935, the first term 197, and number of terms 21; what is the last term? *Ans.* 273.

5. I owe a debt of $7200; I wish to cancel it in 16 payments, increasing regularly at each payment, the first being $300; required the last payment. *Ans.* $600.

850. Since there are five quantities in Arithmetical Series, any three of which being given the other two may be found, there are twenty distinct cases.

851. The rules for the eight simple cases are expressed in the following formulas:

1. $l=a+(n-1)d.$	1. $S=\frac{a+l}{2}\times n.$
2. $a=l-(n-1)d.$	2. $n=\frac{2S}{a+l}.$
3. $d=\frac{l-a}{n-1}.$	3. $a=\frac{2S}{n}-l.$
4. $n=\frac{l-a}{d}+1.$	4. $l=\frac{2S}{n}-a.$

GEOMETRICAL PROGRESSION.

852. A **Geometrical Progression** is a series of numbers which vary by a common multiplier; as, 2, 6, 18, 54, etc.

853. The **Rate** or *Ratio* is the common multiplier; thus, in the above series, the rate is 3.

854. In an **Ascending** series, the rate is greater than a unit; in a *Descending* series, the rate is less than a unit.

855. The **Quantities** considered are five, any three of which being given, the others may be found.

QUANTITIES CONSIDERED.

	Symbols.		*Symbols*
1. The first term,	$a.$	3. The number of terms,	$n.$
2. The last term,	$l.$	4. The rate,	$r.$

5. The sum of the terms, $S.$

CASE I.

856. ***Given, the first term, the rate, and the number of terms, to find the last term.***

1. The first term is 3, the rate 4, and the number of terms 7; required the last term.

SOLUTION.—The 2d term equals 3×4; the 3d term equals 3×4 multiplied by 4, or 3×4^2, which is the 1st term into the 2d power of the rate; the 4th term equals 3×4^2 multiplied by 4, or 3×4^3, which is the first term into the third power of the rate; hence the 7th term equals the first term into the 6th power of the rate, or 3×4^6, which equals 12288.

OPERATION TO FIND THE RULE.

$$2d = 3 \times 4$$
$$3d = 3 \times 4^2$$
$$4th = 3 \times 4^3$$
$$\text{hence, } 7th = 3 \times 4^6 = 12{,}288$$

Rule.—*To find the last term, multiply the first term by the rate raised to a power one less than the number of terms.*

NOTE.—This rule may be expressed by the formula $l = ar^{n-1}$.

2. The first term is 4, rate 5, and the number of terms 7; required the last term. *Ans.* 62,500.

3. The first term is 24, rate $\frac{1}{2}$, and number of terms 10; required the last term. *Ans.* $\frac{3}{64}$.

4 The first term of a progression is $\frac{1}{81}$ and the rate 3; required the 8th term. *Ans.* 27.

5. The first term of a progression is $1953\frac{1}{8}$, and the rate $\frac{2}{5}$; required the 6th term. *Ans.* 20.

6. If I were to buy 12 sheep, giving 1 mill for the first, $2\frac{1}{2}$ mills for the second, $6\frac{1}{4}$ mills for the third, etc., what would I pay for the last sheep? *Ans.* $\$23.841\frac{1757}{2048}$.

7. If I were to buy 20 cows, giving $1\frac{1}{2}$ cents for the first cow, 3 cents for the 2d, 6 cents for the 3d, etc., what would be the price of the last cow? *Ans.* $7864.32.

8. What is the amount of $50 at compound interest for 7 years at 6%? *Ans.* $75.18.

9. Required the amount of $100 for 10 years at 5 per cent. compound interest. *Ans.* $162.889.

10. It is said that one stem of the hyoscyamus sometimes produces more than 50,000 seeds; if every seed should produce a fertile plant, how many plants would there be in the fourth crop from a single seed? *Ans.* 6250 quadrillions

CASE II.

857. ***Given, the last term, the number of terms, and the rate, to find the first term.***

1. The last term of a geometrical series of 10 terms is 1536, and the rate 2; what is the first term?

SOLUTION.—From Case I., we have $1536 = a \times 2^9$; hence the first term equals 1536 divided by 2^9, or $1536 \div 512$, which equals 3.

OPERATION.

$$1536 = a \times 2^9$$
$$a = 1536 \div 2^9$$
$$a = 1536 \div 512 = 3.$$

SOLUTION 2D.—Since $l = ar^{n-1}$, by dividing both members by r^{n-1}, we have $a = \frac{l}{r^{n-1}}$; and substituting the values of l, r, and n, in this formula, we have $a = 1536 \div 512 = 3$.

OPERATION.

$$l = ar^{n-1}$$
$$a = \frac{l}{r^{n-1}}$$
$$a = \tfrac{1536}{512} = 3.$$

Rule.—*To find the first term, divide the last term by the rate raised to a power one less than the number of terms.*

2. The seventh term of a geometrical series is 3645, and the rate 3; what is the first term? *Ans.* 5.

3. What sum at compound interest for 7 years at 6% will amount to \$75.18? *Ans.* \$50.

4. If a person travels 6 days, going $15\frac{3}{16}$ miles the last day of the journey, and at a rate $\frac{3}{4}$ as great any one day as the preceding day, how far does he go the first day? *Ans.* 64 miles.

CASE III.

858. ***Given, the extremes and the number of terms, to find the rate.***

1. The extremes of a geometrical series are 5 and 5120, and the number of terms 6; what is the rate?

SOLUTION.—From Case I., we have $5120 = 5 \times r^5$, hence the rate raised to the 5th power equals 5120 divided by 5, or 1024; factoring 1024 according to Art. 167, we have $r = 4$.

OPERATION.

$$5120 = 5 \times r^5$$
$$r^5 = 5120 \div 5$$
$$r^5 = 1024$$
$$r = 4$$

SOLUTION 2D.—Since $l = ar^{n-1}$, by dividing both members by a, we have $r^{n-1} = \frac{l}{a}$, and substituting the values of l, a, and n, in this formula, and factoring, we have $r = 4$.

OPERATION.

$$l = ar^{n-1}$$
$$r^{n-1} = \frac{l}{a}$$
$$r^5 = \tfrac{5120}{5} = 1024$$
$$r = 4$$

Rule.—*To find the rate, divide the last term by the first, and take a root of the quotient one less than the number of terms.*

2. The first term of a series is 5, last term 1280, and number of terms 9; what is the rate? *Ans.* 2.

3. The amount of $90 for 6 years, at compound interest, is $135.0657; what is the rate? *Ans.* 7%.

4. The amount of $240 for 2yr. 3mo., at compound interest payable quarterly, is $286.8221; what is the annual rate? *Ans.* 8%.

CASE IV.

859. ***Given, the extremes and the rate, to find the number of terms.***

1. The last term is 54, first term 2, and rate 3; what is the number of terms?

SOLUTION.—From Case I., we derive $54 = 2 \times 3^{n-1}$; dividing by 2, we have $3^{n-1} = 27$, that is, a power of 3 *one less than the number of terms* equals 27; hence if we take out the factor 3 from 27 until we reach 1, the number of such divisors plus 1 will equal the number of terms, which is 3.

OPERATION.

$$54 = 2 \times 3^{n-1}$$
$$3^{n-1} = \frac{54}{2} = 27$$
$$\begin{array}{r|l} 3 & 27 \\ \hline 3 & 9 \\ \hline 3 & 3 \\ \hline & 1 \end{array}$$
$$3 + 1 = 4.$$

SOLUTION 2D.—Since $l = ar^{n-1}$, r^{n-1} will equal $\frac{l}{a}$, and substituting, we have $3^{n-1} = 27$, and by factoring, $n - 1 = 3$, and $n = 4$.

OPERATION.

$$l = ar^{n-1}$$
$$r^{n-1} = \frac{a}{l}$$
$$3^{n-1} = 27 = 3^3$$
$$n = 4$$

Rule.—*Divide the last term by the first, divide this quotient by the rate, and thus continue in successive division, until the quotient is* 1; *the number of divisors, plus* 1, *will be the number of terms.*

2. The first term is 8, the last term 512, and the rate 4; required the number of terms. *Ans.* 4.

3. The first term is 4, the last term 78732, the rate 3; required the number of terms. *Ans.* 10

4. The first term is $\frac{1}{9}$, the last term $\frac{1}{294912}$, the rate $\frac{1}{8}$; what is the number of terms? *Ans.* 6.

CASE V.

860. ***To insert a given number of geometrical means between two given numbers.***

1. Insert 3 geometrical means between 3 and 768.

SOLUTION.—Since there are 3 means, there are 3+2, or 5 terms in the series; hence by Case III., the rate equals $\sqrt[4]{768\div3}$, or $\sqrt[4]{256}$, which equals 4; and multiplying the first term by the rate, we have 12, 48, 192, as the required means.

OPERATION.

$768\div3=256$

$\sqrt[4]{256}=4$, rate

$\therefore$ 12, 48, 192.

Rule.—*Take the given numbers as the extremes, and the number of means plus 2 as the number of terms; find the rate by Case III., multiply this by the smaller number for the first mean, and thus complete the series.*

2. Insert three geometrical means between 5 and 405.

Ans. 15, 45, 135.

3. Insert 6 geometrical means between 7 and 546875.

Ans. 35, 175, 875, 4375, 21875, 109375.

4. Insert four geometrical means between $1\frac{1}{2}$ and $\frac{16}{81}$, and write the series. *Ans.* $1\frac{1}{2}$, 1, $\frac{2}{3}$, $\frac{4}{9}$, $\frac{8}{27}$, $\frac{16}{81}$.

5. If two means be found between the successive terms of the series 1, 8, 64, 512, 4096, what will the new series be?

Ans. 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096.

CASE VI.

861. ***Given, the first term, the rate, and the last term or the number of terms, to find the sum of the terms.***

1. The first term is 3, rate 4, and number of terms 5; required the sum of the terms.

SOLUTION.—Writing the series expressing the sum, multiplying it by the rate, and taking the difference of the two series, we have 3 times the *sum* equals $768\times4-3$, hence the *sum* equals $(768\times4-3)\div3$, or 1023. In this solution we observe that we have the *last term multiplied by the rate*, the product *diminished by* the *first term*, and the *difference* divided by the rate minus one; hence the following rule:

OPERATION TO FIND THE RULE.

$$\begin{aligned}\text{Sum}&=3+12+48+192+768\\ \text{Sum}\times4&=\underline{\quad 12+48+192+768+768\times4}\\ \text{Sum}\times3&=768\times4-3\\ \text{Sum}&=\frac{768\times4-3}{3}=1023.\end{aligned}$$

Rule.—*To find the sum, multiply the last term by the rate, subtract the first term, and divide the remainder by the rate diminished by unity.*

NOTES.—1. This rule is expressed in the following formula: $S = \frac{lr-a}{r-1}$.

2. In a descending series, we subtract the product of the last term and the rate from the first term, and divide the remainder by 1 minus the rate.

3. If the number of terms is given, we first find the last term by Case I.

EXAMPLES FOR PRACTICE.

Find the sum

2. Of 2, 6, 18, 54, etc., to 10 terms. *Ans.* 59048.
3. Of 3, 6, 12, 24, etc., to 12 terms. *Ans.* 12285.
4. Of 78732, 26244, etc., to 10 terms. *Ans.* 118096.
5. Of 3, 6, $4\frac{1}{2}$, etc., to 11 terms. *Ans.* $30\frac{84997}{131072}$.
6. Of $\frac{3}{4}$, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{9}$, etc., to 9 terms. *Ans.* $2\frac{1675}{8748}$.
7. Of $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}$, etc., to 10 terms. *Ans.* $\frac{341}{512}$.
8. Suppose a person to begin trading on a capital of \$2000, and to increase his capital by $\frac{1}{4}$ of itself each year for 12 years, what would be the final amount of his capital? *Ans.* \$137,519.15.

9. If a lady, married on the 1st of January, received \$1 from her father, with a promise of \$5 on the 1st of February, \$25 on the 1st of March, and continuing at the same rate for a year, what was her dowry? *Ans.* \$61,035,156.

10. If 10 stones are laid in a line, the first being 4 feet from a heap, the second 12 feet, the third 36 feet, and so on in a geometrical series, how far will a boy walk who picks them up and puts them on the heap one by one, if he starts from the heap in the first place? *Ans.* 236192 ft.

11. A man, inquiring the price of a horse, and being told it was \$350, said it was too much, whereupon the owner said, "The horse has 24 nails in his shoes; if you will give me 1 mill for the first nail, 2 mills for the second, and so on, doubling the price at every nail, you shall have him, on condition that if you are dissatisfied with your bargain, you shall pay my first price." The purchaser consented; which bargain was he likely to agree to finally, and how much would he pay by the last condition? *Ans.* \$16,777.215.

CASE VII.

862. ***Given, the sum of the terms and any two of these three, the first term, the last term, or the rate, to find the remaining one.***

863. From the formula for the sum we readily obtain the three following formulas, which the pupils will derive and state in the form of rules:

$$r=\frac{S-a}{S-l};\ l=\frac{(r-1)S+a}{r};\ a=lr-(r-1)S.$$

EXAMPLES FOR PRACTICE.

1. Given, the first term 12, the last term 26244, and sum of the series 39360, to find the rate. *Ans.* 3.

2. Given, the first term $\frac{2}{3}$, the last term $\frac{256}{6561}$, and sum of series $1\frac{6049}{6561}$, to find the rate. *Ans.* $\frac{2}{3}$.

3. Given, the rate 5, the first term 8, and the sum of the series 156248, to find the last term. *Ans.* 125000.

4. The rate is $\frac{1}{4}$, the first term 200, and the sum of the series $266\frac{333}{512}$; what is the last term? *Ans.* $\frac{25}{512}$.

5. The last term is 196608, the rate 8, and the sum of the series 224694; what is the first term? *Ans.* 6.

6. The last term is $136\frac{11}{16}$, the rate $\frac{3}{4}$, and the sum of the series $3685\frac{15}{16}$; what is the first term? *Ans.* 1024.

864. Since there are five quantities in Geometrical Progression, any three of which being given, the other two may be found, there are twenty distinct cases.

865. The rules for the eight simple cases are expressed in the following formulas:

1. $l=ar^{n-1}$.
2. $a=\frac{l}{r^{n-1}}$.
3. $r=\sqrt[n-1]{\frac{l}{a}}$
4. $n=\frac{\log l-\log a}{\log r}+1$.

1. $S=\frac{lr-a}{r-1}$.
2. $l=\frac{S(r-1)+a}{r}$.
3. $a=lr-S(r-1)$.
4. $r=\frac{S-a}{S-l}$.

NOTE.—For the formulas by which every one of the twenty possible cases may be solved, see *Elementary Algebra.*

INFINITE SERIES.

866. An **Infinite Series** is a series in which the number of terms is infinite.

867. In a descending series of an infinite number of terms, the last term becomes so small that it is considered zero; the formula, $S=\frac{a-rl}{1-r}$ becomes $S=\frac{a}{1-r}$; hence we have the following

Rule.—*To find the sum of an infinite series, divide the first term by unity diminished by the rate.*

1. What is the sum of the infinite series $2+\frac{2}{3}+\frac{2}{9}+\frac{2}{27}$, etc.?

SOLUTION.—In this series, the first term is 2, and the rate $\frac{1}{3}$, and the last term may be regarded as zero, hence the series equals 2 divided by $1-\frac{1}{3}$, or $2\div\frac{2}{3}$, which equals 3.

OPERATION.

$$\text{Sum}=\frac{2}{1-\frac{1}{3}}=3, \textit{ Ans}$$

EXAMPLES FOR PRACTICE.

Find the sum of the following infinite series:

2. Of 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, etc. *Ans.* 2.
3. Of $\frac{1}{2}$, $\frac{3}{8}$, $\frac{9}{32}$, $\frac{27}{128}$, etc. *Ans.* 2.
4. Of 1, $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, etc. *Ans.* $1\frac{1}{2}$.
5. Of $\frac{2}{3}$, $\frac{10}{27}$, $\frac{50}{243}$, etc. *Ans.* $1\frac{1}{2}$.
6. Of $.\dot{4}\dot{5}=.4545$ etc. $=\frac{45}{100}+\frac{45}{10000}$, etc. *Ans.* $\frac{5}{11}$.
7. Of $.\dot{2}1\dot{6}$ and $.\dot{4}15\dot{8}$. *Ans.* $\frac{8}{37}$; $\frac{42}{101}$.
8. Of $.35\dot{1}3\dot{5}$ and $.9\dot{2}8571\dot{4}$. *Ans.* $\frac{13}{37}$; $\frac{13}{14}$.
9. $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}$, etc. *Ans.* $\frac{2}{3}$.

10. If a body should move $\frac{1}{2}$ a mile the 1st second, $\frac{1}{4}$ of a mile the 2d second, and so on until it stops, how far would it move? *Ans.* 1 mile.

11. A ball dropped from the ceiling of a room 12 ft. high, bounds back 6 ft., then falling, bounds back 3 ft., and so on; how far will it move before coming to rest? *Ans.* 36 ft.

12. A fox and hound, 16 rods apart, run so that when the hound has run the 16 rods, the fox has run 4 rods, and when the hound has run these 4 rods, the fox has run 1 rod, etc.; how far will the hound run to catch the fox?

Ans. $21\frac{1}{3}$ rods.

SECTION XII.

HIGHER PERCENTAGE.

COMPOUND INTEREST.

868. Compound Interest is interest on both principal and interest, when the interest is not paid when due.

Compound interest assumes that if the borrower does not pay the interest when due, it is proper that he should pay interest for it until paid Some regard it as just, but it has not the sanction of law.

869. Compound Interest, like Simple Interest, may be treated under four cases.

CASE I.

870. ***Given, the principal, the rate, and the time, to find the compound interest or amount.***

1. What is the compound interest of $500 for 3 years, at 5%?

SOLUTION.—Multiplying by the rate per cent., we find the interest for 1 year to be $25; adding this to the principal, we find the amount to be $525, which is the principal for the second year; multiplying the new principal by the rate, we find the interest for the second year to be $26.25, and adding this to the 2d principal, we find the amount for the 2d year to be $551.25; and so proceeding, we find the amount for 3 years to be $578.81, from which we subtract the first principal, and the remainder, $78.81, is the compound interest. Hence the following

OPERATION.

$500	
.05	
25.00	= Int. 1st yr.
500	
525.00	= Amt. 1st yr.
.05	
26.25	= Int. 2d yr.
525	
551.25	= Amt. 2d year.
.05	
27.5625	= Int. 3d yr.
551.25	
578.81	= Amt. 3d yr.
500	
$78 81	= Com. Int. for 3 yr.

Rule.—I. *Find the amount of the principal for the first period of the time for which interest is reckoned, and make this the principal for the second period.*

II. *Find the amount of this principal for the next period, and thus continue till the end of the given time.*

III. *Subtract the given principal from the last amount, and the result will be the compound interest.*

NOTES.—1. When the interest is due semi-annually or quarterly, we find the interest for such time and proceed as above directed.

2. When the time is for years, months, and days, find the amount for the years, then compute the interest on this for the months and days, and add to the last amount before subtracting.

2. What is the compound interest of $650 for 5 years 3 months? *Ans.* $232.89.

3. What is the compound amount of $5340 for 4 yr. 3 mo. 8 da. at 7%? *Ans.* $7133.03.

4. What is the compound interest of $5000 at 10% for 2 years, payable quarterly? *Ans.* $1092.01.

5. What is the amount of $8350 for 5 yr. 7 mo. 24 da. at 8%, payable semi-annually? *Ans.* $13008.69.

6. Find the compound interest of $1800, invested at 7% for 3 years, and then at 8% for 2 years. *Ans.* $772.

871. The calculation of compound interest is facilitated by the use of a table, for which see Appendix.

Rule.—*Find from the table the amount for the given number of periods at the given rate, and multiply this amount by the principal. If there is any remaining time, find the amount of this product at the given rate for the time; the result will be the compound amount, from which subtract the given principal for the compound interest.*

NOTES.—1. If the time exceeds the limits of the table, calculate the amount for a convenient length of time by the table, take this amount as a principal, and calculate the amount for the remaining time.

2. If partial payments are made on notes bearing compound interest, the amount of the principal must first be found, and the sum of the amounts of the indorsements subtracted from it.

1. What is the compound interest of $7500 for 25 years, at 8%? *Ans.* $43,863.56.

2. What is the compound interest of $5760 for 15 yr. 4 mo. 24 da., at 10%? *Ans.* $19,263.39.

3. What is the amount of $664 for 30 yr. at 6%, payable semi-annually? *Ans.* $3911.77.

4. What is the compound interest of $100 for 40 years at 8%, interest payable quarterly? *Ans.* $2276.98.

5. What is the difference between the simple and compound interest of $400 for 33 yr. 4 mo.? *Ans.* $1590.96.

6. What sum in 15 yr. 2 mo. 27 da., at 6 per cent. simple interest, will amount to the same as $5000 for the same rate and time at compound interest, payable semi-annually?

Ans. $6431.07.

7. A gentleman deposits in a savings bank, at the birth of his son, $1000 to be paid him when he comes of age, interest at 6% compounded semi-annually; what will the deposit amount to at the time it is due? *Ans.* $3460.70.

8. Mr. Adams left $20,000 to be equally divided between his son and daughter, directing that the daughter, who was 8 yr. 6 mo 18 da. old, should receive her share when she was 18 years old, and the son, who was 10 yr. 3 mo. 15 da. old, should receive his when he was 21; what will each receive, if the money is invested in a savings bank at 4 per cent. compounded semi-annually?

Ans. Son, $15282.97; daughter, $14539.55.

9. $700. NEW YORK, MAY 19, 1876.

Three months after date, I promise to pay James Wilkins, or order, Seven Hundred Dollars, for value received, with compound interest at 7%. GEORGE BOOTH.

Indorsements: Dec. 15, 1876, $100; May 19, 1877, $300; Sept. 30, 1877, $150.

What was due May 19, 1878? *Ans.* $213.47.

CASE II.

872. ***Given, the compound interest or amount, the time, and the rate, to find the principal.***

1. What principal, at 6 per cent. compound interest, will yield $1007.26 in 7 years?

SOLUTION.—The compound interest of $1 for 7 years at 6% is $0.50363+, and $1007.26 is the compound interest of as many dollars as $0.50363 is contained times in $1007.26, which is $2000.

OPERATION.

$1007.26 ÷ .50363 = $2000.

Rule.—*Divide the given interest or amount by the interest or amount of* $1 *for the given rate and time, to find the principal.*

Rule.—*To find the rate, divide the last term by the first, and take a root of the quotient one less than the number of terms.*

2. The first term of a series is 5, last term 1280, and number of terms 9; what is the rate? *Ans.* 2.

3. The amount of \$90 for 6 years, at compound interest, is \$135.0657; what is the rate? *Ans.* 7%.

4. The amount of \$240 for 2yr. 3mo., at compound interest payable quarterly, is \$286.8221; what is the annual rate? *Ans.* 8%.

CASE IV.

859. ***Given, the extremes and the rate, to find the number of terms.***

1. The last term is 54, first term 2, and rate 3; what is the number of terms?

SOLUTION.—From Case I., we derive $54 = 2 \times 3^{n-1}$; dividing by 2, we have $3^{n-1} = 27$, that is, a power of 3 *one less than the number of terms* equals 27; hence if we take out the factor 3 from 27 until we reach 1, the number of such divisors plus 1 will equal the number of terms, which is 3.

OPERATION.

$$54 = 2 \times 3^{n-1}$$
$$3^{n-1} = \tfrac{54}{2} = 27$$
$$\begin{array}{r} 3\underline{)27} \\ 3\underline{)9} \\ 3\underline{)3} \\ 1 \end{array}$$
$$3 + 1 = 4.$$

SOLUTION 2D.—Since $l = ar^{n-1}$, r^{n-1} will equal $\frac{l}{a}$, and substituting, we have $3^{n-1} = 27$, and by factoring, $n - 1 = 3$, and $n = 4$.

OPERATION.

$$l = ar^{n-1}$$
$$r^{n-1} = \frac{a}{l}$$
$$3^{n-1} = 27 = 3^3$$
$$n = 4$$

Rule.—*Divide the last term by the first, divide this quotient by the rate, and thus continue in successive division, until the quotient is* 1; *the number of divisors, plus* 1, *will be the number of terms.*

2. The first term is 8, the last term 512, and the rate 4; required the number of terms. *Ans.* 4.

3. The first term is 4, the last term 78732, the rate 3; required the number of terms. *Ans.* 10

4. The first term is $\frac{1}{9}$, the last term $\frac{1}{2949112}$, the rate $\frac{1}{8}$; *what is the* number of terms? *Ans.* 6.

CASE V.

860. ***To insert a given number of geometrical means between two given numbers.***

1. Insert 3 geometrical means between 3 and 768.

SOLUTION.—Since there are 3 means, there are $3+2$, or 5 terms in the series; hence by Case III., the rate equals $\sqrt[4]{768 \div 3}$, or $\sqrt[4]{256}$, which equals 4; and multiplying the first term by the rate, we have 12, 48, 192, as the required means.

OPERATION.

$768 \div 3 = 256$

$\sqrt[4]{256} = 4$, rate

$\therefore$ 12, 48, 192.

Rule.—*Take the given numbers as the extremes, and the number of means plus 2 as the number of terms; find the rate by Case III., multiply this by the smaller number for the first mean, and thus complete the series.*

2. Insert three geometrical means between 5 and 405.

Ans. 15, 45, 135.

3. Insert 6 geometrical means between 7 and 546875.

Ans. 35, 175, 875, 4375, 21875, 109375.

4. Insert four geometrical means between $1\frac{1}{2}$ and $\frac{16}{81}$, and write the series. *Ans.* $1\frac{1}{2}$, 1, $\frac{2}{3}$, $\frac{4}{9}$, $\frac{8}{27}$, $\frac{16}{81}$.

5. If two means be found between the successive terms of the series 1, 8, 64, 512, 4096, what will the new series be?

Ans. 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096.

CASE VI.

861. ***Given, the first term, the rate, and the last term or the number of terms, to find the sum of the terms.***

1. The first term is 3, rate 4, and number of terms 5; required the sum of the terms.

OPERATION TO FIND THE RULE.

$$\begin{aligned} \text{Sum} &= 3+12+48+192+768 \\ \text{Sum}\times 4 &= \phantom{3+{}}12+48+192+768+768\times 4 \\ \hline \text{Sum}\times 3 &= 768\times 4-3 \\ \text{Sum} &= \frac{768\times 4-3}{3} = 1023. \end{aligned}$$

SOLUTION.—Writing the series expressing the sum, multiplying it by the rate, and taking the difference of the two series, we have 3 times the *sum* equals $768\times 4-3$, hence the *sum* equals $(768\times 4-3)\div 3$, or 1023. In this solution we observe that we have the *last term multiplied by the rate*, the product diminished by the *first term*, and the *difference* divided by the rate minus one; hence the following rule:

Rule.—*To find the sum, multiply the last term by the rate, subtract the first term, and divide the remainder by the rate diminished by unity.*

NOTES.—1. This rule is expressed in the following formula: $S=\frac{lr-a}{r-1}$.

2. In a descending series, we subtract the product of the last term and the rate from the first term, and divide the remainder by 1 minus the rate.

3. If the number of terms is given, we first find the last term by Case I.

EXAMPLES FOR PRACTICE.

Find the sum

2. Of 2, 6, 18, 54, etc., to 10 terms. *Ans.* 59048.
3. Of 3, 6, 12, 24, etc., to 12 terms. *Ans.* 12285.
4. Of 78732, 26244, etc., to 10 terms. *Ans.* 118096.
5. Of 8, 6, $4\frac{1}{2}$, etc., to 11 terms. *Ans.* $30\frac{84997}{131072}$.
6. Of $\frac{3}{4}$, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{9}$, etc., to 9 terms. *Ans.* $2\frac{1675}{8748}$.
7. Of $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}$, etc., to 10 terms. *Ans.* $\frac{341}{512}$.
8. Suppose a person to begin trading on a capital of $2000, and to increase his capital by $\frac{1}{4}$ of itself each year for 12 years, what would be the final amount of his capital? *Ans.* $137,519.15.
9. If a lady, married on the 1st of January, received $1 from her father, with a promise of $5 on the 1st of February, $25 on the 1st of March, and continuing at the same rate for a year, what was her dowry? *Ans.* $61,035,156.
10. If 10 stones are laid in a line, the first being 4 feet from a heap, the second 12 feet, the third 36 feet, and so on in a geometrical series, how far will a boy walk who picks them up and puts them on the heap one by one, if he starts from the heap in the first place? *Ans.* 236192 ft.
11. A man, inquiring the price of a horse, and being told it was $350, said it was too much, whereupon the owner said, "The horse has 24 nails in his shoes; if you will give me 1 mill for the first nail, 2 mills for the second, and so on, doubling the price at every nail, you shall have him, on condition that if you are dissatisfied with your bargain, you shall pay my first price." The purchaser consented; which bargain was he likely to agree to finally, and how much would he pay by the last condition? *Ans.* $16,777.215.

CASE VII.

862. ***Given, the sum of the terms and any two of these three, the first term, the last term, or the rate, to find the remaining one.***

863. From the formula for the sum we readily obtain the three following formulas, which the pupils will derive and state in the form of rules:

$$r=\frac{S-a}{S-l};\ l=\frac{(r-1)S+a}{r};\ a=lr-(r-1)S.$$

EXAMPLES FOR PRACTICE.

1. Given, the first term 12, the last term 26244, and sum of the series 39360, to find the rate. *Ans.* 3.

2. Given, the first term $\frac{2}{3}$, the last term $\frac{256}{6561}$, and sum of series $1\frac{6049}{6561}$, to find the rate. *Ans.* $\frac{2}{3}$.

3. Given, the rate 5, the first term 8, and the sum of the series 156248, to find the last term. *Ans.* 125000.

4. The rate is $\frac{1}{4}$, the first term 200, and the sum of the series $266\frac{333}{512}$; what is the last term? *Ans.* $\frac{25}{512}$.

5. The last term is 196608, the rate 8, and the sum of the series 224694; what is the first term? *Ans.* 6.

6. The last term is $136\frac{11}{16}$, the rate $\frac{3}{4}$, and the sum of the series $3685\frac{15}{16}$; what is the first term? *Ans.* 1024.

864. Since there are five quantities in Geometrical Progression, any three of which being given, the other two may be found, there are twenty distinct cases.

865. The rules for the eight simple cases are expressed in the following formulas:

1. $l=ar^{n-1}$.	1. $S=\dfrac{lr-a}{r-1}$.
2. $a=\dfrac{l}{r^{n-1}}$.	2. $l=\dfrac{S(r-1)+a}{r}$.
3. $r=\sqrt[n-1]{\dfrac{l}{a}}$	3. $a=lr-S(r-1)$.
4. $n=\dfrac{\log l-\log a}{\log r}+1$.	4. $r=\dfrac{S-a}{S-l}$.

NOTE.—For the formulas by which every one of the twenty possible cases may be solved, see *Elementary Algebra*.

INFINITE SERIES.

866. An **Infinite Series** is a series in which the number of terms is infinite.

867. In a descending series of an infinite number of terms, the last term becomes so small that it is considered zero; the formula, $S=\frac{a-rl}{1-r}$ becomes $S=\frac{a}{1-r}$; hence we have the following

Rule.—*To find the sum of an infinite series, divide the first term by unity diminished by the rate.*

1. What is the sum of the infinite series $2+\frac{2}{3}+\frac{2}{9}+\frac{2}{27}$, etc.?

SOLUTION.—In this series, the first term is 2, and the rate $\frac{1}{3}$, and the last term may be regarded as zero, hence the series equals 2 divided by $1-\frac{1}{3}$, or $2\div\frac{2}{3}$, which equals 3.

OPERATION.

$$\text{Sum}=\frac{2}{1-\frac{1}{3}}=3, \textit{ Ans}$$

EXAMPLES FOR PRACTICE.

Find the sum of the following infinite series:

2. Of 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, etc. *Ans.* 2.
3. Of $\frac{1}{2}$, $\frac{3}{8}$, $\frac{9}{32}$, $\frac{27}{128}$, etc. *Ans.* 2.
4. Of 1, $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, etc. *Ans.* $1\frac{1}{2}$.
5. Of $\frac{2}{3}$, $\frac{10}{27}$, $\frac{50}{243}$, etc. *Ans.* $1\frac{1}{2}$.
6. Of $.\dot{4}\dot{5}=.4545$ etc. $=\frac{45}{100}+\frac{45}{10000}$, etc. *Ans.* $\frac{5}{11}$.
7. Of $.\dot{2}1\dot{6}$ and $.\dot{4}15\dot{8}$. *Ans.* $\frac{8}{37}$; $\frac{42}{101}$.
8. Of $.35\dot{1}3\dot{5}$ and $.9\dot{2}8571\dot{4}$. *Ans.* $\frac{13}{37}$; $\frac{13}{14}$.
9. $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}$, etc. *Ans.* $\frac{2}{3}$.

10. If a body should move $\frac{1}{2}$ a mile the 1st second, $\frac{1}{4}$ of a mile the 2d second, and so on until it stops, how far would it move? *Ans.* 1 mile.

11. A ball dropped from the ceiling of a room 12 ft. high, bounds back 6 ft., then falling, bounds back 3 ft., and so on; how far will it move before coming to rest? *Ans.* 36 ft.

12. A fox and hound, 16 rods apart, run so that when the hound has run the 16 rods, the fox has run 4 rods, and when the hound has run these 4 rods, the fox has run 1 rod, etc.; how far will the hound run to catch the fox?

Ans. $21\frac{1}{3}$ rods.

SECTION XII.

HIGHER PERCENTAGE.

COMPOUND INTEREST.

868. Compound Interest is interest on both principal and interest, when the interest is not paid when due.

Compound interest assumes that if the borrower does not pay the interest when due, it is proper that he should pay interest for it until paid Some regard it as just, but it has not the sanction of law.

869. Compound Interest, like Simple Interest, may be treated under four cases.

CASE I.

870. ***Given, the principal, the rate, and the time, to find the compound interest or amount.***

1. What is the compound interest of $500 for 3 years, at 5%?

SOLUTION.—Multiplying by the rate per cent., we find the interest for 1 year to be $25; adding this to the principal, we find the amount to be $525, which is the principal for the second year; multiplying the new principal by the rate, we find the interest for the second year to be $26.25, and adding this to the 2d principal, we find the amount for the 2d year to be $551.25; and so proceeding, we find the amount for 3 years to be $578.81, from which we subtract the first principal, and the remainder, $78.81, is the compound interest. Hence the following

OPERATION.

$500	
.05	
25.00	= Int. 1st yr.
500	
525.00	= Amt. 1st yr.
.05	
26.25	= Int. 2d yr.
525	
551.25	= Amt. 2d year.
.05	
27.5625	= Int. 3d yr.
551.25	
578.81	= Amt. 3d yr.
500	
$78 81	= Com. Int. for 3 yr.

Rule.—I. *Find the amount of the principal for the first period of the time for which interest is reckoned, and make this the principal for the second period.*

II. *Find the amount of this principal for the next period, and thus continue till the end of the given time.*

III. *Subtract the given principal from the last amount, and the result will be the compound interest.*

NOTES.—1. When the interest is due semi-annually or quarterly, we find the interest for such time and proceed as above directed.

2. When the time is for years, months, and days, find the amount for the years, then compute the interest on this for the months and days, and add to the last amount before subtracting.

2. What is the compound interest of $650 for 5 years 3 months? *Ans.* **$232.89.**

3. What is the compound amount of $5340 for 4 yr. 3 mo. 8 da. at 7%? *Ans.* **$7133.03.**

4. What is the compound interest of $5000 at 10% for 2 years, payable quarterly? *Ans.* **$1092.01.**

5. What is the amount of $8350 for 5 yr. 7 mo. 24 da. at 8%, payable semi-annually? *Ans.* **$13008.69.**

6. Find the compound interest of $1800, invested at 7% for 3 years, and then at 8% for 2 years. *Ans.* **$772.**

871. The calculation of compound interest is facilitated by the use of a table, for which see Appendix.

Rule.—*Find from the table the amount for the given number of periods at the given rate, and multiply this amount by the principal. If there is any remaining time, find the amount of this product at the given rate for the time; the result will be the compound amount, from which subtract the given principal for the compound interest.*

NOTES.—1. If the time exceeds the limits of the table, calculate the amount for a convenient length of time by the table, take this amount as a principal, and calculate the amount for the remaining time.

2. If partial payments are made on notes bearing compound interest, the amount of the principal must first be found, and the sum of the amounts of the indorsements subtracted from it.

1. What is the compound interest of $7500 for 25 years, at 8%? *Ans.* **$43,863.56.**

2. What is the compound interest of $5760 for 15 yr. 4 mo. 24 da., at 10%? *Ans.* **$19,263.39.**

3. What is the amount of $664 for 30 yr. at 6%, payable semi-annually? *Ans.* **$3911.77.**

4. What is the compound interest of $100 for 40 years at 8%, interest payable quarterly? *Ans.* **$2276.98.**

5. What is the difference between the simple and compound interest of $400 for 33 yr. 4 mo.? *Ans.* $1590.96.

6. What sum in 15 yr. 2 mo. 27 da., at 6 per cent. simple interest, will amount to the same as $5000 for the same rate and time at compound interest, payable semi-annually?

Ans. $6431.07.

7. A gentleman deposits in a savings bank, at the birth of his son, $1000 to be paid him when he comes of age, interest at 6% compounded semi-annually; what will the deposit amount to at the time it is due? *Ans.* $3460.70.

8. Mr. Adams left $20,000 to be equally divided between his son and daughter, directing that the daughter, who was 8 yr. 6 mo 18 da. old, should receive her share when she was 18 years old, and the son, who was 10 yr. 3 mo. 15 da. old, should receive his when he was 21; what will each receive, if the money is invested in a savings bank at 4 per cent. compounded semi-annually?

Ans. Son, $15282.97; daughter, $14539.55.

9. $700. NEW YORK, MAY 19, 1876.

Three months after date, I promise to pay James Wilkins, or order, Seven Hundred Dollars, for value received, with compound interest at 7%. GEORGE BOOTH.

Indorsements: Dec. 15, 1876, $100; May 19, 1877, $300; Sept. 30, 1877, $150.

What was due May 19, 1878? *Ans.* $213.47.

CASE II.

872. ***Given, the compound interest or amount, the time, and the rate, to find the principal.***

1. What principal, at 6 per cent. compound interest, will yield $1007.26 in 7 years?

SOLUTION.—The compound interest of $1 for 7 years at 6% is $0.50363+, and $1007.26 is the compound interest of as many dollars as $0.50363 is contained times in $1007.26, which is $2000.

OPERATION.

$1007.26 ÷ .50363 = $2000.

Rule.—*Divide the given interest or amount by the interest or amount of* $1 *for the given rate and time, to find the principal.*

NOTE.—If the given amount is due at some future time, the principal is its *present worth* at compound interest, and the difference between the amount and the present worth is the *compound interest*.

2. What principal will yield \$31,086.78 interest in 40 yr. at 8%? *Ans.* \$1500.

3. What principal in 35 yr. 7 mo. 21 da. at 5%, will amount to \$4098.95? *Ans.* \$720.

4. What principal in 25 yr. 3 mo. 16 da. at 7%, payable semi-annually, will yield \$26,508.22 interest? *Ans.* \$5640.

5. What is the present worth of \$8445.69, due 23 yr. 4 mo. 12 da. hence, at 10%, payable quarterly? *Ans.* \$840.

6. What is the difference between the present worth at simple and compound interest of \$15,860.85, due in 15 yr., at 8%? *Ans.* \$2209.48.

CASE III.

873. ***Given, the principal, the rate, and the compound interest or amount, to find the time.***

1. In what time will \$2560 at 5%, amount to \$5588.16?

SOLUTION.—Since \$5588.16 is the amount of \$2560 at 5% in a certain time, the amount of \$1 in the same time will be $\frac{1}{2560}$ of \$5588.16, which is \$2.182875, which we find in the table under 5%, corresponds to 16 years.

OPERATION.

\$5588.16 ÷ 2560 = \$2.1828
hence time = 16 yr.

Rule.—*Divide the amount by the given principal, and the quotient will be the amount of* \$1 *for the required time. Find this amount in the table under the given rate; the number of years opposite will be the required time.*

NOTE.—If the amount obtained by division cannot be exactly found in the table, the number next less than that amount under the given rate, will correspond to the years or periods. Then take the difference between the two amounts, and also the difference between the latter amount and the next larger in the table; the ratio of these two differences will be the fractional part of a year or period, which may be expressed in months and days.

2. In what time will \$600 amount to \$1187.96 at 5% compound interest? *Ans.* 14 years.

3. In what time will \$4800 draw \$7351.52 compound interest, at 7%, payable semi-annually? *Ans.* 13 yr. 6 mo.

4. In what time will \$3750 amount to \$10,535.73, at 10% compound interest, payable quarterly?

Ans. 10 yr. 5 mo. 15 da.

5. In what time will any sum of money double itself, at compound interest, at 4, 5, 6, 7, 8, or 10%?

Ans. 17 yr. 8 mo. 1 da.; 14 yr. 2 mo. 13 da.; 11 yr. 10 mo. 21 da.; 10 yr. 2 mo. 26 da.; 9 yr. 2 da.; 7 yr. 3 mo. 5 da.

6. A gentleman deposited $750 in a savings bank for the benefit of his daughter on her fifth birthday, arranging that it should remain at a semi-annual interest of 4% until it amounted to $2000. How old is the daughter when she receives her money? *Ans.* 17 yr. 6 mo. 1 da.

CASE IV.

874. ***Given, the principal, the compound interest or amount, and the time, to find the rate.***

1. At what rate will $600 amount to $1213.422 in 16 years?

OPERATION.

$1213.422 ÷ 600 = $2.02237, which corresponds to $4\frac{1}{2}$%.

SOLUTION.—If $600 amount to $1213.422 in 16 years at the required rate, $1 for the same time and rate will amount to $\frac{1}{600}$ of $1213.422, which is $2.02237, which we find in the table opposite 16 years at $4\frac{1}{2}$%.

Rule.—*Divide the amount by the given principal, and the quotient will be the amount of* $1 *for the given time at the required rate. Find this amount in the table opposite the given time; the rate at the top of the column will be the required rate.*

NOTE.—If the given time contains months and days, or more than an exact number of periods, find in the table opposite the given number of years or periods, the number next less than the amount found by division, and upon this reckon the interest for the remaining time. The result should correspond with the amount obtained by division, if the rate per cent. required is among those given in the table.

2. At what rate will $1000 amount to $2518.17 in 12 yr.? *Ans.* 8%.

3. At what rate will $5640 amount to $17,815.72 in 17 years? *Ans.* 7%.

4. At what rate will $37,800 amount to $77,483.12 in 14 yr. 6 mo. 12 da., interest payable semi-annually? *Ans.* 5%.

5. At what rate will $5849.16 amount to $11,047.999 in 5 yr. 4 mo. 16 da., interest payable quarterly? *Ans.* 12%.

6. At what rate will any sum double itself at compound interest in 10, 15, 20, 25, or 30 years?

Ans. 7+%; $4\frac{1}{2}$+%; $3\frac{1}{2}$+%; $2\frac{1}{2}$+%; 2+%.

ANNUITIES.

875. An **Annuity** is a sum of money to be paid annually, or at some other regular interval of time.

876. An **Annuity Certain** begins and ends at fixed times. A *Perpetuity* is an annuity which continues for ever.

877. A **Contingent Annuity** begins or ends with some uncertain event, such as the birth or death of one or more persons.

878. An **Immediate Annuity** begins immediately. A *Deferred Annuity*, or Annuity in Reversion, begins at some future time.

879. An **Annuity in Arrears,** or **Forborne,** is one on which the payments were not made when due.

880. The **Final Value** is the sum of the amounts of all the payments on interest from the time each is due to the end of the annuity.

881. The **Present Value** is such a sum as put at interest for the given time and rate, will amount to the final value.

NOTES.—1. An annuity is a periodical income. Such incomes may be secured by the payment of a certain sum of money, and may be obtained of *Trust Companies*. It is a popular form of investment in the National Debt of England.

2. The advantage of such an investment is that a larger rate of interest is received, since the capital invested is not to be returned. An old person may receive a very high rate on such an investment.

ANNUITIES AT SIMPLE INTEREST.

CASE I.

882. ***To find the amount or final value of an annuity at simple interest.***

1. What will be the amount or final value of an annuity of $500 in 6 years at 4 per cent.?

OPERATION.

$$l = \$500 + \$20 \times 5 = \$600$$

$$S = \frac{500 + 600}{2} \times 6 = \$3300$$

SOLUTION.—If left unpaid until the end of 6 years, the last payment will be $500 without interest; the 5th payment will have become $500 plus the interest for one year, which is $520; the 4th payment will have become $500 plus the interest for 2 years, or $540; in the same way we find the third payment is $560, the second payment, $580, and the first pay-

ment, $600; the sum of these payments will be the final value, or amount, which we find is $3300. These sums form an arithmetical series, of which the first term is the annuity, or $500, the common difference is the interest for one year, the number of terms is the time in years, and the sum of the terms is the final value. Hence the following

Rule.—*Take the annuity for the first term, the interest for 1 year for the common difference, and the time for the number of terms, and find the last term and then the sum of the terms; this sum will be the final value.*

NOTE.—When the payments are made semi-annually, quarterly, etc., the number of such periods will be the number of terms, and interest for such time will be the common difference.

2. What is the final value of an annuity of $1500 for 9 yr. at 6%? *Ans.* $16,740.

3. What is the final value of an annuity of $600 for 12 yr., payable semi-annually, at 7%? *Ans.* $10,098.

4. Mrs. Wright has an annuity of $480, payable semi-annually, but owing to a financial crisis, it remains unpaid for 5 yr. 7 mo.; what should she receive at the end of the time, int. 8%? *Ans.* $3185.60.

5. Thomas Chase has an annuity of $900 for 7 yr. 9 mo., payable quarterly; if he puts each payment on interest at 7%, what will be the final amount? *Ans.* $8805.93¾.

6. Mr. Thompson rented a house for $600 a year, payable quarterly; if he does not pay his rent for 5 yr. 3 mo., what will be the amount due, at 6%? *Ans.* $3622.50.

CASE II.

883. ***To find the present value of an annuity at simple interest.***

1. What is the present value of an annuity of $300 for 5 yr. at 6%?

OPERATION.

Final value = $1680.
$1680 ÷ 1.30 = $1292.31—

SOLUTION.—Since the present worth of an annuity is the present worth of the final value, we first find the final value by Case I., and then find the present worth of this by Art. 649, Simple Interest. The final value we find is $1680, and the present value of $1680 is $1292.31—. Hence we have the following

Rule.—*Find the final value of the annuity by Case I., and then find the present worth of that sum.*

2. What is the present worth of an annuity of $2000 for 9 years at 6%? *Ans.* $14,493.51.

3. What is the present value of an annuity of $900 for 7 yr., payable semi-annually, at 5%? *Ans.* $5425.

4. An old man received a pension of $960 for 10 years, payable quarterly; for what could he have sold it when it began, interest 6%? *Ans.* $7755.

5. A house was rented for $75 a month for 3 years; what sum would pay the entire rent in advance, interest being reckoned at 6%? *Ans.* $2488.35.

NOTE.—Since the amounts form an arithmetical series, the time, rate, and yearly payment may be found by the different cases under Arithmetical Progression, and it is therefore unnecessary to treat them separately here.

ANNUITIES AT COMPOUND INTEREST.

884. Annuities are usually reckoned at compound interest instead of simple interest.

CASE I.

885. ***To find the final value of an annuity certain at compound interest.***

1. What is the final value of an annuity of $500 for 3 yr. at 6%?

OPERATION.

$500 ÷ 0.06 = $8333.33⅓

$0.191016, Comp. Int. of $1.
8333.33⅓
$1591.80

SOLUTION.—At 6%, $1 gives an annual income of $0.06, hence to give an income of $500 it will require as many times $1 as .06 is contained times in $500, which is $8333.33⅓; and if the annuity remains unpaid for 3 years at 6%, the amount due will be the compound interest of $8333.33⅓ for 3 years at 6%, which we find to be $1591.80. Hence the following

Rule.—*Divide the annuity by the rate, and find the compound interest of the quotient for the given time and rate.*

NOTE.—Use the table in Appendix for finding the compound interest. This case could be solved by Geometrical Progression, but much work is saved by using the table of Compound Interest.

2. What is the final value of an annuity of $300 for 19 yr. at 7%? *Ans.* $11,213.689.

3. What is the amount of an annuity of $700 for 7 yr. at 5%, payable semi-annually? *Ans.* $5781.63.

4. What is the amount of a quarterly salary of $150, in arrears for $6\frac{1}{2}$ yr., interest compounded at 2% quarterly?
Ans. $5050.635.

5. What is the amount of a salary of $800 a year in arrears for 20 years, the interest being compounded biennially, at 5%?
Ans. $26137.38.

6. Mr. Jones spends $50 a year in cigars; what would this amount to from his 21st to his 61st birthday, at 6% compound interest?
Ans. $7738.098.

7. A gentleman, on the day his son was 10 years old, deposited $25 in the savings bank in his name, and did the same every year; what was the amount when the son was of age, at 7% compound interest?
Ans. $447.21.

CASE II.

886. ***To find the present value of an annuity certain at compound interest.***

1. Find the present value of an annuity of $500 for 3 yr. at 6%.

OPERATION.

Final value = $1591.80
$1591.80 ÷ $1.191016 = $1336.51—.

SOLUTION.—The final value of this annuity, as found by Case I., is $1591.80; and the present worth of this sum is the present worth required. The compound amount of $1 for the given rate and time, as given in the table, is $1.191016; and the present worth of this amount is $1; therefore the present worth of $1591.80 is as many times $1 as $1.191016 is contained times in $1591.80, which is $1336.51. Hence the following

Rule.—*Find the final value as in the preceding case, and divide this sum by the amount of $1 at compound interest for the given rate and time.*

NOTE.—Much labor can often be saved by the use of a table, giving the present value of an annuity of $1, which will be found in the Appendix, though the examples can be solved by the table of compound interest.

2. What is the present value of an annuity of $700 for 10 years, int. 7%?
Ans. $4916.507.

3. What is the present worth of an annuity of $75 a quarter, in arrears for 25 years, int. 10%?
Ans. $2746.057.

4. A lady purchased an annuity of $700 a year for 10 years, at 6%; what sum was deposited?
Ans. $5152.06.

5. A capitalist sold an annuity of $560, payable quarterly and running 15 yr. 3 mo., for $4000; does he gain or lose, and how much, money being worth 10%, payable semi-annually? *Ans.* Loses $436.76.

6. A young man buys a farm for $5000, which he agrees to pay in 12 years by annual installments; the owner being pressed for money offers to take $4000 down; which is better for the purchaser, at 6%? *Ans.* 1st, $506.73.

CASE III.

887. *To find the present value of a perpetuity.*

1. What is the present value of a perpetual lease of $500 a year, at 5%?

SOLUTION.—The interest on $1 at the given rate, is $0.05, and to produce an interest of $500, it will require as many times $1 as $0.05 is contained times in $500, which is $10,000. Hence the

OPERATION.

.05)500.00

$10,000

Rule.—*Divide the given annuity by the interest of $1 for the given interval; the quotient will be the present value.*

2. What is the present worth of a perpetuity of $900 a year at 6%? *Ans.* $15,000.

3. What is the value of a ground-rent of $750 annually, at 6%? *Ans.* $12,500.

4. What is the present worth of a perpetuity of $8000 a year, payable semi-annually, at 8%? *Ans.* $100,000.

5. What is the present value of a perpetual leasehold of $1200 a year, payable semi-annually, interest at 6%, payable annually? *Ans.* $20,300.

REMARK.—Find the annual payment by allowing interest on the first semi-annual payment.

CASE IV.

888. *To find the present value of a deferred annuity.*

1. What is the present value of an annuity of $1000, to commence in 10 years, and run 20 years, at 6%?

SOLUTION.—From the table we find the value of the annuity at its commencement to be $11469.921; we then divide

OPERATION.

$11469.921 = Pres. val. for 20 yr.

$11469.921 ÷ 1.790847 = $6404.746.

this amount by \$1.7908477. the compound amount of \$1 for 10 years at 6%, which gives us the present worth of \$11469.921, which is \$6404.746, which is the present value of the deferred annuity. Hence we have the following

Rule.—*Find the value of the annuity at the time it commences, and then find the present worth of that value for the time it is deferred.*

NOTES.—1. This rule applies also to perpetuities.

2. This case may be solved by finding the present worth of the annuity for the time it is deferred, and also from the present time until it terminates; the difference of these two amounts will be the present value required. Many problems can be more easily solved in this manner, but the rule given above is more general.

2. Find the reversion of an annuity of \$750, to begin in 7 years and continue 10 years, int. at 7%. *Ans.* \$3280.45.

3. What is the value of a perpetuity of \$600, to commence in 5 years, at 8%? *Ans.* \$5104.37.

4. What must be paid for the reversion of a lease of \$600 a year, payable quarterly, commencing 6 years hence and continuing 18 years, int. 10%, payable annually?

Ans. \$2881.85.

5. A gentleman leaves his widow an estate of \$800 a year for 15 years, the reversion of it for 20 years to his daughter, and the final possession of it forever to his son. If they wish to sell the estate, what would be the value of each one's share, int. at 6%?

Ans. \$7769.799; \$3828.797; \$1734.736.

6. What is the present worth of the reversion of a perpetuity of \$750 a year, deferred for 100 years, interest at 6%? *Ans.* \$36.84.

CASE V.

889. ***Given, the rate, the time, and the present or final value of an annuity, to find the payment.***

1. The present value of an annuity for 9 years is \$7500, and interest 6%; what is the payment?

OPERATION.

$7500 ÷ 6.801692 = $1102.666.

SOLUTION.—The present value of an annuity of \$1 for 9 years at 6%, is \$6.801692; and the required annuity will be as many times \$1, as \$6.801692 is contained times in \$7500, which is \$1102.666. *Hence the following*

Rule.—*Divide the given present or final value by the present or final value of* **$1** *for the given rate and time, and the quotient will be the payment.*

2. If I pay $8741.03 for an annuity running 15 years, what is the annual payment, int. 6%? *Ans.* **$900.**

3. An annuity which is in arrears 11 years, amounts to $7906.61; what is the payment, interest 8%? *Ans.* **$475.**

4. A gentleman wishing to borrow money, offers as security an annuity deferred 5 years, running 8 years; it is valued at $1490.11; what is the payment, interest 7%? *Ans.* **$350.**

5. A pension, payable quarterly, but in arrears for 5 yr. 6 mo., amounts to $2719.72; what is the payment, interest reckoned at 8% payable annually? *Ans.* **$100.**

CASE VI.

890. ***Given, the present value, the rate, and the payment, to find the time.***

1. An annuity of $400 was bought for $2944.03 at 6%; how long did it run?

SOLUTION.—We first divide the present value, $2944.03, by the payment to find the present value of an annuity of $1, which is $7.3600+. Looking in the table under 6%, we find this amount opposite 10 years; therefore 10 years is the required time. Hence the following

OPERATION.

$2944.03 ÷ $400 = 7.3600+.
time = 10 yr.

Rule.—*Divide the present value by the payment, to find the present value of an annuity of* **$1** *for the given rate and required time; the time corresponding to this quantity in the table will be the time required.*

NOTES.—When the *amount* is given instead of the present value, divide the payment by the rate, which gives the sum of which the annuity is the interest; divide the amount by this quotient, and we have the compound interest of $1, and adding $1, we have the compound amount of $1, from which we can find the time by the table of compound interest.

2. This case is useful in finding the time required to extinguish a debt by a *sinking fund*, that is, by installments at regular intervals.

3. If the number obtained by division is not in the table, take the number of years corresponding to the next one below it, and find the balance remaining by taking the difference between the compound amount of the present value and the final value of the payment for the number of years found by the table.

2. How many years has an annuity of $375 to run, if it can be bought for $2633.85, int. 7%? *Ans.* 10 years.

3. In how many years can a debt of $10,000 be discharged by installments of $850 a year, interest being reckoned at 6%? *Ans.* 21 yr.; $1.82 unpaid.

4. In how many years can a debt of $3,000,000,000 be discharged by a sinking fund of $300,000,000, interest being 7%? *Ans.* 17 yr.; $224,380,457.15 unpaid.

5. If an annuity of $5000 has been left unpaid till it amounts to $2,032,642.82, at 7%, how long has it been running? *Ans.* 50 years.

6. If an annuity of $9500 has been left unpaid till it amounts to $2,574,730.85, how long since it commenced, interest 10%? *Ans.* 35 years.

7. On December 1, 1874, the national debt amounted to about $2,138,940,000; how many years would it require to discharge it by a sinking fund of $120,000,000, interest 5%? *Ans.* 45 years; $54,373,863.73 unpaid.

CASE VII.

891. ***Given, the present value, the payment, and the time, to find the rate.***

1. If an annuity of $450 for 15 years costs $3851.765, what is the rate?

SOLUTION.—If we divide $3851.765 by $450, we have the present value of an annuity of $1 for the given time and required rate, which is $8.559479; and looking in the table opposite 15 years, we find this number under 8%, which is therefore the required rate. Hence the following

OPERATION.
$3851.765 ÷ 450 = $8.559479.
rate = 8%.

Rule.—*Divide the present value by the payment, to find the value of an annuity of* $1 *for the given time and required rate; the rate corresponding to this quantity in the table will be the rate required.*

2. The present value of an annuity of $600 for 17 years is $5857.93; what is the rate of interest? *Ans.* 7%.

3. An annuity of $750 for 20 years can be bought for $7363.61; what is the rate of interest? *Ans.* 8%.

4. A lady pays $15,000 for an annuity of $1000 for 28 years; what is the rate of interest? Ans. About 5%.

CONTINGENT ANNUITIES.

892. Contingent Annuities include Life Annuities, Dowers, Pensions, etc. The value of these depends upon the *probability of life.*

893. The **Probability of Life** is the *probability* that a person at any particular age will live a given number of years.

894. A **Table of Life Annuities** shows the sum to be paid at different ages to secure a life-annuity of $1 during the remainder of the life of the annuitant. See Appendix.

CASE I.

895. ***To find the present value of a life-annuity, or dower.***

1. What must be paid for an annuity of $500, by a person aged 52, at 6%?

SOLUTION.—We find in the table that a person aged 52 must pay, at 6%, $10.208 for an annuity of $1; and for an annuity of $500 he must therefore pay 500 times $10.208, or $5104. Hence the following

OPERATION.

$10.208×500 = $5104.

Rule.—*Find in the table the present value of an annuity of $1 at the given age and rate, and multiply this by the given annuity.*

NOTE.—To find the present value of a life-estate or Widow's Dower (which is a life-estate in one-third of her husband's real estate), we calculate the yearly interest, at an agreed rate, of the value of the property, and find the present value of this interest in the same manner as that of a life-annuity.

2. Mr. Williams, who is 54 years old, wishes to buy an annuity of $850; what will it cost at 6%? *Ans.* $8296.85.

3. James Stevens, being pressed by his creditors, makes over to them a life annuity of $1000; what should it be considered worth, if he is 45 years old, and it is paid at the rate of 5%? *Ans.* $12,648.

4. William Turner has a life estate of $10,500; what is the property on which it is paid, and what is its present value, his age being 40 years and the rate of interest 7%?

Ans. Property, $150,000; present value, $113,872.50.

5. A gentleman dies, leaving real estate to the amount of

$90,000; what is his widow's dower, and what is its present value, her age being 65 years, interest at 5% ?

Ans. Dower, $1500; present value, $11,647.50.

CASE II.

896. ***To find how large an annuity can be purchased for a given sum.***

1. How large an annuity can be purchased for $2000 by a person aged 38, interest 6% ?

OPERATION.

$2000÷12.239=$163.41

SOLUTION.—We find from the table that an annuity of $1 for the given age and rate would cost $12.239; hence he could obtain for $2000 an annuity of as many dollars as $12.239 is contained times in $2000, which is $163.41. Hence the following

Rule.—*Find from the table the present value of an annuity of* $1 *for the given age and rate, and divide the given amount by it.*

2. How large an annuity can be purchased for $850 by a person aged 54, at 4% ? *Ans.* $73.11.

3. The present value of a widow's dower is estimated at $15,000; her age is 59 years, and the interest 6% ; what is her dower, and what the value of her husband's estate ?

Ans. Dower, $1758.71 a year; estate, $87,935.25.

4. John Gibbons sells a life-estate for $6500, he being 48 years old; what is the life-estate, and what the value of the property, interest 7% ?

Ans. Life-estate, $646.637 a year; estate, $9237.67.

CASE III.

897. ***To find the present value of the reversion of a given annuity.***

1. Find the present value of the reversion of an annuity of $400 a year for 50 years, after the death of a person aged 35 years, at 6%.

OPERATION.

$$\begin{array}{r} \$15.761861 \\ 12.573 \\ \hline 3.188861 \\ 400 \\ \hline \$1275.544 \end{array}$$

SOLUTION.—The present value of an annuity of $1 for 50 years at 6%, as found by the table, is $15.761861; and the present value of an annuity on the life of a person 35 years old at 6%, is $12.573; hence the difference of these two numbers, which is $3.188861, will be what remains of an annuity of $1 after the death of the possessor. Multiplying $3.188861 by 400, we have the reversion of an annuity of $400, which is $1275.54.

Rule.—*Find the present value of an annuity of $1 for the whole time, and then find its value during the given life; the difference of these two sums, multiplied by the given annuity, will be the present value of the reversion.*

NOTE.—The present value of the reversion of a life-estate or dower is most easily found by deducting the present value of the life-estate or dower from the value of the property.

2. Find the present value of the reversion of an annuity of $750 for 50 years at 5%, after the death of a person aged 62 years. *Ans.* $7326.69.

3. What is the present value of the reversion of a perpetuity of $1000, after the death of a person aged 56 years, interest 7%? *Ans.* $5690.71.

4. Henry Morris inherits an estate of $50,000 from his uncle, but it is burdened with the dower of the widow, 45 years old; what is the present value of his estate, including the reversion of the dower, int. 6%? *Ans.* $38,572.

5. Samuel Ellis is heir to an entailed estate of $75,000, after the death of his father, aged 75 years; what is the present value of his reversion, at 4%? *Ans.* $59,282.97.

CASE IV.

898. ***Given, the annuity, its present value, and the rate, to find the age of the annuitant.***

1. A, receiving a legacy of $6378.60, buys an annuity of $600 at 6%; what was his age?

OPERATION.

$6378.60 ÷ 600 = $10.631
time = 50 yr.

SOLUTION.— If $6378.60 is the present value of an annuity of $600, the present value of an annuity of $1 will be $\frac{1}{600}$ of $6378.60, which is $10.361; looking in the table under 6%, we find this number opposite 50 years; hence A's age was 50 years. Hence the following

Rule.—*Divide the present value of the annuity by the annuity, find this amount in the table under the given rate, and the corresponding time will be the required age.*

2. It costs me $6847.50 to buy an annuity of $500, at 5%; what is my age? *Ans.* 38 years.

3. A lady having quarreled with her relatives, in order to prevent them from inheriting her property, spent the whole

amount, which was \$47,600, in buying an annuity of \$10,000, at 6%; what was her age? *Ans.* 75 years.

CASE V.

899. *Given, the annuity, its present value, and the age of the annuitant, to find the rate.*

1. A gentleman 50 years old bought an annuity of \$750 for \$9651.765; what was the rate per cent.?

OPERATION.

$\$9651.765 \div 750 = \$12.86902$

Rate = 4 %

SOLUTION.—If \$9651.765 is the present value of an annuity of \$750, the present value of an annuity of \$1 is $\frac{1}{750}$ of \$9651.765, which is \$12.86902; looking in the table opposite 50 years, we find this amount under 4%; hence the rate was 4%. Hence the

Rule.—*Divide the present value of the annuity by the annuity, find this amount in the table opposite the given time, and the corresponding rate will be the rate required.*

2. A lady 45 years of age buys an annuity of \$400 for \$4158.80; what is the rate? *Ans.* 7%.

3. Mr. Wiggins wishes to purchase an annuity of \$850; it costs him \$7213.95, his age being 62 years; what is the rate per cent.? *Ans.* 5%.

NOTE.—For a further discussion of the theory of annuities, and general formulas for working them, see *Manual.*

INSURANCE.

900. Insurance is a contract of indemnity for loss or damage within a given time. It is of two kinds: *Property Insurance* and *Personal Insurance.*

901. Property Insurance is security against loss by fire or transportation. Insuring anything is called "taking a risk."

902. Property Insurance includes *Fire Insurance, Marine Insurance, Transit Insurance,* and *Stock Insurance.*

Transit Insurance is security against loss by transportation by land, or by both land and water; *Stock Insurance* is indemnity for loss of cattle, etc.

903. Personal Insurance includes *Life Insurance, Accident Insurance,* and *Health Insurance.*

Accident Insurance is indemnity for casualties to travellers and others; *Health Insurance* secures a weekly allowance in case of sickness.

904. The **Insurer,** or **Underwriter,** is the party or company taking the risk. The *Insured,* or *Assured,* is the party protected.

905. The **Policy** is the written agreement or contract between the insurers and the insured.

906. The **Premium** is the sum charged for insurance; it is a certain rate per cent. of the amount insured.

FIRE AND MARINE INSURANCE.

907. Fire Insurance is security against loss by fire; *Marine Insurance* is security against loss by navigation.

908. The **Sum Covered** by insurance is the amount insured on a property.

909. The **Base** is the amount insured on a property. The *Rate* varies with the risk.

The *Rate* of insurance is quoted as so many cents on the $100, or as so much per cent. Policies are renewed annually or at stated periods, and the premium is paid in advance. Risks are usually *rated* per annum. The rate for more than 1 yr. is determined by the following table:

The	rate	for	2 yr.	is	$1\frac{1}{2}$	times	the	annual	rate.
"	"	"	3 "	"	2	"	"	"	"
"	"	"	4 "	"	$2\frac{1}{2}$	"	"	"	"
"	"	"	5 "	"	3	"	"	"	"
"	"	"	7 "	"	$4\frac{1}{2}$	"	"	"	"

Insurance is generally done by *stock companies.* When an *individual* takes a risk, it is called an "out-door" business. A *Mutual Insurance Company* is one in which the profits and losses are shared by those who are insured.

To prevent fraud, companies will seldom insure the full value of property. In cases of loss, the *underwriters* may either replace the property insured, or pay its value. Only the amount of actual loss can be recovered; and often claims are *adjusted* for a part of the amount insured.

Before issuing a policy, the company has the property carefully surveyed and described with respect to the dangers to which it is exposed; and any deception in this respect, or subsequent change which increases the risk, vitiates the policy.

910. Short Rate Tables are tables prepared for reckoning the insurance when the time is less than one year.

The rate in short periods is quoted *per annum,* and the actual rate for a short period is given in the table. Such a table is given in the appendix, and is used in solving some of the problems in Cases I. and IV.

911. Perpetual Policies are sometimes issued, the rate being usually equal to that of *ten* annual premiums.

In *Perpetual Policies* the premium is considered merely a deposit with the Insurance Company; for at any time, at the instance of either party, the policy may be cancelled, and 90% of the premium or deposit must be returned to the policy holder.

912. The **Quantities** considered are: 1. The *Amount Insured;* 2. The *Rate of Insurance;* 3. The *Premium;* 4. The *Valuation of Property.*

CASE I.

913. ***Given, the amount insured and the rate, to find the premium.***

1. A man insured his store for $6850, at $1\frac{1}{4}\%$; required the premium.

SOLUTION.—The premium on $6850 at $1\frac{1}{4}\%$ is $.01\frac{1}{4}$ times $6850, which is $85.62½.

OPERATION.

$\$6850 \times .01\frac{1}{4} = \$85.62\frac{1}{2}$

Rule.—*Multiply the amount insured by the rate, to find the premium.*

2. Funk & Co. insured $\frac{3}{4}$ of a vessel worth $32,000, at $1\frac{3}{4}\%$, and $\frac{2}{3}$ of the cargo worth $28,500, at $1\frac{1}{4}\%$; required the premium. *Ans.* $657.50.

3. A man secures a policy of insurance on his house for $2500, furniture $1200, library $550, the policy costing $1.25; what is the whole cost, premium $\frac{3}{4}\%$? *Ans.* $33.12½.

4. My agent in Liverpool notifies me that he has shipped goods valued at £573 12 s. 6 d. I have insured them in New York at $2\frac{1}{4}\%$. What does the insurance cost, the policy being $1.25, and the pound sterling valued at $4.8665? *Ans.* $64.06.

5. An insurance company took a risk of $40,000, at $\frac{3}{4}\%$, reinsured $\frac{1}{2}$ of it with another office at $\frac{4}{5}\%$, and $\frac{1}{4}$ of it with another at $\frac{7}{8}\%$; what did the company clear by reinsuring? *Ans.* $52.50.

6. What is the premium on a $1500 policy, dated Nov. 19, 1877, and expiring March 25th, 1878, annual rate on the risk being $.65 on $100? *Ans.* $5.85.

7. R. Brown effects a perpetual insurance on his dwelling

to the amount of $9000; what is the deposit premium, the annual rate on his risk being $\frac{3}{10}\%$? *Ans.* $270.

8. Mr. Garland orders insurance as follows: $7500 on grain storage for 1 mo., $7500 on do. for 2 mo., $7500 on do. for 3 mo., and $7500 on do. for 4 mo., all in same warehouse, the annual rate being $.75 on the hundred dollars; also at the same time orders a policy for $3000 on his dwelling and $1500 on his barn, for 5 years, annual rate $\frac{1}{2}\%$; for what must he draw his check to the company? *Ans.* $145.50.

9. Mr. Warfel takes out an insurance of $12000 for 3 mo., on cotton stored in a warehouse, rated at 1% per annum; at the expiration of this time, not having sold, he has the policy renewed for 1 mo. longer; what would he have saved by taking out the insurance for 4 mo. at first? *Ans.* $12.

10. Mr. Michener takes out an insurance on his property for 6 months, annual rate $\frac{6}{10}\%$; next day he meets with a total loss by fire; how much does he save by not taking a perpetual policy as advised, the amount of insurance being $4000? *Ans.* $223.20.

NOTE.—The rates in the 7th and the following examples are found in the short rate tables in the Appendix.

CASE II.

914. *Given, the rate and the premium or the value of the property, to find the amount insured.*

1. A man paid $\$174.37\frac{1}{2}$ to insure the transportation of a lot of goods, at $1\frac{1}{4}\%$; required the value of the goods.

SOLUTION.—At a premium of $1\frac{1}{4}\%$, $.01\frac{1}{4}$ times the amount insured equals the premium, which is $\$174.37\frac{1}{2}$; hence the amount insured equals $\$174.37\frac{1}{2} \div .01\frac{1}{4}$, or $13,950.

OPERATION.

$$\frac{174.37\frac{1}{2}}{.01\frac{1}{4}} = \$13,950$$

Rule.—*Divide the premium by the rate, to find the amount insured.*

NOTE.—To find what amount must be insured to cover the premium in case of loss, we *divide the valuation of the property by* 1 *minus the rate.*

2. The premium for insuring $\frac{4}{5}$ of the value of a house, at $1\frac{1}{2}\%$, is $60; required the value of the house. *Ans.* $5000.

3. I insured my hotel at $1\frac{3}{4}\%$, paying $1050 premium and $7.75 for policy and survey; what was the amount insured? *Ans.* $60,000.

4. I insured to cover \$3870 on my barn, and the premium, at $3\frac{1}{4}\%$; what was the amount insured? *Ans.* \$4000.

5. A consignment of grain was insured at $3\frac{1}{4}\%$ to cover $\frac{2}{3}$ of the value, \$2850, and the premium; what was the amount insured? *Ans.* \$2945.73.

6. A cargo of French silks was insured at $5\frac{1}{4}\%$ to cover $\frac{4}{5}$ of its value; the premium was \$105; what was the value of the silk? *Ans.* \$2500.

7. Took a risk at $1\frac{3}{4}\%$; reinsured $\frac{2}{3}$ of it at $2\frac{1}{4}\%$; my share of the premium was \$43; what was the amount of the risk? *Ans.* \$17,200.

8. Took a risk at $2\frac{1}{4}\%$; reinsured $\frac{1}{2}$ of it in a mutual company at a rate equal to 3% of the whole, by which I lost \$37.50; what was the value of the risk? *Ans.* \$5000.

9. Took a risk at $1\frac{3}{4}\%$; reinsured \$8000 of it at 2% and \$6000 of it at $2\frac{1}{4}\%$; my share of the premium was \$55; what amount was insured? *Ans.* \$20,000.

10. I took a risk on a house worth \$40,000, at 2%; reinsured $\frac{1}{2}$ of it for $2\frac{1}{4}\%$ and $\frac{1}{4}$ at $2\frac{1}{2}\%$; in each case the amount covers premium; how much did I gain? *Ans.* \$99.558.

11. Hunter & Bro. pay \$22.50 for a 5-year policy on their stable, annual rate on which is 1%; what is the amount of the policy? *Ans.* \$750.

12. The premium on a perpetual policy of insurance was \$91; what was the amount insured, if the rate charged was $.21\%$ for 5 months? *Ans.* \$2600.

13. A merchant insured his store for $\frac{2}{3}$ of the value, at $1\frac{1}{2}\%$ annually for 7 years; the store was burned down, and his loss was \$4681.87$\frac{1}{2}$; what was the value of the store, and what was the loss of the insurance company?

Ans. Value of store, \$12,375; loss, \$7,693.12$\frac{1}{2}$.

CASE III.

915. ***Given, the premium and the amount insured, to find the rate.***

1. Paid \$478.12$\frac{1}{2}$ for insuring $\frac{3}{4}$ of the value of a church worth \$85,000; required the rate.

SOLUTION.—$\frac{3}{4}$ of \$85,000 is \$63,750; and since the premium equals the amount insured multiplied by the rate, the *rate* equals the premium, \$478.12½, divided by the amount insured, \$63,750, which we find to be .0075, or $\frac{3}{4}$%.

OPERATION.

$\frac{3}{4}$ of \$85,000 = \$63,750

$$\frac{478.125}{63,750} = .0075, \text{ or } \tfrac{3}{4}\%.$$

Rule.—*Divide the premium by the amount insured, to find the rate.*

2. If the premium is \$43.75, and the amount insured \$2500, what is the rate? *Ans.* $1\frac{3}{4}$%.

3. If I pay \$1125 for insuring $\frac{2}{3}$ of a vessel worth \$75,000, what is the rate? *Ans.* $2\frac{1}{4}$%.

4. I paid \$145.50, including the cost of the policy and survey, \$5.50, for insuring \$8000 on my house; required the rate. *Ans.* $1\frac{3}{4}$%.

5. I insured \$15,000 on my house, \$10,000 on my furniture, and \$1750 on my library, for 5 years, paying a premium of \$702.18$\frac{3}{4}$; what was the annual rate? *Ans.* $\frac{1}{8}$%.

6. A draws his check for \$135.60, in favor of an insurance agent for 2 policies as follows: \$135 to apply to the payment of a perpetual policy on his dwelling for \$4500, and the remainder to a 3 months' policy for \$500 on a piano stored therein; what is the annual rate? *Ans.* $\frac{3}{10}$%.

CASE IV.

916. ***To find the return premium on a cancelled policy.***

917. To **Cancel a Policy** is to annul the agreement between the party insured and the insurers.

When the policy is cancelled at the instance of the company, a *pro rata* proportion of the premium paid is returned; when done at the request of the policy holder, the company pays back a return premium governed by what are known as *Short Rate Tables*.

When a partial loss has been paid, the return premium is to the whole premium as the balance of the policy after deducting the partial losses paid is to the whole amount of the policy as first issued.

1. Mr. Carson effects an insurance on his stock of mdse. to the amount of \$8000 for 8 mo., at short rates, his risk being rated at 85¢ on \$100; in consequence of a reduction of stock at the end of 5 mo. he wishes his policy cancelled; to how much return premium is he entitled?

SOLUTION.—The rate for 8 mo. as found in the table is \$.0068, and for 5 mo., \$.0051, hence the return premium is the difference between \$.0068 and \$.0051 multiplied by 8000, or \$13.60.

OPERATION.

$.0068 - .0051 = .0017$
$\$.0017 \times 8000 = \$13.60$

Rule.—*Multiply the amount insured by the difference of the rates for the two periods, to find the return premium.*

2. Mr. Montgomery takes out a perpetual insurance on his house to the amount of \$7500, his risk being rated at $\frac{3}{8}\%$ annually; what is the deposit premium, and if he afterward surrenders his policy for cancellation, how much return premium should he get? *Ans.* \$281.25; \$253.12$\frac{1}{2}$.

3. Mr. Byerly has an annual policy of \$3000 upon his house; at the end of 8 mo. a fire occurs which damages his property to the amount of \$750, which the insurance company pays and indorses the payment on his policy; 3 months afterward he sells his house and surrenders the policy for cancellation in full; what is his return premium, the annual rate being $\frac{8}{10}\%$ on his risk? *Ans.* \$.22$\frac{1}{2}$.

4. A person holds a \$1200 policy on his stock in store; 81 days after date of the policy, he requests a reduction of $\frac{1}{3}$ of this amount on account of a decrease in stock; 50 days after this the insurance company desires to cancel the policy in full. What is the total return premium paid to the assured, the annual rate being $\frac{6}{10}\%$ and the term one year? *Ans.* \$3.36.

5. A perpetual policy of \$6000, annual rate $\frac{1}{4}\%$, is returned for cancellation after a partial loss of \$233.34 has been paid thereunder; how much unearned premium is due the assured? *Ans.* \$129.75.

CASE V.

918. ***To adjust the loss on a risk between several different insurance companies.***

919. When several companies are interested in a risk, a loss is shared by the companies in proportion to the amounts of the several policies.

Companies usually prefer to attach on different items in the same proportion.

1. Jones & Bro. hold a policy of insurance on their store for $7000 in the Phœnix Fire Ins. Co., and also one for $3000 in the Keystone Ins. Co.; a fire damages the property to the amount of $2025; what amount does each of the companies pay?

SOLUTION.—The whole amount insured is $10,000; the amount of the loss to be paid by the Phœnix is to $2025 as $7000 is to $10,000, which we find by proportion gives $1417.50; the amount to be paid by the Keystone is to $2025 as $3000 is to $10,000, which gives $607.50.

OPERATION.

$$x : 2025 :: 7000 : 10000$$
$$x = \frac{2025 \times 7000}{10000} = 1417.50$$
$$x : 2025 :: 3000 : 10000$$
$$x = \frac{2025 \times 3000}{10000} = 607.50.$$

Rule.—*Divide the loss between the several companies in proportion to the amounts of the several policies.*

2. Reese & Co. took out a policy in the Manhattan Insurance Co., covering $2500 on their store, and $1200 on their stock of goods; also a policy in the Ætna Insurance Co., covering $2000 on their store. By a fire, the building is damaged to the amount of $3100, and the stock to the amount of $600; what proportion of the loss does each company bear? *Ans.* Manhattan, 2322.22\frac{2}{9}$; Ætna, 1377.77\frac{7}{9}$.

3. Mr. Harris has a policy in the What Cheer Fire Insurance Co. for $5000, covering $3500 on his mill in Woonsocket and $1500 on the machinery therein; also a policy in the Roger Williams Fire Insurance Co. for $3000, covering $2000 on the mill and $1000 on the machinery. A fire breaking out in the upper part of the building, the property is damaged by fire to the amount of $540 as follows: $440 loss on the building and $100 on the machinery; the machinery was also damaged by water to the amount of $500; how much must be paid by each company?

Ans. What Cheer, $640; Roger Williams, $400.

4. The following was the insurance on a manufactory, to wit, "Orient," $500 on building and $1000 on contents; "Franklin," $750 on building and $1500 on contents; "Amazon," $1000 on building and $2000 on contents; "Lancaster," $1000 on building and $2000 on contents; "Phœnix," $250 on building and $500 on contents; the policies all bear

date April 1, 1876, for term of 1 year, except the "Lancaster," which was for 6 months. A fire occurred on the premises, Oct. 2, 1876, and damage was fixed by appraisers as follows: on building, $775, and on contents, $1900; what amount does each company pay?

Ans. Orient, $535; Franklin, $802.50; Amazon, $1070; Phœnix, $267.50.

LIFE INSURANCE.

920. Life Insurance is a contract by which a company, in consideration of payments made by the insured, stipulates to pay a certain sum of money to his heirs at his death, or to himself if he attains a certain age. Life Insurance companies are either *Stock* or *Mutual.*

921. In **Stock Companies,** the capital is subscribed as in any other corporation, and all profits are divided among the stockholders.

922. In the **Mutual Companies,** the capital is formed by the premiums, and all profits are divided among the insured, and go to decrease the premium or increase the policies.

There are also what are called *Mixed Companies,* which divide the larger part of their profits among the stockholders, and the remainder among the policy holders.

923. The **Policies** of Life Insurance are of the following kinds:

1. *Term Policies,* payable at the death of the insured if it occur within a certain number of years, premium payable annually.
2. *Life Policies,* payable at the death of the insured, premium payable annually during life, or in one, five, or ten annual payments.
3. *Joint Life Policies,* payable at the death of the first of two or more persons, to the survivor, premium payable as in life policies.
4. *Endowment Policies,* payable to the insured at the end of a certain number of years, or to his heirs if he dies sooner, premium payable either annually during the continuance of the policy, or in one, five, or ten annual payments.
5. *Reserve Endowment Policies,* a kind lately introduced, uniting the Life and Endowment plans, the premium being the same as in a life policy, but the insurance terminating at such time as the applicant elects, when an endowment is paid equal to the legal reserve of the policy.
6. *Tontine Savings Fund Policies,* payable like others at death, but in which the excess of premiums received over the claims by death and

expenses, is divided, at the end of a certain fixed period, among the survivors of those insured for that period, and at this time these survivors can either take the money as an endowment or change the form of their policy.

924. The **Premium** is made up of *three elements:* 1. *Reserve,* being that part of the premium, with the interest thereon, which is reserved to pay the policy when it becomes due; 2. *The Cost of Mortality,* being the estimated amount of each one's share of the losses by death each year; 3. The *Loading,* or premium for expenses.

925. The **Rates** of premium, as fixed by different companies, are based on the *probability of life* as determined by a table of mortality, the probable rates of interest, and the loading.

NOTES.—1. Life Insurance has attained such magnitude that several millions are now invested in it in the United States, the number of policy holders being between eight and nine hundred thousand. The combined amount of the policies in force is some $2,000,000,000, and the annual sums disbursed by the companies among the insured are from $50,000,000 to $60,000,000. The largest amount actually paid in the United States on a single life, was $300,000. The largest amount paid *in the world,* was on the life of an English nobleman for $1,250,000.

2. Policies were formerly forfeited on a failure to pay the annual premium, but most companies now arrange some way by which this can be avoided. Some companies allow the policy to run on a certain time, proportioned to the number of premiums that have been paid, and if the insured dies within this time, the amount insured will be paid, minus the premiums omitted.

3. In mutual or mixed companies one half of the premium is sometimes paid by a note and the dividends applied to cancel that note. In other cases, the dividends are added to the policy or used to reduce the premium. Stock companies pay no dividends to policy holders, but their rate of premium is usually from 20 to 30 per cent. less than mutual companies.

4. In case a person wishes to surrender his policy, the company usually returns a part of the premium, but this is often but a small part of what has been paid in. It has lately become a common practice, instead of returning a part of the premium, to give to the person surrendering his policy, a small "paid-up" policy, which guarantees a certain sum at his death without further payments. A much larger sum can be guaranteed in this way than can be paid in cash.

5. A table, given by the Mutual Insurance Company of New York, will be found in the appendix, which corresponds with those of several other companies, and may be considered as a fair representation of the premiums of mutual companies.

926. The **Quantities** considered in Life Insurance, using the tables in our calculations, are: 1. The *Premium* on $1000; 2. The *Gain* or *Loss;* 3. The *Amount* of the *Policy;* 4. The *Age;* 5. The *Period of Insurance.*

CASE I.

927. ***Given, the amount of the policy, the age, and the period of insurance, to find the premium.***

1. What annual premium must a man, aged 35 years, pay for a life policy of $4500?

SOLUTION.—The premium for life at the age of 35, found in the table, is $26.38 for $1000; hence for $4500 it will be 4.500 times $26.38, which is $118.71. Hence the following

OPERATION.

$\$26.38 \times 4.500 = \$118.71$

Rule.—*Find in the table the premium corresponding to the given age and time, and multiply the sum by the amount of the policy, considering all terms of the policy below thousands as decimals.*

2. Mr. Jones, aged 42 years, takes out an endowment policy in the Metropolitan Life Insurance Company for $12,000, payable to himself in ten years, or to his heirs at his death; what premium will he pay? *Ans.* $1291.80.

3. A gentleman insures his life for the benefit of his wife, but his income being somewhat uncertain, he prefers to make a single payment; what must he pay for a policy of $15,000, his age being 50 years? *Ans.* $8506.95.

4. Mr. Stewart wished to insure his life at the age of 35 for $12,000; but times being hard, and his affairs somewhat embarrassed, he did not take out the policy till he was 40 years old; how much more would the premiums amount to for an endowment policy for 20 years than if he had taken it when he first intended? *Ans.* $477.60.

5. James Woodford took out an endowment policy for $7500, payable in 20 years, his age being 30 years; if he lives to receive the endowment, would he have paid more or less if he had taken a policy of the same amount at 35 years of age for 15 years? *Ans.* $353.62½ more.

6. A gentleman, aged 38 years, takes a life policy in the Equitable Life Insurance Company for $8000, and after five payments his premium is reduced one-third by the dividends. If he lives to the age of 70 years, what amount of premiums will he have paid, and how much more than if he had made

20 annual payments, dividends being the same part of premium? *Ans.* 1105.46\frac{2}{3}$.

7. Mr. Detwiler, aged 50 years, takes out an endowment policy for $10,000, payable in 15 years, and dies after making 10 payments; how much would he have saved by taking a life policy for the same amount, premium payable annually? *Ans.* $2941.

CASE II.

928. ***Given, the amount of policy, the age, and the period of insurance, to find the gain or loss by insuring.***

1. A man 30 years of age takes a life policy for $2500, premium payable during life; he dies after making 15 payments; how much will the amount of the policy exceed the payments?

SOLUTION.—Having found the premium, by Case I., to be $56.75, 15 payments will amount to $851.25, and the gain will be the difference between $2500, the amount of the policy, and the amount of the payments, which is $1648.75.

OPERATION.

$22.70× 2.500= $56.75
$56.75× 15= $851.25
$2500−$851.25=$1648.75

Rule.—*Multiply the premium, as found by Case I., by the number of payments, and subtract this product from the amount of the policy.*

NOTE.—If interest is reckoned on the payments, we may obtain their amount as we obtain the final value of an annuity.

2. A gentleman aged 40 years, takes out an endowment policy in the National Life Insurance Company for $5000, payable in 10 years, the annual premium being $89.85 per $1000; reckoning interest at 6% on his payments, will he gain or lose if he lives to receive his endowment?

Ans. Lose $975.025.

3. At 30 years of age, James Headley took out an endowment policy for $18,000, payable in 15 years, his dividend reducing the premium after the third payment 25%; what will be the gain on his policy at maturity, not reckoning interest on premiums? *Ans.* $3577.68.

4. George Holden, at 25 years of age, insures in the Mutual Life Insurance Company of New York, for $30,000, payable to himself in 20 years. If the dividends are added to the policy, and amount to $12,000, how much more will he receive than he has paid, reckoning 7% interest on premiums? *Ans.* He would pay $7634.88 more than he received.

5. At the age of 38 years, Joseph Moore took out a life policy of $5000, premiums to cease in 10 years. He died aged 44 years, 6 months; what was the gain by insuring, reckoning interest on premiums at 6%? *Ans.* $2618.24.

6. Peter Lehr, 39 years of age, took out an endowment policy for $6000, payable in 10 years. In 15 months he died; what was the gain, reckoning interest on premium at 7%, and how much greater profit would it have been to take a life policy, premiums payable during life?

Ans. $4653.89 gain; $964.81 more profit.

7. A clergyman insured for $2000 in 1843, in the Mutual Life Insurance Company of New York, and died in 1869, just after the payment of a premium, his heirs receiving as dividends, $1971.26. Supposing him to have been 45 years old at the issue of the policy, would it have been more profitable to have invested the premiums at 6% compound interest? *Ans.* They would have gained $866.555.

CASE III.

929. ***Given, the age, the time, and the amount of the premiums, to find the amount of the policy.***

1. A gentleman, 32 years old, took out an endowment policy for 15 years, and at its maturity he had paid in premiums $1813.32; what was the amount of his policy?

OPERATION.

$1813.32 ÷ 15 = $120.888

$120.888 ÷ $67.16 = 1.8.

$1000 × 1.8 = $1800.

SOLUTION.—Since the premiums for 15 years amount to $1813.32, the premium for 1 year will be $\frac{1}{15}$ of $1813.32, which is $120.888. From the table we find that the annual premium on an endowment of $1000 at 32 years of age for 15 years is $67.16; and since $67.16 is contained in $120.888, 1.8 times, the amount required must be 1.8 times $1000, or $1800.

Rule.—*Find the annual premium and divide this by the*

premium in the table under the given age and rate, and multiply the quotient by $1000.

NOTE.—If interest is reckoned on premiums, divide the given amount by the amount of the premiums on $1000.

2. William Hoffman, of Lancaster, 25 years old, took out an endowment policy payable in 20 years, but died after making 12 payments, having paid as premiums $2288 64; what was the amount of his policy? *Ans.* $4000.

3. Joseph Duncan took out a life policy at the age of 51 years, and died just after making the 11th payment; his premiums, with interest at 6%, amounted to $10,596.30; what was the amount of his policy? *Ans.* $15,000.

CASE IV.

930. ***Given, the amount of the policy, the premium, and the period of insurance, to find the age of the insured.***

1. John Thompson insured his life for $3000, paying as premium $81.75; what was his age?

SOLUTION.—Since $3000 is 3 times $1000, we divide the premium $81.75 by 3, which gives $27.25, the premium on $1000. Looking in the table, we find $27.25 opposite 36 years, hence 36 years is the required age.

OPERATION.

$81.75÷3=$27.25.
$27.25, prem. for 36 yr.

Rule.—*Divide the annual premium by the ratio of the amount of the policy to* $1000, *and find the quotient in the table; the age opposite this number will be the age required.*

2. Mrs. Mason took out an endowment policy for $2500, payable in 10 years; she paid $2817 as premiums until the policy matured; what was her age at the time of insuring? *Ans.* 50 years.

3. Edward Spencer took out a life policy for $8000, on the single payment plan; his premium was $3735.12; at what age did he insure? *Ans.* 42 years.

4. Lewis Levan took out an endowment policy for $5000 for 15 years, and at the time of maturity, his premiums, if put on interest at 6%, would have amounted to $7713.39; what was his age? *Ans.* 40 years.

CASE V.

931. ***Given, the amount of the policy, the amount of the premiums, and the age, to find the period of insurance.***

1. A gentleman, 43 years of age, was insured for life for $7000; the whole amount of premiums paid was $1717.45; at what age did he die?

SOLUTION.—The premium on $1000, by the table, is $35.05, hence on $7000 it is $245.35; dividing $1717.45 by $245.35, we find the number of payments to have been 7; and since the first was made at the beginning of the period, there have been 6 yearly payments since he was 43 years old, hence his age was 43+6, or 49 years.

OPERATION.

$\$35.05 \times 7 = \$245.35$
$\$1717.45 \div \$245.35 =$
$7 - 1 = 6,\ 43 + 6 = 49$

Rule.—*Find the annual premium on the given sum, and divide the amount of the premiums by this annual premium; the result will be the period of insurance.*

2. An endowment policy for $9000, was taken out by a lady aged 49 years; at its maturity she received $1062.90 less than she had paid in premiums; what was the period of the policy? *Ans.* 10 years.

3. John Wise, at the age of 30, insured his life for $5000, premium to be paid during 20 years; but dying before the expiration of the period, he paid in premiums $607.20 less than if he had made all the payments; how many payments did he make? *Ans.* 16.

4. Nathan Ward had his life insured at the age of 37, premiums payable during life; how long must he live that the amount of premiums paid may exceed the policy?

Ans. Till his 73d year.

5. Peter Long takes out an endowment policy for $11,000, at the age of 42; at its maturity he has paid in premiums $11,635 80; what was the period of the policy?

Ans. 20 years.

NOTE.—For formulas for obtaining the premium, etc., see *Manual*.

BUILDING ASSOCIATIONS.

932. Building Associations are coöperative corporations instituted to receive small deposits at regular periods, and to invest these in loans among the depositors or members, on mortgages given by the borrower.

These associations enable many persons of moderate earnings and incomes to erect or buy buildings, and to invest their savings securely and profitably. The regular installments form the capital of the association, which is loaned to members only. The business is managed directly by the depositors, and the profits are equitably divided among them

933. The **Members** of an association are those who subscribe for shares. They are of two classes, *borrowers*, or those who borrow money of the association, and *non-borrowers*, who subscribe for shares as an investment.

934. The **Shares** are usually issued periodically in *series*, thus producing a constant succession of shares, each series successively reaching its value and being wound up, and a new series taking its place. Many associations have only one series.

When the installments and profits on any series have raised the value of its shares to par, it is wound up by returning to the non-borrowing members the value of their shares (though in some associations the paid-up shares are allowed to remain and draw cash dividends), and to the borrowing members their mortgages and cancelled obligations.

Thus, supposing $200 to be the value of a share and the payments $1 a month, if the capital is accumulated in one hundred months, the non-borrowing member will receive $200 on a share, and the borrowing member's debts will be cancelled, and his mortgage for $200 a share returned. The installments in each case have amounted to only $100, making a profit of $100, or 100% for the time. Many series are closed before their shares are fully equal to $200 in value.

935. The **Dues** are the fixed periodical installments, and are usually $1 a month. *Contingent Dues* for current expenses are assessed annually by some associations. In case of non-payment of dues, fines are levied. It is illegal in Pennsylvania to charge fines on unpaid fines.

At the regular monthly meetings of associations, the aggregate installments or dues, interest, fines, etc., paid in, are loaned to the highest bidder, or sometimes in the order of application, in which latter case there is a fixed or stated premium to be paid by the borrower.

936. The **Premium** is a percentage paid per share, in

excess of interest, on money which is "bought" or borrowed of the association. It is *quoted* for the *beginning of the series.*

937. The **Stated Premium** is the minimum rate fixed by associations, at which money will be sold on shares, each year of a series.

The *Stated Premium* is fixed at \$50, or 25% of a share, for the 1st year; \$45, or 22½% for the 2d year; \$40, or 20% for 3d year, etc.; decreasing 10% yearly to the 7th year, when it becomes \$20, or 10%. Money is seldom loaned after the 7th year, or at a lower "stated premium." The *entire premium* for any year of a series equals the *stated premium* for that year plus the *amount bid.*

Some associations have no *stated premium* to regulate the difference of premium between different series, but deduct, for each expired year of the series, 10% from the *premium bid.* This is avoided by the Installment plan, in which a number of cents a month is bid as premium, thus making no difference in what series the borrower holds shares.

938. There are **Three Modes** of loaning money and fixing the interest, adopted by different associations, called the *Installment Plan*, the *Net Plan*, and the *Gross Plan.*

By the first plan, the *par value* of a share is *loaned* on each share, and the premium is paid in monthly installments, together with the dues and interest. By the second plan, the premium is deducted from the par value, and interest is charged on the net amount of the loan. By the third plan, the premium is deducted from the par value, but interest is charged on the par value of the share.

Thus, by the Installment Plan, the net loan is \$200, the par value of the share and the full amount of the mortgage; the payments are \$1 a month dues, \$1 interest, and — cents premium. By the Net Plan, if the premium is \$50, the net loan is \$150, and payments \$1 a month dues and 75¢ a month interest. By the Gross Plan, the net loan is \$150, but payments are \$1 dues and \$1 interest. The monthly premium in cents by the first plan corresponds nearly to the total premium in dollars on a new series by the other plans, on the basis of 100 months.

In Pennsylvania, where these associations are most numerous, the number of shares at any one time is limited to 5000, and the periodic payments of borrowers to \$2. Thus, by the Installment and Gross Plans, the dues and interest at 6% on \$200, par value of a share, are each \$1 a month, which brings the payments up to the limit, \$2.

If loans are paid before the termination of a series, an equitable part of the premium paid is refunded, by the *Gross* and *Net Plans.* No premium is returned by the *Installment Plan*, since none is paid in advance.

The Installment and Net Plans are more favorable to the borrower than the Gross Plan. Of the three, the Installment Plan is the true one, and merits universal adoption.

939. A **Withdrawal** is made by returning the stock certificates to the association, and making settlement.

In case of withdrawal, a non-borrower receives the dues paid in, and an equitable part of the accrued profits. By the Installment Plan, a borrower pays the difference between the withdrawal value of the shares and the gross amount of the loan. By the Net or Gross Plans, a borrower pays the difference between the sum of the withdrawal value of the shares, increased by the premium for the unexpired years of the series, and the gross amount of the loan.

The *profits* of an association accrue from *interest* and *premiums*. The *True Profit* at any date of a series is the *legal interest* on the payments, plus that part of the profit on premiums which the present value of a share is of the par value, \$200. The *Withdrawal Profit* is the True Profit less a Withdrawal Discount fixed by the Association By-Laws.

NOTE.—Building Associations are not, as often supposed, builders of houses. They are corporations organized *to enable their members to build houses*, or buy them in their individual capacity, and might perhaps as appropriately be called Savings Fund and Loan Associations.

CASE I.

940. *To find the actual cost of any amount of stock.*

1. What would be the annual aggregate dues on 50 shares of stock at \$1 a month per share?

SOLUTION.—Since the dues on 1 share for 1 month are \$1, on 50 shares for 1 month, they will be \$50, and for 1 year 12 times \$50, or \$600.

OPERATION.

$\$1 \times 50 \times 12 = \$600$

Rule.—*Multiply the periodical dues by the number of periods, and to this product add the sum of the fines, if any have been levied.*

2. I buy 10 shares in 1st series, 8 in 2d, and 16 in 3d of Investment Building Association; if these series run out in 8, $8\frac{1}{3}$, and 9 years respectively, how much money in monthly dues will then have been paid in on the three series when closed out? *Ans.* \$3488.

3. Bought 45 shares of Franklin Association, 5 months after the date of issue, for \$252, but was unable to pay up dues for 3 months after purchase; how much did I pay for the year, including fines of 10% on unpaid dues? *Ans.* \$594.

4. At the end of the 4th year of a series, John Doe borrows of the Decatur Building Association, on 20 shares at \$40 premium, Gross Plan, but in two years he becomes insolvent and ceases to pay his dues; if the fines are 5% on unpaid dues, what would be due the association at the end of the seventh year? *Ans.* \$636.

CASE II.

941. ***To find the true premium charged on loans, and refunded on their payment.***

1. Mr. Lee bought a loan on 6 shares of stock 1 year and 1 month old, premium \$72; what is the true premium, if the last annual report gives the value of a share as \$15.50?

SOLUTION.—Since the dues for 1 month have been paid since the report, the accumulated value is \$15.50+\$1 =\$16.50, and the unaccumulated value = \$200 − \$16.50 = \$183.50. Since $\frac{183.50}{200}$ of the par value is unaccumulated, $\frac{183.50}{200}$ of the premium bid must be the premium due, or the true premium; $\frac{183.50}{200}$ of \$72=\$72 × 183.50÷200 = \$66.06.

OPERATION.

$\$15.50+\$1=\$16.50$
$\$200-\$16.50=\$183.50$
$\$72\times183.50\div200=\$66.06$

Rule.—*Multiply the premium bid by the unaccumulated value divided by* 200 (*or by one-half of the unaccumulated value regarded as a rate per cent.*), *and the result will be the True Premium.*

NOTES.—1. This method is more accurate and equitable than that of deducting 10 per cent. for each year, or the "stated premium" method. A money basis, and not a time basis, is the only one that will permit exact results.

2. In finding the Net Loan, *deduct the True Premium;* and in finding the Total Cost of a loan, *multiply the monthly payments by the time in months.*

3. Upon *Payment of Loans*, the *True Premium at the date of payment should be refunded.* The Pennsylvania law requires *one-eighth of the premium to be refunded for every year unexpired of eight.*

2. In June, 1876, I buy a loan on 5 shares of stock, issued April 1873, at a premium of \$65; what amount of cash do I receive, if by the last annual Report a share of this series is worth \$51.80? *Ans.* \$764.05.

3. If the first series of the Centennial Building Association, issued July, 1876, should be published in the sixth annual Report as worth \$140.50, what would be the net amount of a loan made in December, 1882, on 20 shares of this series at a premium of \$95? *Ans.* \$3491.75.

4. If this loan should be returned May, 1884, and the Report of '83 gave the shares at \$165.80, what would be the true premium due the borrower on a share, and the balance due on the loan? *Ans.* \$11.02; \$3779.60

5. The Providen Building Association sold money to Mr

Collins on 23 shares for 67¢ a month premium per share; what was the amount of his loan and what did his monthly payments aggregate for 9 years? *Ans.* $4600; $6632.28.

CASE III.

942. ***To find the amount and the actual cost of a loan to a borrower.***

1. I bought a loan on 12 shares in a new series of Benefit Building Association, Gross plan, at $11 and "stated premium;" if the series runs out in $8\frac{2}{3}$ years, what will be the actual cost of my loan?

OPERATION.

$\$2.00 \times 12 = \$24$, Monthly payment.

$\$24 \times \dfrac{105 \times 104}{4}$¢ $= \$655.20$, Interest 6%.

$\$24 \times 104 = \$2496$, Sum of payments.

$\$2496 + \$655.20 = \$3151.20$, Cost of Loan.

SOLUTION.—The monthly payment on 1 share equals $1 dues and $1 interest, or $2, and on 12 shares the payment is $24. The first installment is on interest 104 months, the second installment 103 months, and so on; hence the interest of a payment of $1 for the different periods equals the interest of $1 for a number of months represented by an arithmetical series whose first term is 1, last term, 104, and number of terms 104, or (Art. 848) $\frac{1}{2}$ of $(104+1) \times 104$. The interest of $1 for 1 month is $\frac{1}{2}$¢, and for the aggregate months, $\frac{1}{2}$ of $105 \times 104 \times \frac{1}{2}$¢ $=$ $105 \times 104 \times \frac{1}{4}$¢ $= \$27.30$; and on $24 it is $\$27.30 \times 24 = \$655.20$. The sum of the payments equals $\$24 \times 104$, or $2496; and the cost of the loan equals $\$2496 + \$655.20$, or $3151.20.

Rule.—I. *Multiply the number of months increased by* 1, *by the number of months, and divide by* 4, *to find the interest at* 6% *on the aggregate monthly payments of* $1.

II. *Multiply the interest on the aggregate payments of* $1, *by the monthly payment, to find the interest on the payments. Find the sum of the payments, and to this sum add the interest; the result will be the cost of the loan.*

NOTES.—1. It is here assumed that the payments draw *simple interest* from their payment to the close of the series. It would be more correct, perhaps, to reckon *annual interest*, or even *compound interest;* but the method given is more convenient.

2. The loan n the Installment plan, is the value of the shares on which it is made, the premium being a part of the monthly payment; in the other plans, the loan equals the value of the share minus the premium.

2. Mr. Smith and Mr. Jones each buy a house for $4000, and Mr. S. gives a 6% bond and mortgage due in 10 years

for the full amount; Mr. J. gives a bond and mortgage to the Investment Building Association for a loan on 20 shares at 75 cents a month premium; if the shares are cancelled in 10 years, who pays the most cash for his house, and how much? *Ans.* Mr. Jones; $200.

3. Mr. Henry and Mr. Williams buy houses for $2460, each paying $500 cash. Mr. H. gives a 6% bond and mortgage for the balance, int. payable annually, which he pays off at the end of 9 years; Mr. W. borrows of a building association a net loan of $1960 at $60 premium, Net plan, in a new series, which "runs out" in 9 years; which house cost the more, reckoning interest on payments? *Ans.* 1st, $1.58.

4. Mr. Brown rents a house for $16 a month for 10 years, and then buys it for $2000; Mr. White buys a house for the same price, and to pay for it obtains a loan from a Building Association, at 65 cents a month premium, on 10 shares of a series which runs out in 10 years; how much less does Mr. B. pay for his house than Mr. W., who pays annually 2% for taxes, and $1\frac{1}{2}$% for repairs, interest reckoned in both cases on monthly payments? *Ans.* $530.15.

CASE IV.

943. ***To find the rate of interest received by a non-borrower.***

1. What rate of interest do I receive on 8 shares of building association, dues $1 per share, if the series runs out in $8\frac{1}{2}$ years?

OPERATION.

$$\$200 - \$102 = \$98$$

$$\frac{103 \times 102}{24} = \text{equated time.}$$

$$\$98 \div \frac{103 \times 102}{24} = 22.39\%$$

SOLUTION.—The installments paid on 1 share for $8\frac{1}{2}$ years, or 102 months, are $102, and the difference between $200, the final value, and $102, the amount paid, equals $98, which is the gain, or interest on the investment. $1, the first payment, is on interest 102 months, the second payment is on interest 101 months, etc.; hence the interest on the payments for the different periods is equivalent to the interest on $1 for a number of months represented by the sum of an arithmetical series whose first term is 102, last term 1, and number of terms 102, or $\frac{1}{2}$ of $(102+1) \times 102$, months $= \frac{1}{24}$ of $(102+1) \times 102$, years; hence the interest on $1 for 1 year, or the *rate*, is $\$98 \div \frac{103 \times 102}{24} = \$.2239+$, or 22.39%

Rule.—I. *Subtract the sum of the installments paid on one share from the value of the share, and the difference will be the interest on the investment.*

II. *Multiply the number of payments by the number of payments increased by one and divide by 24, to find the equated time, or the number of years in which $1 will produce the same interest as the installments.*

III. *Divide the interest on the investment by the equated time; the quotient will be the equated rate per cent.*

2. What equated rate % of profit has been made by the fourth series of the Schuyler Building Association, if at the end of 23 months it is worth $33.26 a share? *Ans.* 44.6%.

3. What rate of interest will a building association pay that runs out in 8 yr.? 9 yr.? *Ans.* 26.8%; 18.76%.

4. If I buy 25 shares of Penn Building Association (new series) paying $1 dues each month and 75¢ a year for contingent expenses, and the series runs out in 10 years, what will be the equated rate of interest on the investment for the given time? *Ans.* 11.21%.

5. Mr. Black buys from a friend 24 shares of Decatur Building Association, during the 8th month of the 4th year of the series, paying $45.50 (estimated value by last Report) and the dues paid since the Report; what rate of interest does he receive on his investment, if the series runs out in $8\frac{2}{3}$ years? *Ans.* $20\frac{59}{100}$% +.

6. Buy 10 shares second series paying $17.58 (estimated value at last report) and dues for 9 months; if at the end of the year the series is valued at $38.30 per share, what is the amount of profit, and what the equated rate of interest? *Ans.* $87.20; $122\frac{1}{25}$%.

CASE V.

944. ***To find the rate of interest paid by a borrower.***

1. I bought a loan of the Penn Building Association on 10 shares, new issue, at $95 premium, Net plan; what rate % of interest will I have paid if the series expires in $8\frac{1}{3}$ yr.?

SOLUTION. — The loan was 10 × ($200—$95) = $1050; the interest on $1050 for 1 mo. is $5.25; and $10 dues + $5.25 int. = $15.25, the monthly payments, which in 100 mo. equal $1525. Now, the interest on the monthly payments (Case III.) is equivalent to the interest on $15.25 for $\frac{101 \times 100}{24}$ years at 6%, or $385.06¼; hence the actual cost of the loan is $1525 + $385.06¼, or $1910.06¼; therefore $1910.06¼—$1050, or $860.06¼, is the interest on the loan for 8⅓ years; and the interest for 1 yr. is $860.06¼ ÷ 8⅓ = $103.2075; hence the rate is $103.2075 ÷ $1050 = .0983—, or $9\frac{83}{100}$%.

OPERATION.

10 × ($200—$95) = $1050, Amt. of loan.
100 × ($10 + $5.25) = $1525, Payment.
$15.25 × $\frac{101 \times 100}{24}$ × .06 = $385.06¼, Interest, 6%.
$1525 + $385.06¼ = $1910.06¼, Entire payment.
$1910.06¼ — $1050 = $860.06¼, Int. on loan 8⅓ yr.
$860.06¼ ÷ 8⅓ = $103.2075, Int. for 1 yr.
$103.2075 ÷ $1050 = .0983—, or $9\frac{83}{100}$%.

Rule.—I. *Find the sum of the installments, and the interest on the installments for the equated time at 6%; their sum will be the entire cost of the loan.*

II. *Subtract the amount of the loan from its entire cost; the remainder will be the interest on the loan for the period, from which the rate is readily found by the method of simple interest.*

2. Mr. Roscoe buys a loan on 10 shares, new series, of Quaker City Building Association, at $75 premium, Gross plan; what equated rate of interest did he pay for his loan, if the series expired in 8¾ years? *Ans.* 12.86%—.

3. Mr. Collins bought of the Provident Building Association a loan on 20 shares, 3d series, at 70¢ a month premium, at the beginning of the 4th year; if it ran out in 8⅓ years, what equated rate of interest did he pay? *Ans.* 4.84%+.

4. I buy two loans of 15 shares each in 1st and 5th series, at the beginning of the 5th series, at $9 and "stated premium," Gross plan; what rates of interest shall I pay if both series run out in 9⅓ years? *Ans.* $6\frac{16}{25}$%+; $11\frac{11}{100}$%+.

5. I bought loans at $63 premium on 17 shares 1st series, worth $80, and 12 shares 3d series, worth $41.70, Net plan, at the beginning of the 5th year of the association; what equated rates of interest do I pay if the first series runs out in 8½ years and the 3d in 9 years? *Ans.* 2.17%; 6.066%.

CASE VI.

945. ***To find the true profit earned and allowed on withdrawals, less the withdrawal discount.***

1. I withdraw at the end of 36 months 28 shares of El Paso Building Association, worth \$54.50; what is the true profit earned and what the withdrawal value, the withdrawal discount being 10%?

OPERATION.

\$54.50 − \$36 = \$18.50, Profits on 1 share.

$\frac{37\times 36}{4}$¢ = \$3.33, Interest, 6%.

\$18.50 − \$3.33 = \$15.17, Rem. profits.

\$15.17 × 54.50 ÷ 200 = \$4.13.

\$4.13 + \$3.33 = \$7.46, True profit.

\$7.46 × 90% = \$6.714, Withdrawal profit.

\$36 + \$6.71 = \$42.71, Value 1 share.

\$42.71 × 28 = \$1195.88, Withdrawal value.

SOLUTION.—Since the profits accrue from interest and premium, we consider both. The value of 1 share, \$54.50, minus \$36, the amount paid on 1 share, equals \$18.50, the profit on 1 share. The interest on installments of \$1 a month for 36 mo. is equivalent to the interest on \$1 for $\frac{1}{2}\times(36+1)\times 36$, months, which equals $\frac{37\times 36}{4}$¢, = \$3.33, interest; \$18.50 profit, minus \$3.33 interest, equals \$15.17; $\frac{54.50}{200}$ of \$15.17 equals \$4.13; \$3.33 + \$4.13 = \$7.46, the true profit; \$7.46, less 10%, withdrawal discount, equals \$6.71, the withdrawal profit; \$36 + \$6.71 = \$42.71, withdrawal value of 1 share, which multiplied by 28 gives \$1195.88, the withdrawal value.

Rule.—I. *Multiply the number of months by the number of months increased by* 1 *and divide by* 4, *for* 6% *interest on the payments, and deduct this from the gross profits.*

II. *Multiply the remaining profits by the present value of a share divided by* 200, *for profits from premiums, and add this to the interest for the True Profit. Subtract the withdrawal discount from the true profit for the withdrawal profit; and add the payments, for the withdrawal value of a share.*

NOTES.—1. To find the amount due the association by a borrowing member on withdrawing, *add the true premium to the withdrawal value of a share, and subtract the result, less all arrearages, from the par value.*

2. Some associations, on a basis of 10 years, allow $\frac{1}{10}$ of the apparent or gross profit for each expired year; others 5 per cent. for every \$10 of apparent value, or a percentage of the profit, or a rate of interest on the dues, or a withdrawal value fixed yearly.

3. In the *Installment Plan*, these rules are not needed, since the *present value* and *true value* of profits and shares are alike, and we merely *subtract the withdrawal discount from the true profit, and add the payments.*

2. At the end of the fifth year of a series estimated at

\$110.70 full value, I withdraw 20 shares, withdrawal discount 10%; what sum do I realize? *Ans.* \$1778.66.

3. Mr. Wrigley subscribed for 10 shares Quaker City B A., fifth series, issued July, 1872. If he withdrew from the Association in May, 1879, what would be the withdrawal value of his stock, discount 10%, if the value given in the Report of 1878 should be \$135.80? *Ans.* \$1280.09.

4. Mr. Smith returns a loan made on 5 shares, new series, at \$58 premium, and withdraws his shares at the end of the third year, withdrawal discount 10%; if a share is worth \$56.20, what part of the loan will he return to the Association? *Ans.* \$575.18.

5. The Mutual Loan Association sells a loan at 28% premium on 20 shares new series; after 3 years and 4 months, the loan was paid and the shares withdrawn, withdrawal discount 10%; what balance of loan was due by the rule, if the shares were given at \$59 in the last Report? and what by Pennsylvania law? *Ans.* \$2251.84; \$2319.04.

CASE VII.

946. ***To find the present value of a share at the close of any period.***

1. A Building Association having a first series of 1490 shares, worth at the end of the 1st year \$13.448, issues at the beginning of the 2d year a second series of 1600 shares; the receipts of the year on both series are, besides dues, \$3950 in premiums, \$1225 in interest, \$40.25 in fines, and \$375 contingent fund for current expenses; what is the value of the shares of each series at the end of the second year?

SOLUTION.—The amount of the dues for the 2d year on both series is 3090 × \$12 = \$37,080; the gross profits equal the sum of the premiums, interest, and fines, which is \$5215.25; the interest on 1 share of 1st series for the 2d year is

OPERATION.

(1490+1600)×\$12 = \$37,080, Total Dues.
\$3950+\$1225+\$40.25 = \$5215.25, Gross Profits.
\$13.448×.06 = \$0.80688, Int. on 1 sh. 1st Series.
\$0.80688×1490 = \$1202.2512, Int. on 1st Series.
\$5215.25—\$1202.25 = \$4013, Net Profits.
\$37,080+\$1202.25 = \$38,282.25, Active Capital.
\$4013÷\$38,282.25 = 10.48%, Rate of Profit.
(\$12+\$0.806)×10.48% = \$1.342, Profit on 1 sh. 1st.
\$12×10.48% = \$1.2576, Profit on 1 sh. 2d Series.
\$13.448+\$0.806+\$12+\$1.342 = \$27.596, Ans.
\$12+\$1.257 = \$13.257, Ans.

$.80688, and on 1490 shares is $1202.25; and the net profits equal gross profits, $5215.25, minus interest, $1202.25, or $4013, which is to be divided among the different items of capital contributed during the year, called the *active capital*, consisting of the interest on the previous year's capital and the dues for the year; the interest and dues equal $38,282.25; dividing the net profits by this sum, we have 10.48%, the *rate* of profit; the profit on 1 share of 1st series equals the sum of interest and dues multiplied by the rate, or $\$12.806 \times 10.48\% = \$1.342$; the profit on 1 share of 2d series equals the dues multiplied by the rate, or $\$12 \times 10.48\% = \$1.257$; then $\$13.448 + \$0.806 + \$12 + \$1.342 = \$27.596$, is the value of a share of 1st series at the end of the 2d year, and $\$12 + \$1.257 = \$13.257$, is the value of a share of the 2d series at the end of the 2d year.

Rule.—I. *Find the legal interest for the term on the values of the old series at the beginning of the term, and deduct this from the profits of the term, for the net profit.*

II. *Divide the net profit by the sum of the dues for the term and the interest on the previous series, to find the rate per cent. of profit.*

III. *Multiply the sum of the interest and dues for the term on 1 share of each series by the rate per cent. of profit, to find the profit on 1 share.*

IV. *Add the previous value of each share, the legal interest on this value, the dues for the term, and the profit on the share, to find the present value of any share of each series.*

NOTE.—The *contingent fund* does not enter into the calculation, as it is usually assessed separately for current expenses.

2. At the beginning of the second year of a Building Association, it has a 1st series of 1350 shares, worth $14.32 a share, and issues a new series of 1500 shares; its receipts for the year on both series are, besides the dues, $3750 in premiums, $1675 in interest, $52.25 in fines, and $350 contingent fund; what is the value of a share of each series at the end of the 2d year? *Ans.* $28.748; $13.464.

3. At the beginning of the 3d year of the same association a new series of 1250 shares is issued; the receipts for the year are, besides dues, $5175 in premiums, $1650 in interest, $49.75 in fines, and $525.50 for contingent fund; what is the value of a share of each series at the end of the 3d year? *Ans.* $43.34; $27.081; $12.758.

SECTION XIII.

PROPERTIES OF NUMBERS.

947. The **Properties of Numbers** are the truths or principles which relate to them.

948. The **Classification of Numbers** is based upon their different peculiarities or properties.

949. All **Numbers** are either *Integral* or *Fractional.* This division is made with reference to their relation to the *Unit.*

950. All **Numbers** are either *Abstract* or *Concrete.* This division is made with reference to their application.

951. All **Numbers** are either *Prime* or *Composite.* This division is made with respect to their composition.

952. All **Numbers** are either *Even* or *Odd.* This division is made with respect to their being or not being a multiple of 2.

953. All **Numbers** are either *Perfect* or *Imperfect.* This division is based upon their relation to the sum of their divisors.

954. The **Properties of Numbers,** as given in this work, embrace the following subjects:

1. Composite Numbers.
2. Prime Numbers.
3. Even and Odd Numbers.
4. Perfect and Imperfect Numbers.
5. Properties of the Number 9.
6. Properties of the Number 11.
7. Properties of the Number 7.
8. Proof by Excess of 9's and 11's.
9. Scales of Notation.

GENERAL PRINCIPLES.

1. *A divisor of two numbers is a divisor of their sum and also of their difference* (Prin. 4, Art. 175).

2. *A divisor of one of two numbers, and not of the other, will divide neither their sum nor their difference.*

For, one number will be a *whole number of times the divisor*, and the other a *mixed number* of times the divisor, and consequently neither their *sum* nor their *difference* will be a whole number of times the divisor, since neither the sum nor the difference of an integer and mixed number can be an integer.

3. *A number which is not a divisor of either of two numbers may or may not divide their sum or their difference.*

Any two such numbers will equal a *number of times the assumed number plus certain remainders*. Now, if the *sum of these remainders* equals the *number*, the *sum of the numbers* is evidently divisible by the *number;* and if the *difference of these remainders is zero*, the difference of the numbers will be divisible by the *number*. In all other cases, neither the sum nor difference of the numbers is divisible by the *number*.

COMPOSITE NUMBERS.

955. A **Composite Number** is one which can be produced by multiplying together two or more numbers, each of which is greater than a unit.

956. The **Principles** of composite numbers are the truths which state their relation to their factors. These principles enable us to determine their factors or divisors.

PRINCIPLES.

1. *A number is divisible by 2 when the right hand term is zero or an even digit.*

If the right hand digit is zero, the number equals a number of *tens;* and, since 10 is divisible by 2, any number of *tens* is divisible by 2.

Any number may be separated into two parts—*a multiple of ten plus the right hand digit*—and when the right hand digit is divisible by 2, both of these parts are divisible by 2, hence their sum, which is the number itself, is divisible by 2 (Prin. 1, Art. 954).

2. *A number is divisible by 3 when the sum of its digits is divisible by 3.*

In Prin. 8, it will be shown that every number consists of a multiple of 9, plus the sum of its digits; hence since a multiple of 9 is divisible by 3, when the sum of the digits is divisible by 3, the number itself is divisible by 3.

3. *A number is divisible by 4 when the two right hand terms are ciphers, or when they express a number which is divisible by 4.*

If the two right hand terms are ciphers, the number equals a number

of *hundreds*, and since 100 is divisible by 4, any number of hundreds is divisible by 4.

Any number may be separated into two parts—*a number of hundreds, plus the number expressed by the two right hand digits* (thus $1232 = 1200 + 32$); and when the number expressed by the two right hand digits is divisible by 4, both of the parts are divisible by 4, hence their sum, which is the number itself, is divisible by 4 (Prin. 1, Art. 954).

4. *A number is divisible by* 5 *when its right hand term is* 0 *or* 5.

If the right hand term is 0, the number is a number of times 10, and since 10 is divisible by 5, the number itself is divisible by 5.

If the right hand term is 5, the entire number will consist of a number of *tens* plus 5, and since both of these are divisible by 5, their sum, which is the number itself, is divisible by 5.

5. *A number is divisible by* 6 *when it is even, and the sum of the digits is divisible by* 3.

Since the number is even it is divisible by 2, and since the sum of the digits is divisible by 3, the number is divisible by 3, and since it contains both 2 and 3, it will contain their product 3×2, or 6 (Prin. 3, Art. 165).

6. *A number is divisible by* 7 *when the sum of the odd numerical periods minus the sum of the even numerical periods is divisible by* 7.

Take any number, as 7936367225. This can be resolved, as shown below, into a multiple of 7, plus the difference between the sums of the odd numerical periods and the even numerical periods. For 1001 is a multiple of 7, 999999 is 999 times 1001, 1000000001 is also a multiple of 1001, and carrying out the number to higher periods, we shall continue to have multiples of 1001, alternately 1 more and 1 less than the number represented by the unit of the period. In the same way it may be shown that any number is equal to a *multiple of* 7 *plus the difference between the odd and even numerical periods;* hence when the difference between those periods is divisible by 7, the number is divisible by 7.

$$7936367225=\left\{\begin{array}{rr} 225 & +225 \\ 367000=367\times(1001-1)=367\times & 1001-367 \\ 936000000=936\times(999999+1)= & 936\times 999999+936 \\ 7000000000=7\times(1000000001-1)= & 7\times 1000000001-7 \end{array}\right.$$

$$\overline{7\times 1000000001+936\times 999999+367\times 1001-7+936-367+225}$$

7. *A number is divisible by* 8 *when the three right hand terms are ciphers, or when the number expressed by them is divisible by* 8.

If the three right hand terms are ciphers, the number equals a number of *thousands*, and since 1000 is divisible by 8, any number of thousands is divisible by 8.

A number may be resolved into a number of *thousands* plus the number expressed by the three right hand digits (thus $17368 = 17000 + 368$); and when both of these parts are divisible by 8, their sum, which is the number itself, is divisible by 8.

8. *A number is divisible by* 9 *when the sum of the digits is divisible by* 9.

Take any number, as 567. This can be resolved, as shown in the margin, into $(5\times 99+6\times 9)+(5+6+7)$, the first part of which is divisible by 9, and the other part is the sum of the digits. In the same way it may be shown that any number is equal to a *multiple of* 9 *plus the sum of the digits;* hence, when the sum of the digits is divisible by 9, the number is divisible by 9.

$$567=\begin{cases} 7= & 7 \\ 60=6\times 10=6\times(9+1)=6\times 9+6 \\ 500=5\times 100=5\times(99+1)=5\times 99+5 \end{cases}$$
$$567= \quad 5\times 99+6\times 9+5+6+7$$

9. *A number is divisible by* 10 *when the unit figure is* 0.

For, such a number equals a number of *tens*, and any number of tens is divisible by 10, hence the number is divisible by 10.

10. *Any number is divisible by* 11, *when the difference between the sums of the digits in the odd places and in the even places is divisible by* 11, *or when this difference is* 0.

Take any number, as 4928. This can be resolved, as shown in the margin, into a *multiple of* 11, *plus the difference between the sum of the digits in the odd places and the even places.* In the same way it may be shown that *any number* equals a *multiple of* 11, *plus the difference between the sums of the digits in the odd and even places;* hence, when the difference between these sums is divisible by 11, or is 0, the number is divisible by 11.

$$4928=\begin{cases} 8= & 8 \\ 20=2\times(11-1)= & 22-2 \\ 900=9\times(99+1)= & 891+9 \\ 4000=4\times(1001-1)= & 4004-4 \end{cases}$$
$$4928=22+891+4004+(8+9)-(2+4)$$

11. *A number is divisible by* 12 *when the sum of the digits is divisible by* 3 *and the number expressed by the two right hand digits is divisible by* 4.

For, since the sum of the digits is divisible by 3, the number is divisible by 3, and since the number expressed by the two right hand digits is divisible by 4, the number is divisible by 4; hence, since the number is divisible by both 3 and 4, it is divisible by their product, or 12.

NOTE.—In a similar manner we can find conditions of divisibility by 14, 15, 16, 18, etc. It will be an interesting exercise for the pupils to state such conditions. The subject, however, is more theoretical than practical.

EXAMPLES FOR PRACTICE.

Name some of the divisors of the following numbers:

1. 24324.	**4.** 23157.	**7.** 40884.
2. 76872.	**5.** 210070.	**8.** 47222.
3. 534258.	**6.** 536148.	**9.** 247968.

PRIME NUMBERS.

957. A **Prime Number** is a number that cannot be produced by multiplying two or more numbers together, each of which is greater than a unit.

958. The **Principles** of prime numbers are the truths which enable us to determine primes.

PRINCIPLES.

1. *A prime number has no integral divisor except itself and unity.*

For, if it had, it would be the product of two numbers, each greater than unity, and hence would not be a prime number.

2. *Every prime number except* 2 *is an odd number.*

For, if it is not odd it is even, but if even it is divisible by 2, and hence not prime; therefore any prime number except 2 must be odd.

3. *The right hand term of every prime number, except* 2 *and* 5, *must be* 1, 3, 7, or 9.

For, if the right hand term is even, the number is divisible by 2, and if it is 5 or 0, it is divisible by 5, in both of which cases it is not prime.

4. *If a number has no integral divisor not exceeding its square root, it is a prime number.*

For, if a number has no divisor less than its square root, it cannot have one greater than its square root, since if it had, the quotient would be a divisor, and it would thus have a divisor less than its square root.

5. *Every prime number greater than* 2 *is a multiple of* 4, *plus* 1, *or minus* 1.

For, if we divide a prime number by 4, the remainder may be either 1, 2, or 3; hence a prime number equals a *number of times* 4,+1, or +2, or +3. But a *number of times* 4, +2 is divisible by 2, and hence is not prime; therefore every prime number must be a *number of times* 4, +1, or a *number of times* 4,+3. But a *number of times* 4,+3 is also a *number of times* 4,—1.

6. *Every prime number greater than* 3 *is a multiple of* 6, *plus or minus* 1.

If we divide a prime number by 6, the remainder must be either 1, 2, 3, 4, or 5; but the remainder cannot be 2, 3, or 4, for then the prime number would equal a *number of times* 6,+2, or a *number of times* 6,+3, or a *number of times* 6,+4, the second of which is divisible by 3, and the others by 2; hence the remainder must be 1 or 5, and consequently every prime number equals a *number of times* 6,+1, or a *number of times* 6,—1.

7. *Every prime number greater than* 5 *is a multiple of* 8, *plus* 1 *or* 3, *or minus* 1 *or* 3.

The demonstration is similar to that of Prin. 6. Let the pupil prove it.

NOTE.—Every prime number is comprehended in one or another of the above propositions, although the converse proposition, that every number in one of those forms is prime, is not true.

EXAMPLES FOR PRACTICE.

Show that these principles are true with the following primes:

1. 71.	4. 257.
2. 97.	5. 839.
3. 163	6. 3209.

FINDING PRIME NUMBERS.

959. A **General Method** of determining prime numbers beyond a certain limit, has not yet been discovered, although much time has been spent in the investigation.

960. The method commonly used consists in writing a series of numbers and sifting out those which are composite, the remaining numbers being prime.

CASE I.

961. *To find all the prime numbers from* 1 *up to any limit.*

1. Find the prime numbers below 100.

METHOD.—Since all the prime numbers except 2 are odd (Prin. 2.), we write the series of odd numbers thus:

$$1,\ 3,\ 5,\ 7,\ \overset{3}{9},\ 11,\ 13,\ \overset{3.5}{15},\ 17,\ 19,\ \overset{3.7}{21},\ 23,\ \overset{5}{25},\ \overset{3}{27},\ 29,\ 31,\ \overset{3}{33},\ \overset{5.7}{35},\ 37,$$
$$\overset{3}{39},\ 41,\ 43,\ \overset{3.5}{45},\ 47,\ \overset{7}{49},\ \overset{3}{51},\ 53,\ \overset{5}{55},\ \overset{3}{57},\ 59,\ 61,\ \overset{3.7}{63},\ \overset{5}{65},\ 67,\ \overset{3}{69},\ 71,\ 73,$$
$$\overset{3.5}{75},\ \overset{7}{77},\ 79,\ \overset{3}{81},\ 83,\ \overset{5}{85},\ \overset{3}{87},\ 89,\ \overset{7}{91},\ \overset{3}{93},\ \overset{5}{95},\ 97,\ \overset{3}{99}.$$

Now, since this series increases by 2, the third term from 3 is $3+3\times 2$, which is divisible by 3, hence every third term after 3 is divisible by 3, and is therefore composite We will therefore place the figure 3 over every third term. We see, by a similar course of reasoning, that every fifth term after 5 is divisible by 5, and is therefore composite; and will therefore place the figure 5 over every 5th number. Proceeding in the same manner with 7, the numbers unmarked, together with the number 2, will be the prime numbers below 100. Hence all the prime numbers below 100 are 1, 2, 3, 5, 7, 11, 13, etc., to 97.

NOTE.—This method of finding prime numbers originated with Eratosthenes, a Greek mathematician. He inscribed the series of odd numbers upon parchment, and then cut out the composite numbers, leaving the primes. The parchment with its holes resembled a *sieve;* hence the method was called *Eratosthenes' sieve.*

EXAMPLES FOR PRACTICE.

2. Find all the primes from 1 up to 127.
3. Find all the primes from 1 up to 181.
4. Find all the primes below 300.
5. Find all the primes between 300 and 400.

CASE II.

962. *To ascertain if a given number is prime.*

Rule.—I. *Search for the number in the table, if contained within its limits; if it is found there it is prime, if not, it is composite.*

II. *Divide the number by the successive primes; if an exact divisor is found, the number is composite; if we continue the division until the quotient is less than the divisor without finding an exact divisor, the number is prime.*

NOTE.—The *Table of Prime Numbers* will be found on the following page.

EXAMPLES FOR PRACTICE.

Determine which of the following numbers are prime:

1. 273.	5. 3413.
2. 649.	6. 3853.
3. 2671.	7. 4001.
4. 3063.	8. 4049.

NOTES.—1. Several remarkable formulas have been discovered, which contain many prime numbers. Thus, the formula x^2+x+41, by making successively $x=0$, 1, 2, 3, 4, etc., will give the series 41, 43, 47, 53, 61, 71, etc., the first forty terms of which are prime numbers. This formula is mentioned by Euler.

2. The formula x^2+x+17 gives seventeen of its first terms prime; and the formula $2x^2+29$ gives twenty-nine of its first terms prime. Fermat asserted that the formula 2^m+1 is always a prime when m is taken any term in the series 1, 2, 4, 8, 16, etc.: but Euler found that $2^{32}+1$, which equals 641×6700417, is not a prime.

3. One of the most celebrated theorems for investigating primes is that discovered by Fermat, known as *Fermat's Theorem*. This formula may be stated thus: If p be a prime number, the $(p-1)^{th}$ power of every number prime to p will, when diminished by unity, be exactly divisible by p. Thus 25^6-1 is exactly divisible by 7. For a fuller discussion of the subject, see the author's *Philosophy of Arithmetic*.

TABLE OF PRIME NUMBERS.

963. A Table of Prime Numbers is a list of the prime numbers from 1 up to any given limit.

964. The following table contains the prime numbers from 1 up to 3407.

TABLE OF PRIMES.

1	173	409	659	941	1223	1511	1811	2129	2423	2741	3079
2	179	419	661	947	1229	1523	1823	2131	2437	2749	3083
3	181	421	673	953	1231	1531	1831	2137	2441	2753	3089
5	191	431	677	967	1237	1543	1847	2141	2447	2767	3109
7	193	433	683	971	1249	1549	1861	2143	2459	2777	3119
11	197	439	691	977	1259	1553	1867	2153	2467	2789	3121
13	199	443	701	983	1277	1559	1871	2161	2473	2791	3137
17	211	449	709	991	1279	1567	1873	2179	2477	2797	3163
19	223	457	719	997	1283	1571	1877	2203	2503	2801	3167
23	227	461	727	1009	1289	1579	1879	2207	2521	2803	3169
29	229	463	733	1013	1291	1583	1889	2213	2531	2819	3181
31	233	467	739	1019	1297	1597	1901	2221	2539	2833	3187
37	239	479	743	1021	1301	1601	1907	2237	2543	2837	3191
41	241	487	751	1031	1303	1607	1913	2239	2549	2843	3203
43	251	491	757	1033	1307	1609	1931	2243	2551	2851	3209
47	257	499	761	1039	1319	1613	1933	2251	2557	2857	3217
53	263	503	769	1049	1321	1619	1949	2267	2579	2861	3221
59	269	509	773	1051	1327	1621	1951	2269	2591	2879	3229
61	271	521	787	1061	1361	1627	1973	2271	2593	2887	3251
67	277	523	797	1063	1367	1637	1979	2283	2609	2897	3253
71	281	541	809	1069	1373	1657	1987	2287	2617	2903	3257
73	283	547	811	1087	1381	1663	1993	2293	2621	2909	3259
79	293	557	821	1091	1399	1667	1997	2297	2633	2917	3271
83	307	563	823	1093	1409	1669	1999	2309	2647	2927	3299
89	311	569	827	1097	1423	1693	2003	2311	2657	2939	3301
97	313	571	829	1103	1427	1697	2011	2333	2659	2953	3307
101	317	577	839	1109	1429	1699	2017	2339	2663	2957	3313
103	331	587	853	1117	1433	1709	2027	2341	2671	2963	3319
107	337	593	857	1123	1439	1721	2029	2347	2677	2969	3323
109	347	599	859	1129	1447	1723	2039	2351	2683	2971	3329
113	349	601	863	1151	1451	1733	2053	2357	2687	2999	3331
127	353	607	877	1153	1453	1741	2063	2371	2689	3001	3343
131	359	613	881	1163	1459	1747	2069	2377	2693	3011	3347
137	367	617	883	1171	1471	1753	2081	2381	2699	3019	3359
139	373	619	887	1181	1481	1759	2083	2383	2707	3023	3361
149	379	631	907	1187	1483	1777	2087	2389	2711	3037	3371
151	383	641	911	1193	1487	1783	2089	2393	2713	3041	3373
157	389	643	919	1201	1489	1787	2099	2399	2719	3049	3389
163	397	647	929	1213	1493	1789	2111	2411	2729	3061	3391
167	401	653	937	1217	1499	1801	2113	2417	2731	3067	3407

EVEN AND ODD NUMBERS.

965. An **Even Number** is one that is exactly divisible by 2; as, 2, 4, 6, etc.

966. An **Odd Number** is one that is not exactly divisible by 2; as, 1, 3, 5, 7, etc.

967. The **Even Numbers** are divided into the *oddly even* numbers, as 2, 6, 10, 14, etc., and the *evenly even* numbers, as 4, 8, 12, 16, etc.

968. The **Odd Numbers** are divided into the *evenly odd* numbers, as 1, 5, 9, 13, etc., and the *oddly odd* numbers, as 3, 7, 11, 15, etc.

NOTE.—The form of an even number is $2n$; the form of an odd number is $2n+1$, in which n represents any integer. In the *evenly even numbers*, n (in $2n$) is *even*; in the *oddly even numbers*, n is *odd*. In the *evenly odd*, n (in $2n+1$) is *even*; in the *oddly odd*, n is *odd*.

PRINCIPLES.

1. *Every even number equals a* NUMBER OF 2'S, *and every odd number equals a* NUMBER OF 2'S, *plus* 1.

For, since an even number is divisible by 2, it is evidently equal to a *number of 2's*; and since an odd number is not exactly divisible by 2, there will be a remainder of 1; hence an odd number equals a number of 2's, plus 1.

2. *The sum or difference of two even numbers is even.*

For, since both numbers equal a *number of 2's*, their sum is a *number of 2's* plus another *number of 2's*, which equals a *number of 2's*; hence the sum is an *even number*. Their difference equals a *number of 2's* minus another *number of 2's*, which equals a *number of 2's*; hence their difference is an *even number*.

3. *The sum or difference of two odd numbers is even.*

For, each number equals a *number of 2's*,+1, hence their sum equals a *number of 2's*,+2, or an *exact number of 2's*; hence their sum is *even*. Their difference equals an *exact number of 2's*; hence their difference is *even*.

4. *The sum or difference of an even number and an odd number is odd.*

For, the even number equals a *number of 2's*, and the odd number equals a *number of 2's*,+1; hence their *sum* and *difference* will equal a *number of 2's*,+1, and be an *odd number*.

5. *The product of two even numbers is an even number.*

For, since both of them contain the factor 2, their product will contain the factor 2, and therefore be *even*.

6. *The product of two odd numbers is an odd number.*

For, since neither of them contains the factor 2, their product will not contain the factor 2, and will therefore be *odd*.

7. *The product of an even and an odd number is an even number.*

For, since one of the numbers contains the factor 2, the product of the two numbers will contain the factor 2, and will therefore be *even*.

8. *If an even number is exactly divisible by an odd number, the quotient will be even.*

For, the divisor multiplied by the quotient equals the dividend, hence when the dividend is *even* and the divisor *odd*, the quotient must be *even*, since an *odd* number multiplied by an *even* number will give an *even number*.

9. *If an odd number is exactly divisible by an odd number, the quotient is odd.*

For, since an odd number must be multiplied by an odd number to produce an odd number, the quotient must be *odd* that the product of it and the divisor may equal the *odd* dividend.

10. *If an even number is exactly divisible by an even number, the quotient may be even or odd.*

For, an even number multiplied by either an even or an odd number will produce an even number, hence the quotient may be *even* or *odd*.

11. *An odd number is not exactly divisible by an even number, and the remainder is odd.*

Since an even number multiplied by no integral number will produce an odd number, an odd number is not exactly divisible by an even number. The remainder is odd, since it is the difference between an odd number and an even number.

12. *If an even number is not exactly divisible by another even number, the remainder is even.*

For, the remainder will be the difference between the dividend and a number of times the divisor, that is, the difference between two even numbers, which is even.

13. *If an even number is not exactly divisible by an odd number, then when the quotient is even the remainder is even, and when the quotient is odd the remainder is odd.*

14. *If an odd number is not exactly divisible by an odd number, then when the quotient is odd the remainder is even, and when the quotient is even the remainder is odd*

NOTE.—Let the pupil be required to demonstrate the last two principles.

PERFECT AND IMPERFECT NUMBERS.

969. A **Perfect Number** is one which is equal to the sum of all its divisors except itself; thus, $6=1+2+3$; $28=1+2+4+7+14$.

970. An **Imperfect Number** is one which is not equal to the sum of all its divisors; Imperfect Numbers are *Abundant* or *Defective.*

971. An **Abundant Number** is one the sum of whose divisors exceeds the number itself; as, $18<1+2+3+6+9$.

972. A **Defective Number** is one the sum of whose divisors is less than the number itself; as, $16>1+2+4+8$.

973. Two numbers are called *Amicable Numbers*, when each is equal to the sum of the divisors of the other; thus, 284 and 220.

NOTES.—1. Every number of the form $(2^{n-1})(2^n-1)$, the latter factor being a prime number, is a *perfect number*. The difficulty in finding perfect numbers consists in finding primes of the form of 2^n-1, which is very laborious. Substituting 2 for n in the formula just given, we have $2\times(2^2-1)=6$, the first perfect number; the second is $2^2\times(2^3-1)=28$.

2. The following are the first eight perfect numbers: 6, 28, 496, 8128 33550336, 8589869056, 137438691328, 2305843008139952128. It will be noticed that each number ends in 6 or 28.

3. The formulas for finding *amicable numbers* are $A=2^{n+1}d$ and $B=2^{n+1}bc$, in which n is an integer and b, c, and d are prime numbers satisfying the following conditions:

1st, $b=3\times2^n-1$; 2d, $c=6\times2^n-1$; 3d, $d=18\times2^{2n}-1$.

If we make $n=1$ we find $b=5$, $c=11$, $d=71$; substituting these in the above formulas, we have $A=4\times71=284$, and $B=4\times5\times11=220$, the first pair of amicable numbers. The next two pairs are 17296, 18416, and 9363584, 9437056.

4. Figurate Numbers are formed from an arithmetical progression whose first term is unity, and common difference integral, by taking successively the sum of the first two, the first three, the first four, etc., terms; and then forming in the same manner another series from the one just obtained and so on. For a discussion of Figurate Numbers, see *Philosophy of Arithmetic*

EXAMPLES FOR PRACTICE.

1. Find the third perfect number by the formula ($n=5$).
2. Find the fourth perfect number by the formula ($n=7$)
3. Show that 496 and 8128 are perfect numbers.
4. Find the second pair of amicable numbers ($n=3$).
5. Show that 220 and 284 are amicable numbers.
6. Show that 17296 and 18416 are amicable numbers.

PROPERTIES OF THE NUMBER 9.

974. The **Properties** of the **Number Nine** are the truths growing out of its relation to the decimal scale.

975. These properties are presented in the following principles:

PRINCIPLES.

1. *A number divided by* 9 *leaves the same remainder as the sum of the digits divided by* 9.

Take any number, as 6854, and analyze it as in the margin, and we see that it consists of two parts; the first part a *multiple of* 9 and the second part the *sum of the digits*. Now the first part is divisible by 9, hence the only remainder that can arise by dividing by 9, must arise from dividing the sum of the digits. Therefore, etc.

$$6854=\begin{cases} 4= & & & 4 \\ 50=5\times 10 & =5\times(9+1) & =5\times 9 & +5 \\ 800=8\times 100 & =8\times(99+1) & =8\times 99 & +8 \\ 6000=6\times 1000 & =6\times(999+1) & =6\times 999 & +6 \end{cases}$$

$$\therefore 6854=\underbrace{5\times 9+8\times 99+6\times 999}_{\text{Multiple of 9.}}+\underbrace{4+5+8+6}_{\text{Sum of the digits.}}$$

2. *A number is exactly divisible by* 9 *when the sum of its digits is divisible by* 9.

3. *The difference between any number and the sum of its digits is divisible by* 9.

4. *A number divided by* 9 *will leave the same remainder if the order of the figures is changed.*

5. *The difference between two numbers, the sums of whose digits are equal, is exactly divisible by* 9.

NOTE.—Pupils should be required to show how the last four principles are derived from the first.

EXAMPLES FOR PRACTICE.

1. Illustrate Principles 1, 2, 3, and 4, with 8703; with 31685.

2. Illustrate Prin. 5 with the numbers 3671 and 5264.

3. If I invert the digits of 74 and subtract the resulting number 47 from 74, the sum of the digits of the remainder will equal 9; explain it.

4. Prove that this is true of any other number of two integral digits in which the difference is a number expressed by two digits.

EXCESS OF 9's.

976. The **Excess of 9's** in a number is the remainder after dividing it by 9. It is found by the following rule:

Rule.—*Add the digits, dropping* 9 *from the sum when this equals or exceeds* 9, *and thus continue with the excess.*

Thus, to find the excess of 9's in 6789, begin at the left and say "6 and 7 are 13, excess 4 and 8 are 12, excess 3 and 9 are 12, excess 3."

Find the excess of 9's in

1. 5680638. *Ans.* 0.
2. 30675284. *Ans.* 8.
3. 75638216. *Ans.* 2.
4. 419672586. *Ans.* 3.

PROPERTIES OF THE NUMBER 11.

977. The **Number 11** has also some peculiar properties which are presented in the following

PRINCIPLES.

1. *Every number is a multiple of* 11, *plus the sum of the digits in the odd places, minus the sum of the digits in the even places.*

Take any number, as 65478, and analyze it as in the margin, and we see that it consists of two parts, the first a multiple of 11, and the second the sum of the digits in the odd places minus the sum of the digits in the even places.

$$65478=\begin{cases} 8= & & +8\\ 70= & 7\times(11-1)= & 77-7\\ 400= & 4\times(99+1)= & 396+4\\ 5000= & 5\times(1001-1)= & 5005-5\\ 60000= & 6\times(9999+1)= & 59994+6\end{cases}$$

$$\overbrace{77+396+5005+59994}^{\text{Multiple of 11.}}+\overbrace{8+4+6}^{\text{Digits in odd places.}}-\overbrace{7+5}^{\text{Digits in even places.}}$$

2. *A number is exactly divisible by* 11 *when the sum of the digits in the odd places is equal to the sum of the digits in the even places.*

3. *A number is exactly divisible by* 11 *when the difference between the sums of the digits in the odd places and the even places is a multiple of* 11.

4. *A number increased by the sum of the digits in the even places and diminished by the sum of the digits in the odd places, is exactly divisible by* 11.

5. *The excess of* 11's *in any number is not changed by adding any multiple of* 11 *to the sum of the digits of either order.*

EXCESS OF 11's.

978. The **Excess of 11's** in a number is the remainder after dividing it by 11. It may be found as follows:

Rule.—*Subtract the term on the left from the next term on the right, the remainder from the next, and thus continue with all the terms, adding* 11 *to each minuend when less than the subtrahend.*

Thus, take the number 24658 and subtract as directed, and the series of remainders will be 4—2, 6—4+2, 5—6+4—2, 8—5+6—4+2 = (8+6+2)—(5+4), which we see equals the sum of the digits in the odd places minus the sum of the digits in the even places. These remainders can be reduced as we proceed, remembering to add 11 to any minuend when it is less than the subtrahend, which will not affect the excess. (Prin. 5.)

1. Find the excess of 11's in 273849.

SOLUTION.—2 from 7 leaves 5, 3+11 are 14, 5 from 14 leaves 9, 8+11 are 19, 9 from 19 leaves 10, 4+11 are 15, 10 from 15 leaves 5, 5 from 9 leaves 4; hence the excess is 4.

Find the excess of 11's in

2. 37210856. *Ans.* 1.

3. 73285673. *Ans.* 10.

4. 25738564. *Ans.* 5

5. 472153869. *Ans.* 0.

PROPERTIES OF THE NUMBER 7.

979. The **Number Seven** has also some peculiar properties, which are presented in the following

PRINCIPLES.

1. *Every number is a multiple of* 7, *plus the sum of the numbers formed by taking its double numerical periods.*

Take any number, as 6945391657, and analyze it, as below, and we see that it consists of two parts, the first a multiple of 7, since 999999 is a multiple of 7, and the second the sums of the numbers expressed by the double periods; and this, it will be readily seen, is general.

$$6945391657 = \begin{cases} 391657 = & 391657 \\ 6945 \times (999999+1) = & 6945 \times 999999 + 6945 \end{cases}$$

$$\underbrace{6945 \times 999999}_{\text{Multiple of 7}} + \underbrace{6945 + 391657}_{\text{Sum of periods.}}$$

2. *A number divided by* 7 *gives the same remainder as the sum of its double numerical periods divided by* 7.

3. *A number is exactly divisible by* 7, *when the sum of the numbers expressed by its double numerical periods is divisible by* 7.

980. There is another interesting property of the *number seven,* which is derived in a similar manner.

PRINCIPLES.

1. *Every number is a multiple of 7, plus the sum of the odd numerical periods, minus the sum of the even numerical periods.*

Take any number, as 6945391657, and analyze it, as in the margin, and we see that

$$6945391657=\left\{\begin{array}{lr} 657= & +657 \\ 391\times(1001-1)= & 55913\times7-391 \\ 945\times(999999+1)= & 134999865\times7+945 \\ 6\times(1000000001-1)= & 857142858\times7-\ \ 6 \end{array}\right.$$

$$(857142858+134999865+55913)\times7+\overline{945+657}-\overline{391-6}$$

it consists of two parts; the first is a multiple of 7 and the second the difference between the sums of the odd and the even numerical periods.

2. *A number is exactly divisible by 7 when the sum of the odd numerical periods is equal to the sum of the even numerical periods.*

3. *A number is exactly divisible by 7 when the difference between the sums of the odd periods and the even periods is divisible by 7.*

4. *A number increased by the sum of the even numerical periods and diminished by the sum of the odd periods, is exactly divisible by 7.*

NOTE.—Other laws are given in the *Philosophy of Arithmetic.*

PROOF OF THE FUNDAMENTAL RULES

BY CASTING OUT NINES OR ELEVENS.

981. The **Fundamental Rules** may be proved by the excess of 9's and 11's.

PROOF OF ADDITION.

982. The **Proof of Addition** by casting out 9's is based upon the following principle:

Prin. *The excess of 9's in the sum of two or more numbers is equal to the excess of 9's in the sum of the excesses of those numbers.*

Each number equals a *multiple of* 9, + *the excess;* hence their sum will equal a *multiple of* 9, + *the sum of the excesses;* consequently the excess of 9's in the *sum of the excesses,* will equal the excess in the sum of the numbers.

NOTE.—To prove by excess of 11's, proceed as in proving by excess of 9's. Pupils may be required to test each problem by 11 also.

1. Find the sum of 275, 463, and 907, and prove the work.

SOLUTION.—The excess of 9's in 275 is 5, in 463 is 4, in 907 is 7, and the excess in the sum of these excesses is 7. The excess in the sum is also 7; hence the work is correct.

OPERATION.

275	excess	5
463	"	4
907	"	7
1645	excess	7

Rule.—I. *Find the excess of 9's in each number, then the excess in the sum of these excesses, and then the excess in the sum of the numbers.*

II. *If the work is correct, the last two excesses will be equal.*

NOTES.—1. We need not write the excess of each number, but can pass from one number to another and write the last excess. We can also add in columns for excess, as well as in rows.

2. This method fails when the digits are misplaced, or when one digit is as much too great as another is too small.

Add and prove the following:

2. 6573+8325+5641+4319+3978+6807.

3. 5432+6431+27944+56352+78698.

4. 46932+79876+85432+65435+57697.

5. 443367+637389+457934+697989+609687.

PROOF OF SUBTRACTION.

983. The **Proof of Subtraction** by casting out **9's** is based upon the following principle:

Prin. *The excess of 9's in the minuend equals the excess of 9's in the sum of the excesses of the subtrahend and remainder*

This is evident from the principle in the previous case, since the minuend equals the sum of the subtrahend and remainder.

1. Subtract 2562 from 4625, and prove the work.

SOLUTION.—The excess of 9's in the minuend is 8, in the subtrahend 6, in the remainder 2, and the excess in the sum of the excesses of the subtrahend and remainder is 2+6, or 8, the same as the excess of the minuend; hence the work is correct.

OPERATION.

4625	excess	8
2562	"	6
2063	excess	2

Rule.—I. *Find the excess of 9's in each of the three terms, and the excess in the sum of the excesses of the subtrahend and remainder.*

II. *If the work is correct, the last excess will equal the excess in the minuend.*

Subtract and prove the following:

2. 4736—2431.	5. 233461—87563.
3. 57973—44567.	6. 446561—345612.
4. 98793—47867.	7. 876543—625781.

PROOF OF MULTIPLICATION.

984. The **Proof of Multiplication** by casting out 9's is based upon the following principle:

Prin. *The excess of 9's in the product of two numbers equals the excess of 9's in the product of the excesses of those numbers.*

Each number is a multiple of 9, plus its excess, hence the product will be a multiple of 9, plus the product of the excesses, and the excess in this product of excesses will therefore evidently be the excess in the product of the two numbers.

1. Multiply 346 by 68.

OPERATION.

```
 346 excess 4
  68   "    5
----       --
2768       20 excess 2.
2076
-----
23528 excess 2.
```

SOLUTION.—The excess in the multiplicand is 4, in the multiplier 5, and in the product of these excesses 2. The excess in the product is also 2; hence the work is correct.

Rule.—I. *Find the excess of 9's in the multiplier and multiplicand, the excess in the product of these excesses, and also the excess in the product of the numbers.*

II. *If the work is correct, the last two excesses will be equal.*

Multiply and prove the following:

2. 6563×736.	5. 68735×5642.
3. 4918×875.	6. 79636×4876.
4. 15978×6353.	7. 387981×3578.

PROOF OF DIVISION.

985. The **Proof of Division** by casting out 9's is based upon the following principle:

Prin. *The excess of 9's in the dividend equals the product of the excesses in the divisor and quotient, plus the excess in the remainder.*

For $D = d \times q + r$. Now the excess of 9's in the product $d \times q$ equals the excess in the product of the excesses of these terms (Prin. Art. 984); and the excess in this product plus the excess in r must equal the excess in the dividend. (Prin. Art. 982.)

1. Divide 2443 by 56 and prove the result.

SOLUTION.—The excess in the divisor is 2; in the quotient 7; in $q \times d$ it is the excess in 2×7 or 14, which is 5; in r, 8; in $q \times d + r$ it is the excess in $5 + 8$, or 13, which is 4; and in the dividend it is 4; hence the work is correct.

OPERATION.

56)2443(43	excess in d, 2
224	excess in q, 7
203	excess in $q \times d$, 5
168	excess in r, 8
35	excess in $q \times d + r$, 4
	excess in D, 4

Rule.—I. *Find the excess of 9's in the divisor and quotient, the excess in the product of these excesses, the excess in the remainder, then the excess in the sum of the last two excesses, and then the excess in the dividend.*

II. *If the work is correct, the last two excesses will be equal.*

Divide and prove the following:

2. $6734 \div 371$.
3. $59453 \div 276$.
4. $679432 \div 4833$.
5. $793742 \div 4242$.
6. $8746391 \div 3792$.
7. $93949598 \div 249801$.

SCALES OF NOTATION.

986. The **Scale** of a system of notation is the law of relation between its successive orders of units.

987. The **Radix** of the scale is the number which expresses the relation of the successive orders.

Any number might have been taken as the basis of the scale of Notation. The use of *ten*, the basis of the decimal scale, originated from the counting of the fingers of the two hands, which was the primitive method of calculation.

988. A scale whose radix is *two* is called *Binary;* three, *Ternary;* four, *Quaternary;* five, *Quinary;* six, *Senary;* seven, *Septenary;* eight, *Octary;* nine, *Nonary;* ten, *Denary;* twelve, *Duodenary* or *Duodecimal*, etc.

989. In expressing a number in any one of these scales, there must be as many significant characters as there are units in the basis of the scale, less 1. Thus in the decimal

scale there are 9, in the octary 7, in the quinary 4, etc. In each the zero, 0, is used to fill vacant places.

990. In expressing numbers in scales higher than the decimal, it is necessary to introduce some new characters; thus Φ may stand for *ten*, and Π for *eleven*.

In order to use any scale of notation with facility, the *names of numbers* should also be based on the same scale. Thus, in the quinary scale we should count *one, two, three, four, five, one and five, two and five*, etc., to *two fives*, and then *two fives and one, two fives and two*, etc.

Not having these names, we may read by powers of the radix. Thus, 4234 in the quinary scale may be read, *four 5's cubed, two 5's squared, three 5's and 4 units.* The scale in which a number is expressed may be indicated by writing the radix as a subscript.

CASE I.

991. ***To pass from any scale to the decimal scale.***

1. 2432_5 is a number in the quinary scale; express the same number in the decimal scale.

SOLUTION.—The given number consists of 2 *ones*, 3 *fives*, 4 *fives squared*, and 2 *fives cubed*. *Two fives cubed* equal two hundred and fifty; 4 *fives squared* equal one hundred; 3 *fives* equal fifteen; 2 *ones* equal 2 ones; the sum of all is *three hundred* and *sixty-seven*, which expressed in the decimal scale is 367.

OPERATION.

$$\begin{array}{r} 5^3\times 2 = 250 \\ 5^2\times 4 = 100 \\ 5\ \times 3 = \ \ 15 \\ 1\ \times 2 = \ \ \ \ 2 \\ \hline 367 \end{array}$$

Change each of the following to the decimal scale:

2. 3204_5; 6035_8; 21032_4; 2534_8. *Ans.* 429; 3101, etc.

3. 101101_2; 785036_9; $37\Phi208_{11}$; $20\Pi6\Phi38_{12}$.
Ans. 45; 469509; 599200; 6211916.

CASE II.

992. ***To pass from the decimal scale to any other scale.***

1. 45789 is a number in the decimal scale; express the same number in the quinary scale.

OPERATION.

$$\begin{array}{r} 5)\underline{45789} \\ 5)\underline{\ 9157+4} \\ 5)\underline{\ 1831+2} \\ 5)\underline{\ \ 366+1} \\ 5)\underline{\ \ \ 73+1} \\ 5)\underline{\ \ \ 14+3} \end{array}$$

SOLUTION.—To express any number in the quinary scale, we ascertain how many *fives*, how many *fives squared*, how many *fives cubed*, etc., the number contains. Dividing by *five*, we ascertain the number of *fives* and *units;* dividing the number of 5's by 5, we ascertain the number of 5's *squared;* dividing these by 5, we ascertain the number of 5's *cubed*, etc. In this manner we find 45789 equals 4 *ones*, 2 *fives*, 1 *five squared*, 1 *five cubed*, etc., which written in the quinary scale gives 2431124_5.

2. Express 3478 and 79437 in the octary scale.

Ans. 6626_8 ; 233115_8.

3. Express 54321 and 33787 in the senary scale.

Ans. 655253_6 ; 420231_6.

4. Express 67893 and 59466 in the duodecimal scale.

Ans. 33359_{12} ; $2\Phi 4\Pi 6_{12}$

CASE III.

993. ***To pass from one scale to another, neither being the decimal scale.***

1. 3464_8 is a number in the octary scale; express the same number in the quinary scale.

OPERATION.

$$\begin{array}{r} 5)\underline{3464} \\ 5)\underline{\ 560+4} \\ 5)\underline{\ 111+3} \\ 5)\underline{\ \ 16+3} \\ 2+4 \end{array}$$

24334_5, *Ans.*

SOLUTION.—Remembering that the given number is in the scale of eight, and dividing successively by 5, we find the number contains 4 *ones*, 3 *fives*, 3 *fives squared*, 4 *fives cubed*, and 2 *fives fourth power*, which, expressed in the quinary scale, gives the number 24334_5.

NOTE.—In making the division, it must be remembered that the number divided is in the octary scale, and hence any remainder, instead of being so many tens, is so many *eights*.

2. Reduce 2433_7 and 10111_2 to the quaternary scale.

Ans. 32022_4 ; 113_4.

3. Reduce 157742_9 and 34581_{11} to the ternary scale.

Ans. 4221211102_3; 2112111200_3.

4. Reduce 30321_4 and 45324_6 to the nonary scale.

Ans. 1116_9 ; 8677_9.

CASE IV.

994. ***To perform arithmetical operations on numbers in any scale.***

1. Add 2367_8, 5062_8, 75064_8. *Ans.* 104535_8.
2. Subtract $75\Phi 8\Pi_{12}$ from $\Phi 28\Pi 6_{12}$. *Ans.* $28\Phi 27_{12}$
3. Multiply $54\Phi 8_{12}$ by $3\Pi 7_{12}$. *Ans.* 1953768_{12}.
4. Divide 1953768_{12} by $3\Pi 7_{12}$. *Ans.* $54\Phi 8_{12}$.
5. Extract the square root of $\Pi 5301_{12}$. *Ans.* 347_{12}.
6. Add 2312_4, 4324_5, 54341_6, 37346_8, $2\Phi 49\Pi_{12}$.

Ans. 243525_8.

NOTE.—For a fuller discussion of this subject, see *Philosophy of Arithmetic.*

SECTION XIV.

MENSURATION.

995. Mensuration treats of the measurement of geometrical magnitudes.

996. Geometrical Magnitudes consist of the *Line, Surface, Volume,* and *Angle.*

997. A **Line** is that which has length without breadth or thickness. Lines are either *straight* or *curved.*

998. A **Straight Line** is one that has the same direction at every point.

999. A **Curved Line** is one that changes its direction at every point. The word *line* used alone means a *straight line.*

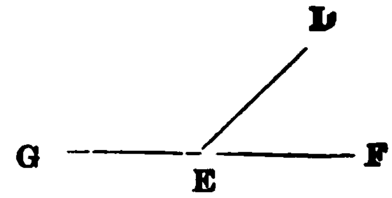
1000. Parallel Lines are those which have the same direction. Parallel lines, it is thus seen, will never meet.

1001. One line is said to be *perpendicular* to another when the adjacent angles formed by the two lines are equal.

1002. An **Angle** is the opening between two lines which diverge from a common point.

1003. A **Right Angle** is an angle formed by one line perpendicular to another; as, ABC.

1004. An **Acute Angle** is an angle less than a right angle; as, DEF. An **Obtuse Angle** is one larger than a right angle; as, DEG.



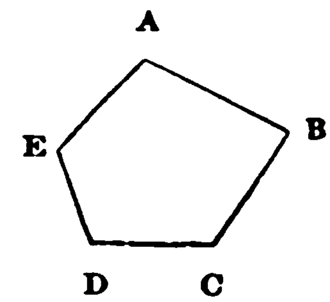
MENSURATION OF SURFACES.

1005. A **Surface** is that which has length and breadth without thickness. Surfaces are *plane* or *curved.*

1006. A **Plane Surface** is a surface such that if any two of its points be joined by a straight line, every part of that line will lie in the surface.

1007. A **Plane Figure** is a plane surface bounded by lines either straight or curved.

1008. A **Polygon** is a figure bounded by straight lines; as, ABCDE. A Polygon of three sides is called a *Triangle*, of four sides, a *Quadrilateral*, etc.



1009. A **Diagonal** of a polygon is a line joining the vertices of two angles not consecutive.

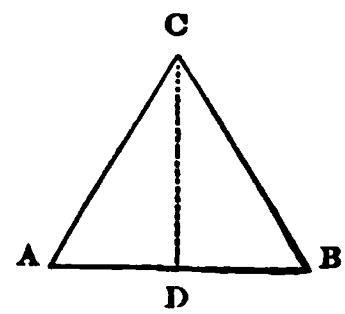
1010. The **Perimeter** of a polygon is the sum of its sides.

1011. The **Area** of a plane figure is the number of square units in its surface.

NOTE.—The principles of mensuration are derived from geometry; it is customary to give their application to practical purposes in arithmetic.

THE TRIANGLE.

1012. A **Triangle** is a polygon of three sides and three angles; as, ABC.



1013. The **Base** is the side upon which it seems to stand; as, AB. The *Altitude* is a line perpendicular to the base, drawn from the angle opposite; as, CD.

1014. A **Right-Angled Triangle** is a triangle which has one right angle; when one angle is obtuse, it is called *obtuse-angled;* when all the angles are acute, it is called *acute-angled.*

1015. An **Equilateral Triangle** is a triangle which has its three sides equal; when two sides are equal it is called *isosceles;* when its sides are unequal it is called *scalene.*

Rule.—*To find the area of a triangle, multiply the base by one-half of the altitude.*

NOTE.—If the three sides are given and not the altitude, *take half the sum of the sides, subtract from it each side separately, multiply the half sum and these remainders together, and take the square root of the product.*

1. The base of a triangle is 25 feet and the altitude 16 feet; what is the area? *Ans.* 200 sq. ft.

2. The base of a triangular lot is 370 yards and the altitude is 915 feet; how many acres does it contain?
Ans. 11.658+ acres.

3. It requires 252 square feet of boards to cover the gables of a house, and the height is 10 feet 6 inches; what is the base? *Ans.* 24 feet.

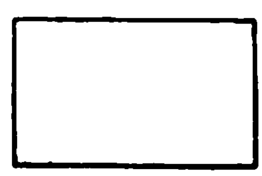
4. The three sides of a field are 56 rods, 72 rods, and 98 rods respectively; what is the area? *Ans.* 12 A. 70.28 P.

5. The perimeter of a piece of land in the form of an equilateral triangle is 624 rods; what is the area?
Ans. 117 A. 13.861 P.

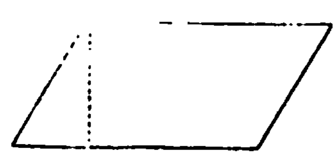
THE QUADRILATERAL.

1016. A **Quadrilateral** is a polygon having four sides and therefore four angles. There are three classes, the *parallelogram*, *trapezoid*, and *trapezium.*

1017. A **Parallelogram** is a quadrilateral whose opposite sides are parallel. The *altitude* is the perpendicular distance between its opposite sides.



1018. A parallelogram which is right-angled is called a *Rectangle.* When the four sides are equal it is called a *Square.*



1019. An oblique-angled parallelogram is called a *Rhomboid.* An equilateral rhomboid is called a *Rhombus.*

Rule.—*To find the area of a parallelogram, multiply the base by the altitude.*

1. What is the area of a parallelogram whose base is 35 feet and altitude 15 feet? *Ans.* 525 sq. ft.

2. What is the altitude of a rhomboid whose base is 63 inches and area 3087 sq. in? *Ans.* 49 inches.

3. What is the difference in the area of two farms, one being 520 rd. long and 65 rd. wide, and the other 95 chains long and 45 chains wide? *Ans.* 216 A. 40 P.

4. A carpenter had a plank 20 inches wide, from which he wished to saw off 10 square feet; what will be the length of the piece sawed off? Ans. 6 ft.

1020. A **Trapezoid** is a quadrilateral which has two of its sides parallel. Its *altitude* is the perpendicular distance between its parallel sides.

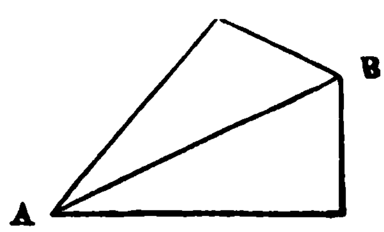
Rule.—*To find the area of a trapezoid, multiply one-half the sum of the parallel sides by the altitude.*

1. Required the area of a trapezoidal garden, one side 52 ft., the other 75 ft., and altitude 40 ft. *Ans.* 2540 sq. ft.

2. What is the surface of a plank 18 in. wide at one end, 25 in. at the other, and 16 ft. long? *Ans.* 28$\frac{2}{3}$ sq. ft.

3. Owing to the irregularity of the streets, a house lot measured 42 ft. front, and only 33 ft. back; its depth was 50 ft.; what were the contents of the lot? *Ans.* 1875 sq. ft.

4. There are two fields, one a rectangle 56 feet long and 42 feet wide, and the other a trapezoid, one side being 80 feet and the other 88 feet, and the altitude 28 feet; what is the difference in their areas? *Ans.* They are equal.



1021. A **Trapezium** is a quadrilateral which has none of its sides parallel. A *diagonal*, AB, divides the trapezium into two triangles.

Rule.—*To find the area of a trapezium, divide the trapezium into two triangles by a diagonal, find the area of each triangle, and take the sum.*

1. Required the area of a trapezium, whose diagonal measures 156 ft., and the altitudes of the two triangles are 45 and 54 feet respectively. *Ans.* 7722 sq. ft.

2. The diagonal of a tract of land in the form of a trapezium measures 75 chains, and the length of the sides are 35, 50, 70, and 85 chains respectively; what is the area?

Ans. 322 A. 109.328 P.

REGULAR AND IRREGULAR POLYGONS.

1022. A **Regular Polygon** is one whose sides and angles are respectively equal.

Rule I.—*To find the area of a regular polygon, multiply*

half the perimeter by the perpendicular let fall from the centre on one of the sides.

Rule II.—*Square the side of the polygon, and multiply by the tabular area set opposite the polygon.*

TABLE OF AREAS.

Triangle . . .	0.4330127	Octagon . . .	4.8284271
Square . . .	1.0000000	Nonagon . . .	6.1818242
Pentagon . . .	1.7204774	Decagon . . .	7.6942088
Hexagon . . .	2.5980762	Undecagon . . .	9.3656404
Heptagon . . .	3.6339124	Dodecagon . .	11.1961524

1. What is the area of a regular hexagon, whose side is 12 feet and perpendicular 10.39 feet? *Ans.* 374.04 sq. ft.

2. What is the area of a regular pentagon, whose side is 15 feet? *Ans.* 387.107+sq. ft.

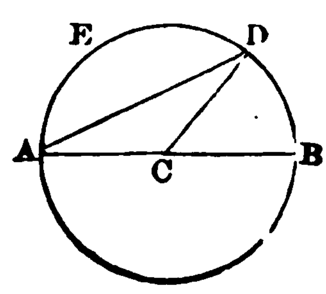
3. A gentleman has an octagonal summer house, the side of which measures 9 feet; how much ground does it cover? *Ans.* 391.102 sq. ft.

1023. Rule.—*To find the area of an irregular polygon, draw diagonals dividing the polygon into triangles, find the area of these triangles, and take their sum.*

1. What is the area of an irregular pentagon, whose diagonals are 125 and 130 inches, and the perpendiculars on the first diagonal are 20 inches and 35.7 inches, and on the second 20 inches? *Ans.* 4781.25 sq. in.

2. In a hexagonal field, the first side is 42 chains, the second 35, the third 27, the fourth 37, the fifth 35, the sixth 32; the diagonal from the first angle to the third is 49, from the first to the fourth 40, from the first to the fifth 45; what is the area? *Ans.* 251A. 134.368P.

THE CIRCLE.



1024. A **Circle** is a plane figure bounded by a curved line, every point of which is equally distant from a point within called the *centre.*

1025. The curved line is called the *circumference,* and a line passing through the centre and end-

ing in the circumference is the *diameter*, as AB. Half the diameter is called the *radius*, as BC or CD.

1026. An **Arc** is any part of the circumference, as AD or BD.

1027. A **Chord** is a straight line joining the extremities of an arc, as AD.

1028. A **Segment** is a portion of the circle included between an arc and its chord, as AED.

1029. A **Sector** is a portion of the circle included by an arc and the radii drawn to its extremities, as DCB.

Rule.—*To find the circumference of a circle, multiply the diameter by* 3.1416.

1. What is the circumference of a flower-bed whose diameter is 36 inches? *Ans.* 113.0976 inches.

2. If the diameter of the earth is 7912 miles, what is its circumference? *Ans.* 24856.339+ miles.

3. Which requires the most fence, a circular field 15 rods in diameter, or a square one whose side is 14 rods?
Ans. The latter, 8.876 rods.

1030. Rule.—*To find the diameter of a circle, multiply the circumference by* .3183.

1. A carriage wheel is 7 feet in circumference; what is its diameter? *Ans.* 2.2281 feet.

2. A circular park is 3 miles in circumference; what is its diameter? *Ans.* .9549 miles.

1031. Rule I.—*The area of a circle equals the circumference multiplied by one-fourth of the diameter, or the square of the circumference multiplied by* .07958.

Rule II.—*The area of a circle equals the square of the radius multiplied by* 3.1416, *or the square of the diameter multiplied by* .785398.

NOTE.—The area will vary slightly, for the different rules.

1. The diameter of a circle is 16 and circumference 50.2656; what is the area? *Ans.* 201.0624.

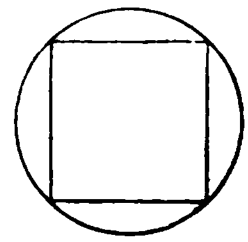
2. The circumference of a circular pond is 144 feet; what is its area? *Ans.* 1650.17 sq. ft.

3. If a cow is fastened by a chain 10 feet long to a stake in a field, how large an area will be within her reach?

Ans. 314.16 sq. ft.

4. There is a circular park 240 rods in diameter, and within it a private garden, also circular, 110 rods in diameter; how much of the park is open to the public?

Ans. 223 A. 55.609 P.



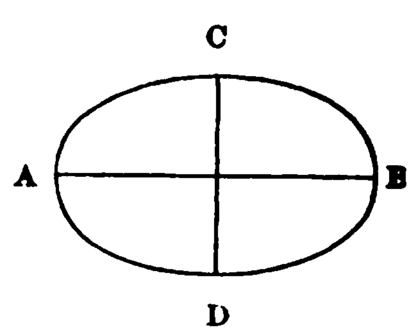
1032. A square is inscribed in a circle when the vertex of each of its angles is in the circumference.

Rule.—*To find the side of an inscribed square, multiply the diameter by* .707106, *or multiply the circumference by* .225079.

1. The end of a round stick of timber is 3 feet in diameter; what will be the side of the largest square stick that can be hewn from it? *Ans.* 2.12+ft.

2. The circumference of a circular garden is 320 feet; what is the area of the largest square garden that can be inclosed in it? *Ans.* 5187.6 sq. ft.

THE ELLIPSE.



1033. An **Ellipse** is a plane figure bounded by a curved line, the sum of the distances from every point of which to two fixed points is equal to the line drawn through these points and terminated by the curve. The two fixed points are called *foci;* the line through the foci is the *transverse* axis, and a line perpendicular to this passing through the centre and terminated by the curve, is the *conjugate* axis.

Rule.—*To find the area of an ellipse, we multiply the semi-axes together, and that product by* 3.1416.

1. Required the area of an elliptical mirror whose length is 7 feet and breadth 3.5 feet. *Ans.* 19.2423 sq. ft.

2. The axes of an ellipse are 100 inches and 60 inches;

what is the difference in area between the ellipse and a circle having a diameter equal to the conjugate axis?

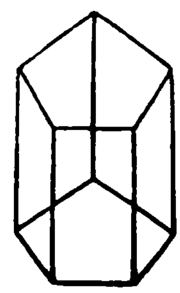
Ans. 1884.96 sq. in.

MENSURATION OF VOLUMES.

1034. A **Volume** is that which has length, breadth, and thickness. Volumes include the *Prism*, the *Pyramid*, the *Cylinder*, the *Cone*, the *Sphere*, etc.

THE PRISM.

1035. A **Prism** is a volume whose ends are equal polygons and whose sides are parallelograms.



1036. The polygons are called *bases*, the parallelograms form the *convex surface*, and the prism takes its name from the form of its bases.

1037. The **Parallelopipedon** is a prism whose bases are parallelograms. A *Cube* is a parallelopipedon all of whose sides are squares.

Rnle.—*To find the convex surface of a prism, multiply the perimeter of the base by the height.*

NOTE.—To find the entire surface we add the area of the bases.

1. What is the convex surface of a triangular prism, the sides of whose base are 10, 12, and 18 inches respectively, and its height 25 inches? *Ans.* 1000 sq. in.

2. What is the convex surface of a parallelopipedon, the sides of whose base are 12 and 15 inches, and the height 42 inches? *Ans.* 2268 sq. in.

3. What is the entire surface of a regular hexagonal prism, one side of the base being 25 inches, and the height 32 inches? *Ans.* 8047.594+sq. in.

1038. Rule.—*To find the contents of a prism, multiply the area of the base by the altitude of the prism.*

1. Required the contents of a triangular prism, the sides of the base being 12, 12, and 9 inches, and the height 36 inches. *Ans.* 1802.124 cu. in.

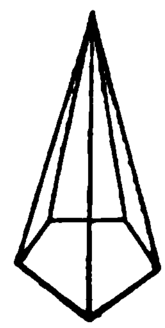
2. What are the contents of a parallelopipedon, the side

of whose base is 17 inches, the altitude of the base 13 inches, and altitude of prism 25 inches? *Ans.* 5525 cu. in.

3. Required the contents of a pentagonal prism, the side of the base being 20 inches and the altitude of the prism being 16 inches. *Ans.* 31656.784+cu. inches.

THE PYRAMID.

1039. A **Pyramid** is a volume bounded by a polygon and several triangles meeting in a common point. The polygon is called the *base*, and the triangles form the *convex surface.*



1040. The point at the top is called the *vertex*, the distance from the vertex to the base is the *altitude*, and from the vertex to the middle of a side is the *slant height.*

Rule.—*To find the convex surface of a pyramid, multiply the perimeter of the base by one-half of the slant height.*

1. What is the convex surface of a triangular pyramid, whose sides are each 16 ft. and slant height 26 ft.?

Ans. 624 ft.

2. Required the convex surface of the pyramid of Cheops in Egypt, one side measuring 763.4 feet, and the slant height being about 612 feet. *Ans.* 934401.6 sq. ft.

3. What is the entire surface of an octagonal pyramid, the side of the base being 64 feet and the slant height 75 feet? *Ans.* 38977.237+sq. ft.

1041. Rule.—*To find the contents of a pyramid, multiply the area of the base by one-third of the altitude.*

1. What are the contents of a triangular pyramid, the sides of which are 65, 75, and 85 feet, and the altitude 96 feet?

Ans. 75119.904 cu. ft.

2. Required the contents of a heptagonal pyramid, each side of the base being 56.52 feet, and the altitude 19.89 feet

Ans. 76964.825+cu. ft.

3. Required the contents of a decagonal pyramid, each side of the base being 9 ft. 6 in., and the altitude 52 feet.

Ans. 12036.307 cu. ft.

THE CYLINDER.

1042. The **Cylinder** is a round body of uniform diameter, with circles for its ends. The two circular ends are called *bases*.

1043. The **Altitude** of a cylinder is the distance from the centre of one base to the centre of the other.

Rule.—*To find the convex surface of a cylinder, multiply the circumference of the base by the altitude.*

1. What is the convex surface of a cylinder, whose altitude is 15 ft. and diameter of base 9 ft.? *Ans.* 424.116 sq. ft.

2. The warm air pipes of a furnace are 11 inches in diameter and 246 feet in length; how many square feet of tin do they contain? *Ans.* 708.43 sq. ft.

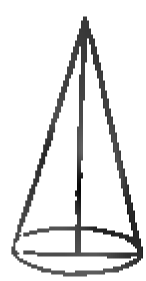
1044. Rule.—*To find the contents of a cylinder, multiply the area of the base by the altitude.*

1. Required the contents of a cylindrical stick of wood 2 ft 6 in. in diameter, and 4 ft. 9 in. long. *Ans.* 23.316 cu. ft.

2. Required the contents of a wire $\frac{1}{4}$ of an inch in diameter and 20 feet long. *Ans.* 11.78 cu. in.

3. Required the number of cubic feet of iron in a water-pipe 8 inches in diameter on the inside, and $8\frac{1}{2}$ inches on the outside, the length of the pipe being 650 yards. *Ans.* 87.74+cu. ft.

THE CONE.



1045. A **Cone** is a volume whose base is a circle, and whose convex surface tapers uniformly to a point called a *vertex*.

1046. The **Altitude** of a cone is the distance from the vertex to the centre of the base, and the *slant height* is the distance from the vertex to the circumference of the base.

Rule.—*To find the convex surface of a cone, multiply the circumference of the base by one-half of the slant height.*

1. What is the convex surface of a cone, slant height 45 in., circumference of base 72 in.? *Ans.* 1620 sq. in.

2. There is a conical haystack whose slant height is 7.6 feet, and the diameter of the base 5.5 ft.; how many square yards of canvas will cover it? *Ans.* 7.29+sq. yd.

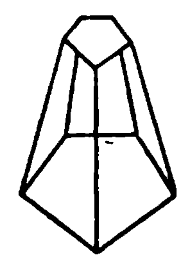
3. The distance to the top of a certain mountain is $2\frac{1}{2}$ miles, and the circumference of its base 7.35 miles; what is its surface, supposing it to be nearly a perfect cone? *Ans.* 9.1875 sq. miles.

1047. Rule.—*To find the contents of a cone, multiply the area of the base by one-third of the altitude.*

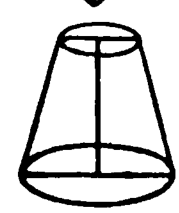
1. What are the contents of a sugar-loaf, the diameter of whose base is 9 inches and whose height is 20 inches? *Ans.* 424.116 cu. in.

2. How many cubic feet in a conical hay-stack, 6.6 ft. high and 25 ft. in circumference? *Ans.* 109.4225 cu. ft.

THE FRUSTUM OF A PYRAMID AND CONE.



1048. The **Frustum of a Pyramid** is the part of a pyramid which remains after cutting off the top by a plane parallel to the base.



1049. The **Frustum of a Cone** is the part of a cone which remains after cutting off the top by a plane parallel to the base.

Rule.—*To find the convex surface of a frustum, take the sum of the perimeters or circumferences of the two bases, and multiply it by one-half of the slant height.*

1. Required the convex surface of the frustum of a triangular pyramid, the side of the upper base being 3 ft., of the lower 5 ft., and the slant height 8 ft. *Ans.* 96 sq. ft.

2. Required the convex surface of the frustum of a cone, the diameters of the bases being 6 and 10 feet respectively, and the slant height 12 ft. 3 in. *Ans.* 307.8768 sq. ft.

1050. Rule.—*To find the contents of a frustum, take the sum of the two bases and the square root of their product, and multiply the sum by one-third of the altitude of the frustum.*

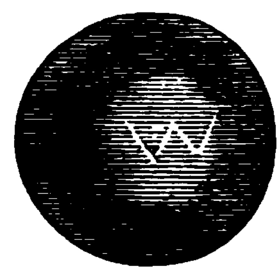
1. What are the contents of the frustum of a square pyramid, the sides of whose bases are 18 and 25 feet, and the altitude 15 feet? *Ans.* 6995 sq. ft.

2. How many cubic feet in a log 45 feet in length, the radius of one end being $2\frac{1}{2}$ feet and of the other $7\frac{1}{2}$ ft.?

Ans. 3828.825 cu. ft.

THE SPHERE.

1051. A **Sphere** is a volume bounded by a curved surface, every point of which is equally distant from a point within called the *centre.*



1052. The **Diameter** of a sphere is a line passing through its centre and ending in the surface. The *radius* is half the diameter.

Rule.—*To find the surface of a sphere, multiply the circumference by the diameter; or square the radius and multiply by* 4 *times* 3.1416.

1. Required the surface of a sphere whose diameter is 36 inches. *Ans.* 4071.5136 sq. in.

2. The circumference of the earth is nearly 25000 miles; what is its surface? *Ans.* 198937500 sq. miles.

1053. Rule.—*To find the contents of a sphere, multiply the cube of the diameter by $\frac{1}{6}$ of* 3.1416, *or by* .5236.

1. Required the contents of a sphere whose diameter is 84 inches. *Ans.* 310339.8144 cu. in.

2. What is the weight of a cannon ball 9 inches in diameter, the metal weighing 6953 oz. per cubic foot?

Ans. 1535.8742 oz.

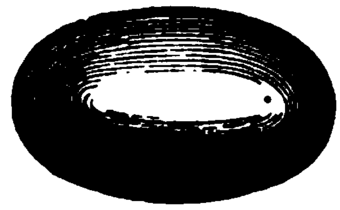
1054. Rule.—*To find the edge of a cube which may be cut from a given sphere, square the diameter, divide by* 3, *and extract the square root of the quotient.*

1. Required the edge of a cube that can be cut out of a sphere whose radius is 12 inches. *Ans.* 13.856 in.

2. Required the contents of a cube inscribed in a sphere having a circumference of 15.7085 inches.

Ans. 24.05+cu in.

THE SPHEROID



1055. A Spheroid is a volume formed by the revolution of an ellipse about one of its axes.

1056. A revolution about the longer axis forms a *prolate spheroid;* about the shorter *axis,* an *oblate spheroid.*

Rule.—*To find the contents of a spheroid, multiply the square of the revolving axis by the fixed axis, and that product by* $\frac{1}{6}$ *of* 3.1416, *or by* .5236.

1. What are the contents of a balloon in the shape of a prolate spheroid, the longer axis being 15 feet and the shorter 10 feet? *Ans.* 785.4 cu. ft.

2. The earth is an oblate spheroid, the longer axis being about 7925 miles and the shorter 7898 miles; what are its contents? *Ans.* 259,725,929,424.5 cu. miles.

IRREGULAR BODIES.

1057. Rule.—*To find the contents of an irregular body, immerse the body in a vessel of known dimensions, containing water; note the rise in the water, and calculate accordingly.*

1. A stone was thrown into an empty cylindrical vessel, which was then filled with water; when the stone was taken out, the water fell 4.75 in.; what was the volume of the stone, the diameter of the vessel being 9 in.? *Ans.* 302.18+cu. in.

2. A lump of iron ore being put into a vessel 1 cubic foot in capacity, it was found that it took $2\frac{1}{4}$ gallons to fill the vessel; required the volume of the ore. *Ans.* $1208\frac{1}{4}$ cu. in.

GAUGING.

1058. Gauging is the process of ascertaining the capacity of casks and other vessels.

1059. Barrels and casks differ from cylinders in bulging out in the middle. It is necessary, therefore, first to ascertain the approximate mean diameter of the cask or barrel, and the capacity can then be obtained like that of a cylinder

Rule I.—*To find the mean diameter of a barrel or cask add to the head diameter $\frac{2}{3}$, or, if the staves are not much curved, $\frac{3}{5}$, of the difference between the head and bung diameters.*

Rule II.—*To find the capacity in gallons, multiply the square of the mean diameter by the length (both expressed in inches), and this product by* .0034.

NOTE.—The contents of a cylinder are found (Art. 1044) by multiplying together the length, the square of the diameter, and .7854. To reduce to gallons, we divide this product by 231 (Art. 505), or, which is the same thing, multiply the length and the square of the mean diameter, by $(.7854 \div 231)$ or .0034.

1. What is the capacity in gallons of a cask whose head diameter is 30 inches, bung diameter 38 inches, and length 42 inches? *Ans.* 178.2778 gal.

2. How many gallons in a barrel of cider, with staves slightly curved, the head diameter being 2 ft., the bung diameter 2 ft. 3 in., and the length 2 ft. 10 in.?

Ans. 76.947 gal.

LUMBERMEN'S PRACTICAL RULE.

1060. In lumbering it is convenient to be able to determine the amount of square-edged inch-boards that can be sawed from a round log. The most convenient method of doing this is by the following rule, known as *Doyle's Rule:*

Rule.—*From the diameter in inches subtract* 4; *the square of the remainder will be the number of square feet of inch boards yielded by a log* 16 *feet in length.*

NOTE.—This is quite a close approximation to a scientific rule; and though it favors the buyer in small logs and the seller in large ones, yet, since logs are often crooked, no rule averages a more correct result.

1. How many square feet of lumber can be cut from a log 44 in. in diameter and 24 ft. long?

SOLUTION.—No. of square feet $= 40 \times 40 \times \frac{3}{2} = 2400$.

2. How many square feet of square-edged lumber in a log 12 in. in diameter and 18 ft. long? *Ans.* 72 sq. ft.

3. What is the yield of a log 36 in. in diameter and 20 ft. long? *Ans.* 1280 sq. ft.

NOTE.—*Doyle's Rule* is the basis of the tables in Scribner's *Lumber and Log Book*, which is a recognized standard among lumbermen.

SECTION XV.

ARITHMETICAL ANALYSIS.

1061. We present a few problems and solutions under the head of *Arithmetical Analysis*.

NOTE.—For an analysis of many of the old problems which present such excellent combinations of conditions as to be regarded as classic, see the author's *Normal Written Arithmetic*.

CASE I.

1. If an article had cost 20% less, the gain would have been 30% more; what was the gain per cent.?

SOLUTION.—The second cost is 100% — 20%, or 80% of the first cost. If on 100% the amount is a certain *rate*, on 1% the rate will be 100 times as great, and on 80% it will be $\frac{1}{80}$ of 100, or $\frac{5}{4}$ times as great; hence $\frac{5}{4} - \frac{4}{4}$, or $\frac{1}{4} = 30\%$, the difference in the rate, and $\frac{4}{4} = 120\%$, the rate at first cost; hence the gain per cent. was 20.

2. If an article had cost me 10% less, the gain would have been 12% more; what was the gain per cent.?

Ans. 8%.

3. If the cost had been 4% less, the gain would have been $4\frac{2}{3}$% more; what was the gain per cent.?

Ans. 12%.

CASE II.

1. If an article had cost 20% more, the gain would have been 25% less; what was the gain per cent.?

SOLUTION.—The second cost is 120% of the first cost, and therefore on it the amount will be $\frac{5}{6}$ as great a rate per cent. as on the first cost; hence $\frac{6}{6} - \frac{5}{6}$, or $\frac{1}{6} = 25\%$, the difference in the rates, hence $\frac{6}{6} = 150\%$, the rate at first cost, and the gain is 50%.

2. If the cost of certain goods had been 25% more, the gain would have been 30% less; what was the gain per cent.? *Ans.* 50%.

3. If an invoice of calicoes had cost 15% more, the gain would have been 12% less; what was the gain per cent.?

Ans. 8% loss

4. If I had paid 10% more for my fall stock, the profit would have been 10% less; what was the gain per cent.?

Ans. 10%.

CASE III.

1. A merchant sold cloth at 20% gain, but had it cost $49 more, he would have lost 15% by selling at the same price; what did the goods cost?

SOLUTION.—The cloth was sold for 120%, or $\frac{6}{5}$ of the cost, but had it cost $49 more, it would have been sold for 85%, or $\frac{17}{20}$ of the cost; hence $\frac{6}{5}$ of the first cost equals $\frac{17}{20}$ of the second cost, and $\frac{24}{17}$ of the first cost equals $\frac{20}{20}$ of the second cost; but the difference between the first cost and the second cost is $49; hence $\frac{24}{17}-\frac{17}{17}$, or $\frac{7}{17}$ of the first cost equals $49, and the cost was $119.

2. A quantity of goods were sold at 25% gain, but if they had cost $40 less, the gain at the same selling price would have been 35%; what was the cost of the goods?

Ans. $540.

3. A farmer lost 10% on his wheat crop, but if it had cost him $50 more he would have lost 20%; what was the cost of the crop? *Ans.* $400.

4. A commission merchant sold flour for his principal at a loss of 10%, but if the flour had cost $1 a barrel less, he would have gained 5%; what was the cost of the flour per barrel? *Ans.* $7.

CASE IV.

1. A father willed $43,500 to his two sons, A and B, aged 12 and 15 years respectively, to be divided in such a manner that the two parts, on interest at 6%, would amount to equal sums when they became of age; what were the parts?

SOLUTION.—A's money was on interest 9 years, and B's 6 years. For 6 years at 6%, $\frac{34}{25}$ of the principal equals the amount; hence $\frac{34}{25}$ of B's *share* equals his *amount;* and in the same way we see that $\frac{77}{50}$ of A's *share* equals his *amount.* Now, since the amounts are equal, $\frac{77}{50}$ of A's share equals $\frac{34}{25}$ of B's, from which we find B's share $=\frac{77}{68}$ of A's; hence $\frac{68}{68}$ of A's $+\frac{77}{68}$ of A's, or $\frac{145}{68}$ of A's $=$ $43,500; $\frac{1}{68}$ of A's $=$ $300, $\frac{68}{68}=$ $20,400, and $\frac{77}{68}$ of A's, or B's $=$ $23,100.

2. A gentleman divided $84,700 among his three sons, aged 11, 14, and 17 years respectively, so that the different shares, being on interest at 5%, should amount to equal sums when they became of age; what were the shares?

Ans. $25,200; $28,000; $31,500.

3. A gentleman put out $49,103 on interest at 7% for the benefit of his three sons, aged 16, 17, and 18 years respectively, dividing it in such a manner that each, as he became of age, should receive the same amount; what were the shares of each? *Ans.* $15,488 ; $16,335 ; $17,280.

CASE V.

1. It is between 10 and 11 o'clock, and the minute-hand of the clock is $\frac{1}{2}$ as far after 12 as the hour-hand is before it. What is the time of day?

SOLUTION.—At 10 o'clock the hour-hand was at 10 and the minute-hand at 12. Since that time, the minute-hand has moved $\frac{1}{2}$ of the hour-hand's distance from 12, and the hour-hand $\frac{1}{12}$ of $\frac{1}{2}$ of that distance, or $\frac{1}{24}$ of that distance. Then $\frac{24}{24}+\frac{1}{24}=\frac{25}{24}$ of that distance=the distance from 10 to 12=10 minute-spaces, and $\frac{1}{24}=\frac{1}{25}$ of 10 minute-spaces, and $\frac{12}{24}$ or $\frac{1}{2}$ of the hour-hand's distance, which is the minute-hand's distance $=\frac{12}{25}$ of 10 minute-spaces$=4\frac{4}{5}$ spaces, and the time is 4 min. 48 seconds after 10 o'clock.

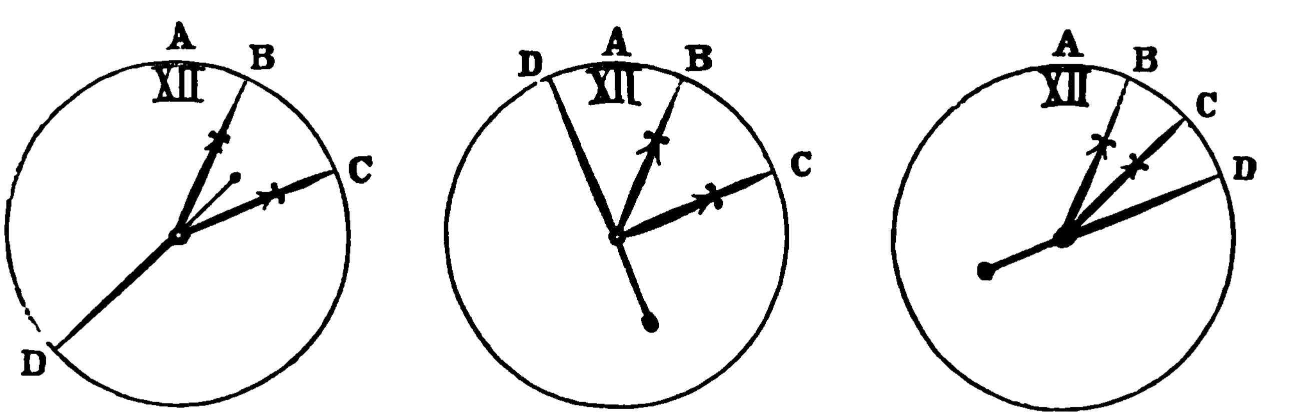
2. It is between 1 and 2 o'clock, and the minute-hand is as far past 2 as the hour-hand is before it; required the time. *Ans.* 13 min. $50\frac{10}{13}$ sec. after 1.

3. It is between 3 and 4 o'clock, and the minute-hand is $\frac{1}{3}$ as far before 12 as the hour-hand is after it; required the time. *Ans.* 53 min. $30\frac{30}{37}$ sec. past 3.

CASE VI.

1. Suppose the hour-hand, minute-hand, and second-hand of a clock all turn upon the same centre; they will be together at 12 o'clock; how long before each hand respectively will be half-way between the other two?

1 2 3



SOLUTION.—CASE 1. Let A be 12 on the dial, and B, C, and D, the positions of the hour-hand, minute-hand, and second-hand respectively,

the second-hand being equally distant from the other two. Then, while the hour-hand passes over a certain space, the minute-hand passes over 12 times that space, and the second-hand 720 times the space. Hence AC $=12$ AB, and ACD$=720$AB. But BD$=$CD$=720$AB-12AB$=708$AB, and BC$=$AC$-$AB$=11$AB; hence the whole circumference$=$CD$+$BD$+$BC$=708$AB$+708$AB$+11$AB$=1427$AB; hence AC, the space passed over by the minute-hand, is $\frac{12}{1427}$ of 3600 seconds, or $30\frac{390}{1427}$ seconds.

CASE 2. The hour-hand being between the other two hands, we have BD$=$BC$=11$AB; hence AD$=10$AB, and AD$+$ACD$=10$AB$+720$AB$=730$AB, the whole circumference; therefore AC$=\frac{12}{730}$ of 3600 seconds, or $59\frac{13}{73}$ seconds.

CASE 3. When the minute-hand is half-way between the other two, the second-hand will have gone once round the face of the clock; hence, since CD$=$BC$=11$AB, AD$=$AC$+$CD$=23$AB, and the circumference$+23$AB$=720$AB; hence the circumference$=697$AB, and 12AB$=\frac{12}{697}$ of 3600 seconds, or $61\frac{683}{697}$ seconds.

2. The three hands turning upon the same centre, how long will it be after 12 o'clock before the hour and second hands, the minute and second hands, and the hour and minute hands, will be together again?

Ans. $60\frac{60}{719}$ seconds; $61\frac{1}{59}$ seconds; $65\frac{5}{11}$ minutes.

3. How long will it be after 12 o'clock before the hour and second hands, the minute and second hands, and the hour and minute hands, are at right angles with each other?

Ans. $15\frac{15}{719}$ seconds; $15\frac{15}{59}$ seconds; $16\frac{4}{11}$ minutes.

4. How long will it be after 12 o'clock before the hour and second hands, the minute and second hands, and the hour and minute hands, are exactly opposite each other?

Ans. $30\frac{30}{719}$ seconds; $30\frac{30}{59}$ seconds; $32\frac{8}{11}$ minutes.

CASE VII.

1. If 6 acres of grass, together with what grows on the 6 acres during the time of grazing, keep 16 oxen 12 weeks, and 9 acres keep 26 oxen 9 weeks, how many oxen will 15 acres keep 10 weeks, the grass growing uniformly all the time?

SOLUTION.—If 6 acres$+$the growth of 6 acres for 12 weeks, keep 16 oxen 12 weeks, one acre$+$the growth of 1 acre for 12 weeks, will support 16 oxen $\frac{1}{6}$ of 12 weeks, or 2 weeks, or 32 oxen 1 week. In the same manner, 1 acre$+$the growth of 1 acre for 9 weeks, will keep 26 oxen 1 week; subtracting these, we have the growth of 1 acre for 3 weeks will keep 6 oxen for 1 week, and the growth of 1 acre for 9 weeks will keep 18 oxen for 1 week; subtracting this latter expression from 1 acre

+the growth of 1 acre for 9 weeks, we have 1 acre, without the growth, will keep 8 oxen 1 week; hence, 15 acres will keep 120 oxen 1 week, or $\frac{1}{10}$ of 120, or 12 oxen 10 weeks; again, since the growth of one acre for 3 weeks will keep 6 oxen 1 week, the growth of 1 acre for 1 week will keep 2 oxen 1 week, hence the growth of 15 acres for 10 weeks will keep 30 oxen for 10 weeks; and adding, we have 15 acres+ the growth of 15 acres for 10 weeks, will keep 42 oxen 10 weeks.

OPERATION.

A.	A.	W.	O.	W.
6+	6 for	12	16	12
9+	9 for	9	26	9
1+	1 for	12	32	1
1+	1 for	9	26	1
	1 for	3	6	1
	1 for	9	18	1
1+	0 for	0	8	1
15+	0 for	0	12	10
	15 for	10	30	10
15+	15 for	10	42	10

2. If 5 acres of grass, together with what grows on them during the time of grazing, keep 20 oxen 10 weeks, and 8 acres keep 29 oxen 16 weeks, how many weeks will 15 acres keep 70 oxen? *Ans.* 6 weeks.

3. If 10 acres of grass keep 48 oxen 15 weeks, and 7 acres keep 34 oxen 14 weeks, how many acres will keep 38 oxen 16 weeks, the grass growing uniformly all the time?

Ans. 8 acres.

CASE VIII.

1. What number divided by 13, leaves 12 for a remainder, by 7 leaves 3, by 6 leaves 5, and by 5 leaves 2?

SOLUTION.—It is evident that 13+12, or 25, will satisfy the first condition. Dividing 25 by 7, the remainder is 4, which is greater than the required remainder; hence we must add to 25 a number which, divided by 7, will leave a remainder that increased by 4, will contain 7 once, with a remainder of 3. This remainder is 6. But the number added must also contain 13, and both conditions are fulfilled by 13 itself; hence 25+13, or 38, fulfills the first two conditions. Dividing 38 by 6, the remainder is 2, which is 3 less than the required remainder; hence we must add to 38 a number which, divided by 6, will leave a remainder of 3, and the number added must also be a multiple of 13 and 7. The least multiple of 13 and 7 is 91, which, divided by 6, leaves 1; hence to get a remainder of 3, we must add 3 times 91, or 273. 38+273 = 311, which satisfies the first three conditions. Continuing the operation in the same manner, we find that 857 is the least number that will satisfy all the conditions. Other numbers can be found by adding to 857 any common multiple of all the divisors.

2. What number divided by 11, leaves 10 remaining, by 9 leaves 7, by 7 leaves 6, by 4 leaves 3? *Ans.* 1231.

3. What number divided by 15, leaves 8 for a remainder, by 13 leaves 7, by 11 leaves 9, by 7 leaves 5? *Ans.* 7313.

SECTION XVI.

MISCELLANEOUS EXAMPLES.

These problems are designed both as a review of the work and a test of the knowledge and arithmetical skill of the pupil. They are to be used in accordance with the needs of the pupil and the judgment of the teacher.

1. If Henry's capital is 20% less than William's, how many % is William's more than Henry's? *Ans.* 25%.

2. I wish to put 20 hogsheads of ale (54 gallons) into 10 empty wine pipes; what must be the capacity of a cask which shall contain what is left over? *Ans.* $58\frac{3}{7}\frac{4}{7}$ wine gal.

3. A bought stock 5% below par value, and sold it 5% above par, and gained $550; what was the par value of the stock? *Ans.* $5500.

4. Mr. Russell bought stocks $2\frac{1}{2}$% above par, and was obliged to sell $2\frac{1}{2}$% below par, and lost $235; what did the stocks cost him? *Ans.* $4817.50.

5. A asked at one time $33\frac{1}{3}$% less for an article than cost, but afterwards sold it for $33\frac{1}{3}$% more than this price; required the loss per cent. *Ans.* $11\frac{1}{9}$%.

6. What must I ask for a house that cost me $7520, that after falling 6% on the price, I may gain $18\frac{3}{4}$% on the cost? *Ans.* $9500.

7. A grocer asked for flour 35% more than cost, but sold it for $66\frac{2}{3}$% of his asking price; required the loss per cent. *Ans.* 10%.

8. What must be asked for a farm which cost $8160, so that after raising the price $33\frac{1}{3}$%, I may gain $8\frac{1}{3}$% on the cost? *Ans.* $6630.

9. A's gain at wholesale is $12\frac{1}{2}$%, and his retail price is 5% more than his wholesale price; required the gain per cent. at retail. *Ans.* $18\frac{1}{8}$%.

10. B's gain at retail is $12\frac{1}{2}$%, and his wholesale price is $2\frac{1}{2}$% less than his retail price; required the gain per cent. at wholesale. *Ans.* $9\frac{11}{16}$%.

11. A barrel of molasses lost 20% by leakage, and the remainder was sold at a gain of 20%; required the gain or loss per cent. *Ans.* 4% loss.

12. A log 19 in. thick is sawn into 15 boards, each $1\frac{1}{8}$ in. thick; what % of the board is wasted? *Ans.* $11\frac{7}{38}$%.

13. A merchant lost 20% of his goods, and sold the remainder for $33\frac{1}{3}$% more than cost, and gained \$250.75; what did his goods cost? *Ans.* \$3761.25.

14. A man lost 25% of a purchase; what must he gain per cent. on the remainder, that he may gain 25% on the whole? *Ans.* $66\frac{2}{3}$%.

15. A bought a house and barn, paying 3 times as much for the house as for the barn; if he had paid $12\frac{1}{2}$% more for the house, it would have cost \$$4725\frac{1}{2}$; what was the cost of each? *Ans.* House, \$4200.44; barn, \$1400.148.

16. A borrowed of B a certain sum; $37\frac{1}{2}$% of the debt is \$45.84, which is $66\frac{2}{3}$% of what has been repaid; how much does A still owe? *Ans.* \$53.48.

17. If 7 horses or 6 cows eat $5\frac{6}{11}$ tons of hay in 17 days, how long will it take 6 horses and 7 cows to eat the same quantity? *Ans.* $8\frac{2}{5}$ days.

18. If stock bought at 10% above par pays 8% on the investment, what per cent. will it pay if bought at 10% discount? *Ans.* $9\frac{7}{9}$%.

19. 12 men can do a piece of work in $8\frac{5}{6}$ days; how long may 3 men remain away, and the work be finished in the same time by their bringing 7 more with them?

Ans. $6\frac{11}{60}$ days.

20. What will it cost to paper the walls of a room 25.5 ft. long, 14 5 ft wide, and 9.25 ft. high, a roll of paper being 8 yards long and $\frac{1}{2}$ a yard wide, and costing 45 cents a roll?

Ans. \$12.15.

21. If an important vote in the English Parliament is taken at 1 h. 30 min. A. M., Jan. 15th, and telegraphed immediately to San Francisco, 122° 26′ 15″ W., at what hour will it be received? *Ans.* 5 h. 20 min. 15 sec. P. M., Jan. 14th.

22. At a certain time between 5 and 6 o'clock, the minute-

hand of a clock was between 6 and 7; within an hour the hands had exactly changed places with each other; when were they in the first position? *Ans.* 5 h. 32 min. $18\frac{6}{13}$ sec.

23. When gold was worth 50% more than currency, what was the value in gold of a ten-dollar bill? When currency was worth 50% less than gold, what was the value in currency of a gold eagle? *Ans.* $\$6\frac{2}{3}$; \$20.

24. A room is 20.5 ft. long, 16.7 ft. wide, and 9.5 ft. high; there are 2 windows 5.75 ft. high, and 3.5 ft. wide, and a door 6.6 ft. high and $3\frac{1}{3}$ ft. wide. What will be the expense of plastering the room at $31\frac{1}{4}$¢ a square yard, and of carpeting it with ingrain carpet a yard wide @$\$1.62\frac{1}{2}$? *Ans.* \$96.08.

25. A goldsmith bought an ingot of gold at \$192 per lb., and sold it at \$16 per ounce, using Avoirdupois weight both times. If the true weight of the ingot was 8 lb. Troy, how much did he gain by the fraud, gold being worth \$16 an ounce? *Ans.* $\$421\frac{53}{175}$.

26. A steamer going from Philadelphia to Liverpool passes over $8\frac{1}{2}$ degrees of longitude on an average in a day; how long is it from noon one day to noon the next day, and how long will it be on the return voyage?

Ans. 23 h. 26 min. out; 24 h. 34 min. return.

27. Four men make regular excursions into the country, between which each stays at home just one day. A is always absent 3 days, B 5 days, and C and D each 7 days. If they all set out on the same day, how many days will elapse before they can all be at home on the same day? *Ans.* 23 days.

28. If $\frac{5}{6}$ of the cost of an article equals $\frac{7}{8}$ of its selling price, what is the loss per cent.? If $\frac{7}{8}$ of the cost of an article equals $\frac{5}{6}$ of its selling price, what is the gain per cent.?

Ans. $4\frac{16}{21}$%; 5%.

29. How many bricks will be required to build a 13 inch wall of average bricks 7 ft. high round a garden containing $2\frac{1}{2}$ acres in the form of a square, and what will be the cost of the bricks at the rate of $\$4\frac{1}{2}$ ℔ M., the mortar being $\frac{1}{4}$ of an inch thick, and no allowance made for corners?

Ans. \$873.18.

30. B drew out of bank 20% of his deposits, then 30% of the remainder, and afterwards 40% of what then remained, and had $420 left; what was his deposit? *Ans.* $1250.

31. A, having a quantity of canal stock, sold 25% of it to B, who sold $33\frac{1}{3}$% of his purchase to C, who sold $37\frac{1}{2}$% of his purchase to D, who received 5 shares; how many had A at first? *Ans.* 160 shares.

32. A dress-pattern having been cut from a piece of silk, there were left $8\frac{3}{8}$ yards, which was $66\frac{2}{3}$% less than the quantity cut off; how many yards were there at first? *Ans.* $33\frac{1}{2}$ yards.

33. A speculator invested $9720 in oil and lost 20%; he then invested the remainder in sugar and gained 25%; he then invested his money in fancy stocks and lost 30%; did he gain or lose, and how much? *Ans.* Lost $2916.

34. A and B receive equal legacies; A spent 75% of his in land, and B lost in speculation as much as equaled $33\frac{1}{3}$% of what both received, and then they had together $350; what was the amount of the legacy, and what had each left? *Ans.* Legacy, $600 each; A, $150; B, $200.

35. A merchant spent equal sums of money in cotton, linen, and woolen goods, and made 10% on the cotton, and 8% on the linen, but lost 25% on the woolens; the whole amount of the sales was $2842.10; what did he pay for each kind of goods? *Ans.* $970.

36. A's gain was 26%, and B's 30%, and A's gain was $27 less than B's; what was the capital of each, if $\frac{2}{3}$ of A's equals $\frac{3}{4}$ of B's? *Ans.* A's $4050; B's $3600.

37. A sold his farm and house for $9000, receiving $\frac{4}{5}$ as much for his house as for his farm; on the farm he gained 7% and on the house he lost 5%; what was the cost of each? *Ans.* House, $4210.526; farm, $4672 897.

38. A broker charged me $2\frac{1}{4}$% for purchasing some uncurrent bank-notes at 15% discount. Three bills, of $20, $50, and $100 respectively, turned out to be worthless, but by selling the rest at par I made $85; what was the face of the notes? Ans. $2000.

39. What is the amount of a note for $765.35, dated Aug 9, 1872, and paid June 12, 1876, interest 7%, payable annually? *Ans.* $992.78.

40. Two steamboats leave Philadelphia and Trenton at the beginning of ebb-tide, going towards each other, their rate of travel being 10 miles an hour, and the tide running $1\frac{1}{2}$ miles an hour; how far from Philadelphia will the boats meet, the distance being about 30 miles? *Ans.* $12\frac{3}{4}$ mi.

41. Three men start from the same point to travel round an island 80 miles in circumference; the first goes 5 miles a day, the second 10 miles a day, and the third goes 10 miles a day in the opposite direction; how long before they will all meet again? *Ans.* 16 days.

42. If the shipment of a coal operator for a year was 1800 long tons, and the cost per long ton was for mining 50¢, hauling and outside labor 48¢, incidentals, wear of tools, machinery, etc., 45¢, royalty on land 50¢, freight by railroad $1.75, capital 48¢, commission on sales 4%, and the sales averaged $$6\frac{1}{4}$ per ordinary ton; what was his profit for the year? *Ans.* $4608.

43. A gentleman, asking the consent of a lady to marry the second of her five daughters, was told that he should have her on condition of finding what was the fortune of each daughter by their father's will, which was as follows: the first four had $50,000; the last four, $66,000; the first and last three, $60,000; the first three and last, $56,000; and the first two and last two, $64,000; what were their fortunes? *Ans.* $8000; $14,000; $10,000; $18,000; $24,000.

44. A company engaged an agent to do business for one month at a salary of $25, giving him goods amounting to $57.54 and $32.17 in cash to start with. The agent bought during the month goods amounting to $59.91. At the end of the month the goods on hand amounted to $31.67, and the amount of sales for the month was $102.97; required the balance of the account. *Ans.* Loss to company, $7.81.

45. In 1869 it is estimated that 550,000,000 feet of lumber was manufactured in the Saginaw Valley, at a profit of

$375,000; what was the rate per cent. of profit, if the capital invested was in mill property $3,754,000, in shingle mill property $295,500, and in tools, teams, slides, etc., estimated at $4 per M. feet of lumber made? *Ans.* 6%+.

46. A tailor sold 11 garments for $77, viz.: coats at $13, pants at $6, vests at $5, and cravats at $3 each; required the number of each. *Ans.* 3, 4, 1, 3.

47. Mr. Johnson invested a certain amount in cotton, and Mr. Wilson invested 3 times as much; Mr. Johnson lost 20%, while Mr. Wilson gained 25%, and the difference between the amounts they received was $885; how much did each invest? *Ans.* $300; $900.

48. A retail merchant sold a quantity of silks for $1078.12½, thereby gaining 20%; the wholesale merchant from whom he bought them made a profit of 15%; and the importer who sold them gained 25%; what did they cost the importer? *Ans.* $625.

49. A traveler journeyed 500 miles in two days, and $\frac{2}{8}$ of the distance he traveled the first day, plus $\frac{1}{8}$ of the distance he traveled the second day, equals $\frac{4}{5}$ of the distance he traveled the first day; how far did he travel each day?

Ans. $241\frac{29}{31}$ miles; $258\frac{2}{31}$ miles.

50. What sum must a man save annually, commencing at 21 years of age, to be worth $50,000 when he is 50 years old, investing his money yearly at 6% compound interest?

Ans. $678.98.

51. A, B, and C formed a partnership; A put in $500 for 8 months, B, $750 for a time unknown, and C, an amount not known for 10 months; what were B's time and C's stock, if A received $580 for his stock and profit, B, $840 for his stock and profit, and C, $720 for his stock and profit? *Ans.* B, 6 months; C, $600.

52. A railroad has been constructed through a farm, making it necessary to build fences at a cost of $750, which must be renewed every 15 years; what should the owner receive to meet this expenditure, at 6% compound interest?

Ans. $1287.03.

53. A gentleman having 4 sons, left the youngest \$5000, the eldest \$7200, and the two others the arithmetical and geometrical means of these sums respectively; what were the shares of the other sons? *Ans.* \$6000; \$6100.

54. Two adjoining farms rent for \$400 a year, rent being paid in the one case semi-annually, and in the other quarterly; what would be the difference in the amount of the rent of each for 25 years, int. 8%? *Ans.* \$689.81.

55. Mr. Smith bought a house for \$3000, agreeing to pay for it in annual payments of \$500 each; but finding himself unable to make the payments, and having the reversion of a perpetuity in 12 years, he makes an agreement with his creditors to pay the whole amount with compound interest at 6% when he enters upon his perpetuity; what amount will then be due? *Ans.* \$6036.59—.

56. Two drovers met on the road, when one said to the other, Give me one of your steers and I shall have twice as many as you then have. But, replied the other, if you give me one of yours, I shall have as many as you. How many had each? *Ans.* 7; 5.

57. A man gave me his note for \$500, payable in 8 years, interest at 6%; if he pays the interest annually, to what rate is this equivalent, if the same amount of interest had been paid at the end of the time? *Ans.* $7\frac{13}{50}$%.

58. A tailor bought 50 yards of broadcloth $1\frac{3}{4}$ yards wide, but on sponging, it shrunk 5% in width, and 5% in length; to line it he bought flannel $1\frac{1}{2}$ yards wide, which shrunk 1 yard for every 20 yards in length and $\frac{1}{16}$ of a yard in width; how many yards of flannel are required? *Ans.* $57\frac{19}{23}$ yd.

59. A planter hired 75 persons for \$90, giving the men \$3, the women $\$1\frac{1}{5}$, the boys $\$\frac{1}{2}$, and the girls $\$\frac{2}{5}$ a day; how many were there of each?

$$Ans. \begin{cases} 21, \ 6, \ 6, \ 42: \ 18, \ 12, \ 36, \ 9: \\ 19, \ 11, \ 18, \ 27: \ 15, \ 24, \ 18, \ 18. \end{cases}$$

60. In turning a cart within a circle, it was observed that the outer wheel made two turns while the inner made but one; the wheels were each three feet high, and the axle-tree

$4\frac{1}{2}$ feet long; what was the circumference of the track described by the outer wheel? *Ans.* 56.548+feet.

61. A person went to a store, borrowed as much money as he had, and spent 16 cents; he then went to a second store, borrowed as much as he then had, and spent 16 cents; he repeated this at a third and fourth store, and then had no money remaining; how much had he at first? *Ans.* 15¢.

62. Three persons took a house in partnership for a year at a rent of $750; at the expiration of three months they took in three more tenants, and at the end of every three months till the expiration of the time they took in four more; how much should one of each class pay?

Ans. $\$125\frac{25}{28}$; $\$63\frac{11}{28}$; $\$32\frac{1}{7}$; $\$13\frac{11}{28}$.

63. $750. MILLERSVILLE, JAN. 22, 1875.

For value received, I promise to pay H. S. Snyder, or order, on demand, Seven Hundred and Fifty Dollars, without defalcation. I. REIMEL.

Indorsements: March 1, $50; June 11, $200; July 10, $25; Sept. 9, $250; Dec. 1, $150.

What is due Jan. 1, 1876? *Ans.* $102.40.

64. A and B set out from the same place and traveled in the same direction, A at the rate of 20 miles a day. After they had been gone $4\frac{1}{6}$ days, A goes as far back as B has traveled in that time; he then turns, and pursuing his journey, overtakes B 25 days from the time they set out; at what rate does B travel? *Ans.* 15 miles a day.

65. A general drew up his brigade in the form of a square and had one man left over; but receiving a reinforcement of 423 men, he was able to increase the side of the square by 4 men; how many men had he at first? *Ans.* 2602.

66. A speculator invests $2120 in grain, including 4% for freight and 2% for commission, then sells the grain at 20% advance on the cost price for a note at 60 days, which he gets discounted at bank at 6%, and repeats this operation every 10 days; how much will his gains amount to in a month, if he invests the whole proceeds each time?

Ans. $859.95

67. I have a garden $21\frac{1}{4}$ rods long and $10\frac{1}{2}$ rods wide; it is surrounded by a fence $7\frac{1}{8}$ feet high; a walk is laid out within the fence which is $7\frac{1}{4}$ feet wide at the sides of the garden, and $6\frac{3}{4}$ feet wide at the ends; how much is left for cultivation? *Ans.* $53518\frac{1}{32}$ sq. ft.

68. A bath-tub will hold 160 gallons of water; it is filled by a faucet discharging 60 gallons in 5 minutes, and emptied by a waste-pipe discharging 46 gallons in 4 minutes; if both are opened at 6 o'clock in the morning, and then the waste pipe closed at 8 o'clock, at what time will the tub be full?
Ans. 8 h. 8 min. 20 sec. A. M.

69. A tree stands exactly opposite the front door of my house, which looks towards the northeast, but the distance cannot be directly measured on account of a small pond lying between them. I therefore measure 60 yards due north from the front door, and then 120 due east, when I find I am exactly 40 yards to the north of the tree; what is the distance from the front door to the tree? *Ans.* 121.65 yards.

70. Wishing to speculate in land, I obtained at a bank on my note payable in 3 months money enough to buy 50 acres at $75 per acre, and also borrowed of a friend sufficient to buy another lot of the same size and value, giving my note payable on demand, interest 6%; I sold both lots in time to take up my note at bank when it became due, and found that the price received for 36 acres would pay for it. What did I gain by my speculation, supposing that I paid my demand note at the same time as that in bank? *Ans.* $2961.94.

71. A land-owner who held his lots for sale at $4 per foot ground rent, agreed with a tricky builder to give him a bonus of $9000 cash with 180 feet of his lots, taking ground-rent at $7 per foot in return. The builder erected 10 houses upon them, costing $2000 each, and then sold them at $1800 cash each, subject to the ground-rent; how much did the builder make by the operation? *Ans.* $7000.

72. Bought $5000 gold @ 113 on Feb. 1, paying a commission of $\frac{1}{4}$%; on Feb. 25, I sold it at $114\frac{5}{8}$; what is my gain, interest at 6%? *Ans.* $33.60.

73. I bought a 6% mortgage for \$2500 at 5% discount, with two years to run; what interest do I get on the money invested if the mortgage is satisfied at maturity? *Ans.* $8\frac{18}{19}$%.

74. A is enrolled as a life-member of the Fairmount Park Art Association upon contributing \$50; at the same time B becomes a member, and for 20 years makes the yearly contribution of \$5; at the end of this period, which has contributed the larger sum, interest included? *Ans.* B, \$53.

75. The "Consolidated Virginia" silver mine rose from 94 to 590; what gain per cent. was this, and what would I have made if I had held 750 shares? *Ans.* $527\frac{31}{47}$%; \$372,000.

76. Bought 50 shares bank stock (\$100) at 108; at the end of 5 yr. 6 mo., having regularly received a semi-annual dividend of 5%, I sold the stock at 110; what did I gain, money worth 6% compound interest? *Ans.* \$1229.01.

77. Buy ground "in fee" for \$400, and build a house costing \$1000, and pay for them by subscribing for 10 shares Building Association, new series, and buying a loan at \$60 premium, Net plan. If I rent the house for \$15 a month, and pay taxes and water rent equal to \$3 a month, what will my house cost me without interest, and what with interest, if the series runs out in 10 years? *Ans.* \$600; \$781.50.

78. Bought a check on a suspended bank at 75%, and exchanged it for railroad bonds of the same nominal value at 85%, bearing 7% interest; what rate of interest do I receive on my investment? *Ans.* $9\frac{1}{3}$%.

79. Four men agree to perform a certain piece of work for \$428. A, B, and C can do it in 24 days; B, C, and D in 30 days; A, C, and D in 35 days; and A, B, and D in 42 days: what share of the money should each receive?

Ans. A, \$92; B, \$140; C, \$188; D, \$8.

80. There are two clocks, one of which gains 30 seconds a day and the other loses 25 seconds. If the pendulums beat together when both clocks indicate exactly 6 o'clock, what time does each show when they next beat together?

Ans. 1st., 26 min. $11\frac{5}{11}$sec. past 6; 2d, 26 min. $10\frac{5}{11}$ sec past 6.

81. A merchant bought 15 pieces of cloth, each containing 20 yards, @ $\$5\frac{1}{4}$, on a credit of 8 mo., and sold them @ $\$6\frac{1}{8}$, on a credit of 3 mo; what was his cash gain at 6%?

Ans. $295 92.

82. Two girls, each 10 years old, receive legacies from a relative; the sum left to the first is invested at 8% simple interest, and the sum left to the second at 6% compound interest, and each investment will amount to $5000 when the owner is 18 years old; what were the legacies?

Ans. 1st, $3048.78; 2d, $3137.06.

83. How many cannon balls, 6 inches in diameter, are contained in a cubical vessel whose side measures 2 feet, and how many gallons of water will it hold after it is filled with the balls? *Ans.* 64 balls; 28.5 gal.

84. How many cannon balls, 8 inches in diameter, are contained in a cubical vessel whose side measures 6 feet, and how many gallons of ale can be poured in after the vessel is filled with balls, each ball containing a hollow 5 inches in diameter, and the opening containing $1\frac{1}{2}$ cubic inches?

Ans. 729 balls; 803.6237 gal.

85. Four men buy a grindstone 32 inches in diameter, with a square hole whose diagonal is 4 inches; if each grinds off his share in turn, how much of the semi-diameter will each one take? *Ans.* 2.108 in.; 2.4905 in.; 3.2165 in.; 6.185 in.

86. A father left $61,248 to be divided among his four sons, aged respectively 16, 14, 12, and 10 years, so that their respective shares being invested at 10% simple interest, shall amount to the same sum when they become 21 years old; what was the share of each?

Ans. $18,088; $15,960; $14,280; $12,920.

87. Two men in Philadelphia hired a carriage for $25 to go to Trenton, a distance of 30 miles, and back, with the privilege of taking in three more persons. At Bristol, 19 miles from Philadelphia, they take in Mr. Jones; at Trenton they take in Mr. Newell; and 15 miles from Philadelphia they take in Mr. Stokes; what is each one's share of the cost? *Ans.* $\$8.29\frac{31}{36}$; $\$4.34\frac{1}{36}$; $\$2.81\frac{1}{4}$; $1.25.

88. Sold for % of consignor on September 3, $5230.43 on 4 months; Sept. 10, $437 for cash; October 15, $3730.37 on 60 days; Sept. 2, paid freight on this stock, $97.46, and our commission for selling is 5%; what is due to consignor, and when shall the proceeds be remitted?

Ans. $8830.45; Dec. 22.

89. Mr. Green took 20 shares in a building association which runs out in 10 years; Mr. Gray bought a 6% mortgage at 3.2% discount, interest payable annually, for a sum which will give the same interest as the periodic payment on 20 shares for 10 years; if the mortgage is paid off in 10 years, which of the two realizes the larger sum on his equal investment? *Ans.* Green, $1797.50.

90. A, B, and C are to travel a distance of 40 miles; A walks at the rate of 1 mile an hour, B 2 miles an hour, and C, with a horse and buggy, goes 8 miles an hour. C, at the start, takes in A and carries him so far that if he returns and meets B, and takes him in, they will get to the end of the journey at the same time that A does; required the distance A and B walk, C rides, and the time of each and all.

Ans. A, $5\frac{35}{41}$ mi.; B, $13\frac{27}{41}$ mi.; C, $80\frac{40}{41}$ mi.; $10\frac{5}{41}$ hours.

91. Paul B. Myers, of Philadelphia, received per steamer Cynthia, one case of cigars, Londres brand, marked PBM#1, containing 1000 1st quality @$55 ℔M., 1300 2d quality @ $45, 700 3d quality @ $35; charge for case, carting, export duties, etc., $12; com. $2\frac{1}{2}$%; the gross weight was 60 lb., tare, 15 lb. Make out the invoice and find the amount of duty @ $2.50 a lb., and 25% *ad valorem*. *Ans.* $151.

92. NÜRNBURG, APRIL 3, 1874

KUHN BROS., Philadelphia,

Bo't of GEO. S. ECKHARDT

F B.	1 case.		
No. 48.	25 Musical Boxes, 4 airs	@ 20 Florin.	Fl.
		Com. 5%.	
	Duty,	Case and Emb.	20
	Mus. Inst.		Fl.
	$ @ 30% = $		

What was the duty in currency, gold being 112?

Ans. $82.95.

93. At what time between 4 and 5 o'clock do the hour and minute hands of a clock point in opposite directions?

Ans. $54\frac{6}{11}$ minutes past 4.

94. At what time between 6 and 7 o'clock do the hour and minute hands make equal angles with a line from 12 to 6?

Ans. $27\frac{9}{13}$ min. past 6.

95. A merchant in St. Petersburg wishes to remit sufficient money to Philadelphia to settle a debt there of $5000. If $1 = 1 rouble 40 copecks, direct exchange, and 1 rouble 25 copecks = 4 reichsmarks in Hamburg, 2 reichsmarks = 1.2 guilders, 12 guilders = £1, £1 = $4.87 in New York, and exchange in New York on Philadelphia is $\frac{1}{4}$% discount; which will be the most advantageous, the direct exchange, or through Hamburg, Amsterdam, London, and New York?

Ans. The circular, 599 roubles, 20.5 copecks.

96. What is the balance of the following account, and when will it become due? *Ans.* Bal., $747.25; due May 20.

DR. A. E. THOMAS IN ACCOUNT WITH R. MAY. CR.

1874.					1874.				
Jan.	10	To mdse. at 3 mos.,	475	00	May	20	By cash,	547	50
Feb.	1	" " " 4 mos.,	497	50	June	9	" "	100	00
Apr.	30	" sundries,	245	00	July	31	" check,	200	00
June	21	" mdse. at 60 days,	377	25					

97. Required the cash balance of the following account, July 1, 1875, interest 6%. *Ans.* $430.57.

DR. JAMES GOULD IN ACCOUNT WITH ROBERT LINCOLN. CR.

1875.					1875.				
Jan.	1	To mdse. on 2 mos.,	275	00	Mar.	12	By cash,	100	00
Mar	15	" " " 3 mos.,	349	75	May	25	" "	250	00
Apr.	10	" " " 4 mos.,	420	00	June	30	" "	173	25
May	1	" " " 30 da.,	127	25	July	1	" "	219	75

98. A farmer having a pair of good oxen, fully shod, agreed to exchange them with one of his neighbors for a valuable horse on the following terms: for the oxen 1 cent should be paid for the first shoe, 2 for the second, and so on in geometrical progression for all the shoes, while for the horse 2 cents should be paid for the first shoe, 4 for the sec-

ond, and so on; which brought the larger price, the oxen or the horse? *Ans.* Oxen, $655.05 more

99. Find the least possible whole number which divided by 32 will leave 25 for a remainder, divided by 25 will leave 19, and divided by 19 will leave 11. *Ans.* 5369.

100. Required the least three numbers which divided by 15 will leave 14 remainder, divided by 14 will leave 13 remainder, and so on to unity.

Ans. 360,359; 720,719; 1,081,079.

101. A grocer offers to take a young man into partnership on condition that if he advances $2000 he will allow him $620 per annum for his services and the use of his money, but if he advances $3000, he will allow $680; what was the per cent. offered for the use of the money, and what the salary? *Ans.* $500; 6%.

102. A teacher divided prizes to the amount of $50 among 100 students, giving to the different grades prizes worth respectively $5, $$2\frac{1}{2}$, $1, and $$\frac{1}{4}$ each; required the number of students in each grade. *Ans.* 1, 3, 18, 78; 1, 1, 24, 74.

103. A gentleman paid $8000 for a house, which he sells after a time for $10,705.808, gaining 6 per cent. compound interest, and invested the money in a perpetuity commencing in 9 years from the time he bought the house; what is the annuity, and how long did he own the house?

Ans. Annuity, $810.98; 5 years.

104. Mr. Framley, having an estate of $8000, disposed of it thus by his will: his wife was to receive one-half, and the remainder to be divided between his two children in such a manner that their shares at 7% simple interest should amount to the same sum when they reached the age of 21 years; their ages, at the time of their father's death, were 10 yr. 3 mo. and 12 yr. 5 mo. respectively; what was each child's share? *Ans.* $$1909.54\frac{138}{503}$; $$2090.45\frac{365}{503}$.

105. A takes 5 shares of a new series in a building association which runs out in $9\frac{1}{2}$ years; B, at the same time, begins to deposit $5 a month with a savings bank, at 4%, compounded semi-annually, interest beginning at the end of

the first semi-annual period; at the end of $9\frac{1}{2}$ years he withdraws his savings and interest; how much more interest does A receive than B, and what is the equated rate % on each investment? *Ans.* \$314.78; A's, 15.74%; B's, 4.22%.

106. Mr. Johnson has a garden 160 feet long and 105 feet wide; he wishes to raise the surface 5 inches by using the earth taken from a ditch 3 feet wide dug around it within the fence, but finds that this earth loses 10% in bulk after being spread over the garden; what must be the depth of the ditch? *Ans.* $4\frac{361}{666}$ ft.

107. Three men buy a grindstone 27 inches in diameter, with a square hole whose diagonal is 3 inches, paying for it \$5.20, of which A paid 20% more than B, and B 10% more than C; what part of the diameter must each grind off, A taking his share first, and then B, and what did each pay?

Ans. A, 5.761 in., \$$2.00\frac{40}{57}$; B, 6.423 in., \$$1.67\frac{43}{171}$; C, 11.816 in., \$$1.52\frac{8}{171}$.

108. Three men, Black, White, and Gray, bought a conical stack of hay, but finding the top and bottom somewhat damaged, it was agreed that Mr. Black, who took the top, should have 10% more than Mr. White, and Mr. Gray should have 8% more than Mr. White. The stack was 18 feet high and 120 feet in circumference, and contained $7\frac{1}{2}$ tons at \$9 a ton. Required the amount of hay, and the number of feet of the height of the stack that each one receives.

Ans. Black, $2\frac{63}{106}$ tons, 12.636 ft.; White, $2\frac{19}{53}$ tons, 3.038 ft.; Gray, $2\frac{29}{53}$ tons, 2.326 ft.

109. Mr. Smith and Mr. Johnson rented 30 acres of pasture for 12 weeks, Mr. Smith to have the grass then on the field, and Mr. Johnson what grew during the time they rented it; how many horses was each limited to pasture, and how much should each pay, if 3 acres will keep 12 horses 4 weeks, and 7 acres will keep 21 horses 8 weeks, the whole rent being \$80?

Ans. $\begin{cases}\text{Smith keeps 20 horses, and pays \$20.}\\ \text{Johnson keeps 60 horses, and pays \$60.}\end{cases}$

110. A boy carrying apples to market was asked how

many he had; he answered that he had not counted them, but that he noticed if he picked them up 2 at a time, there was 1 left; if he picked them up 3 at a time, there were 2 left; if 4 at a time, there were 3 left; if 5 at a time, there were 4 left; if 6 at a time, there were 5 left; but if 7 at a time, there were none left; how many apples did the boy have? *Ans.* 119.

111. A man bought in 1859 a ticket from Jacksonville, Florida, to Boston for $35; stopping at Charleston he paid a hotel bill of 19 s. 8 d., and bought a book for 5 s. 8 d.; at New York he paid 4 s. for a visit to the theatre, 3 s. 6 d. for trifling expenses, and a hotel bill for 4 days at 32 s. per day; in Boston he bought a hat for 32 s, spent $4\frac{1}{2}$ hours in riding out to Mount Auburn and other places of interest at 9 s. an hour, bought some photographs for 2 s. 6 d., and paid a hotel bill of 45 shillings; he was then obliged to borrow from a friend to pay his fare to Chicago; how much did he need to borrow if the fare was $25, and he had started from Jacksonville with $100? *Ans.* 2.36\frac{17}{28}$.

112. On April 24, Mr. Stanton buys 20 shares Pennsylvania R. R. at $66\frac{1}{2}$ regular, and 100 shares at $66\frac{7}{8}$ b 30, depositing $400 as a margin. On May 1 a privileged subscription at par is declared of 1 share for every six registered and one for any fractional part of six shares, which he takes up May 2. On May 30 is paid a semi-annual dividend of 5% on the stock held previous to May 1, the time at which the dividend is declared. On May 10 he sells 116 shares (100 b 30 and 16 new) ex div. @ $63\frac{1}{2}$, and on May 20 he sells 24 shares (20 regular and 4 new) ex div. @ 65. Now reckoning the interest on the money invested and borrowed, and the brokerage at $\frac{1}{4}$%, how much does he clear upon the transaction, and what per cent. does he make upon the money invested? *Ans.* $149.78; 37.87$\frac{35}{48}$+%.

The Connecticut and Vermont rules for Partial Payments are as follows:

CONNECTICUT RULE.

"*Compute the interest to the time of the first payment, if that be one year or more from the time that interest commenced; add it to the principal, and deduct the payment from the sum total. If there be after payments made, compute the interest on the balance due to the next payment, and then deduct the payment as above; and in like manner from one payment to another till all the payments are absorbed;* PROVIDED, *the time between one payment and another be one year or more. But if any payments be made before one year's interest hath accrued, then compute the interest, or the principal sum due to the obligation, for one year, add it to the principal, and compute the interest on the sum paid, from the time it was paid up to the end of the year; add it to the sum paid, and deduct that sum from the principal and interest added as above.**

"*If any payments be made for a less sum than the interest arisen at the time of such payment, no interest is to be computed, but only on the principal sum for any period.*"—Kirby's Reports.

*NOTE.—"If a year does not extend beyond the time of payment; but if it does, then find the amount of the principal remaining unpaid up to the time of settlement, likewise the amount of the payment or payments from the time they were paid to the time of settlement, and deduct the sum of these several amounts from the amount of the principal."

VERMONT RULE.

I. "*When payments are made on notes, bills, or similar obligations, whether payable on demand or at a specified time, with interest, such payments hall be applied;* FIRST, TO LIQUIDATE THE INTEREST *that has accrued at the time of such payments, and* SECONDLY, TO THE EXTINGUISHMENT OF THE PRINCIPAL.

II. "*The annual interests that shall remain unpaid on notes, bills, or similar obligations, whether payable on demand or at a specified time, 'with interest annually,' shall be* SUBJECT *to simple* INTEREST *from the time they become due to the time of final settlement.*

III. "*If payments have not been made in any year, reckoning from the time such annual interest began to accrue, the amount of such payments at the end of the year, with interest thereon from the time of payment, shall be applied;* FIRST, *to* LIQUIDATE *the* SIMPLE INTEREST *that has accrued from the* UNPAID ANNUAL INTERESTS; SECONDLY, *to* LIQUIDATE *the* ANNUAL INTERESTS *that have become due;* THIRDLY, *to the* EXTINGUISHMENT *of the* PRINCIPAL.

NOTE.—The New Hampshire Rule, when partial payments are made on notes "with interest annually," is essentially the same as the preceding. But "where payments are made expressly on account of interest accruing but not then due, they are applied when the interest falls due, *without interest* on such payments."

APPENDIX.

TABLE.

Amount of $1 at Compound Interest in any number of years.

Yr.	2 per cent.	2½ per cent.	3 per cent.	3½ per cent.	4 per cent.	4½ per cent.
1	1.0200 0000	1.0250 0000	1.0300 0000	1.0350 0000	1.0400 0000	1.0450 0000
2	1.0404 0000	1.0506 2500	1.0609 0000	1.0712 2500	1.0816 0000	1.0920 2500
3	1.0612 0800	1.0768 9062	1.0927 2700	1.1087 1787	1.1248 6400	1.1411 6612
4	1.0824 3216	1.1038 1289	1.1255 0881	1.1475 2300	1.1698 5856	1.1925 1860
5	1.1040 8080	1.1314 0821	1.1592 7407	1.1876 8631	1.2166 5290	1.2461 8104
6	1.1261 6242	1.1596 9342	1.1940 5230	1.2292 5533	1.2653 1902	1.3022 6012
7	1.1486 8567	1.1886 8575	1.2298 7387	1.2722 7926	1.3159 3178	1.3608 6183
8	1.1716 5938	1.2184 0290	1.2667 7008	1.3168 0904	1.3685 6905	1.4221 0061
9	1.1950 9257	1.2488 6297	1.3047 7318	1.3628 9735	1.4233 1181	1.4860 9514
10	1.2189 9442	1.2800 8454	1.3439 1638	1.4105 9876	1.4802 4428	1.5529 6942
11	1.2433 7431	1.3120 8666	1.3842 3387	1.4599 6972	1.5394 5406	1.6228 5305
12	1.2682 4179	1.3448 8882	1.4257 6089	1.5110 6866	1.6010 3222	1.6958 8143
13	1.2936 0663	1.3785 1104	1.4685 3371	1.5639 5606	1.6650 7351	1.7721 9610
14	1.3194 7876	1.4129 7382	1.5125 8972	1.6186 9452	1.7316 7645	1.8519 4492
15	1.3458 6834	1.4482 9817	1.5579 6742	1.6753 4883	1.8009 4351	1.9352 8244
16	1.3727 8570	1.4845 0562	1.6047 0644	1.7339 8604	1.8729 8125	2.0223 7015
17	1.4002 4142	1.5216 1826	1.6528 4763	1.7946 7555	1.9479 0050	2.1133 7681
18	1.4282 4625	1.5596 5872	1.7024 3306	1.8574 8920	2.0258 1652	2.2084 7877
19	1.4568 1117	1.5986 5019	1.7535 0605	1.9225 0132	2.1068 4918	2.3078 6031
20	1.4859 4740	1.6386 1644	1.8061 1123	1.9897 8886	2.1911 2314	2.4117 1402
21	1.5156 6634	1.6795 8185	1.8602 9457	2.0594 3147	2.2787 6807	2.5202 4116
22	1.5459 7967	1.7215 7140	1.9161 0341	2.1315 1158	2.3699 1879	2.6336 5201
23	1.5768 9926	1.7646 1068	1.9735 8651	2.2061 1448	2.4647 1555	2.7521 7635
24	1.6084 3725	1.8087 2595	2.0327 9411	2.2833 2849	2.5633 0417	2.8760 1383
25	1.6406 0599	1.8539 4410	2.0937 7793	2.3632 4498	2.6658 3633	3.0054 3446
26	1.6734 1811	1.9002 9270	2.1565 9127	2.4459 5856	2.7724 6979	3.1406 7901
27	1.7068 8648	1.9478 0002	2.2212 8901	2.5315 6711	2.8833 6858	3.2820 0956
28	1.7410 2421	1.9964 9502	2.2879 2768	2.6201 7196	2 9087 0332	3.4296 9999
29	1.7758 4469	2.0464 0739	2.3565 6551	2.7118 7798	3.1186 5145	3.5840 3649
30	1.8113 6158	2.0975 6758	2.4272 6247	2.8067 9370	3.2433 9751	3.7453 1813
31	1.8475 8882	2.1500 0677	2.5000 8035	2.9050 3148	3.3731 3341	3.9138 5745
32	1.8845 4059	2.2037 5694	2.5750 8276	3.0067 0759	3.5080 5875	4.0899 8104
33	1.9222 3140	2.2588 5086	2.6523 3524	3.1119 4235	3.6483 8110	4.2740 3018
34	1.9606 7603	2.3153 2213	2.7319 0530	3.2208 6033	3.7943 1634	4.4663 6154
35	1.9998 8955	2.3732 0519	2.8138 6245	3.3335 9045	3.9460 8899	4.6673 4781
36	2.0398 8734	2.4325 3532	2.8982 7833	3.4502 6611	4.1039 3255	4.8773 7846
37	2.0806 8509	2.4933 4870	2.9852 2668	3.5710 2543	4.2680 8986	5.0968 6040
38	2.1222 9879	2.5556 8242	3.0747 8348	3.6960 1132	4.4388 1345	5.3262 1921
39	2.1647 4477	2.6195 7448	3.1670 2698	3.8253 7171	4.6163 6599	5.5658 9908
40	2.2080 3966	2.6850 6384	3.2620 3779	3.9592 5972	4.8010 2063	5.8163 6454
41	2.2522 0046	2.7521 9043	3.3598 9893	4.0978 3381	4.9930 6145	6.0781 0094
42	2.2972 4447	2.8209 9520	3.4606 9589	4.2412 5799	5.1927 8391	6.3516 1548
43	2.3431 8936	2.8915 2008	3.5645 1677	4.3897 0202	5.4004 9527	6.6374 3818
44	2.3900 5314	2.9638 0808	3.6714 5227	4.5433 4160	5.6165 1508	6.9361 2290
45	2.4378 5421	3.0379 0328	3.7815 9584	4.7023 5855	5.8411 7568	7.2482 4843
46	2.4866 1129	3.1138 5086	3.8950 4372	4.8669 4110	6.0748 2271	7.5744 1961
47	2.5363 4351	3.1916 9713	4.0118 9503	5.0372 8404	6.3178 1562	7.9152 6849
48	2.5870 7039	3.2714 8956	4.1322 5188	5.2135 8898	6.5705 2824	8.2714 5557
49	2.6388 1179	3.3532 7680	4.2562 1944	5.3960 6459	6.8333 4937	8.6436 7107
50	2.6915 8803	3.4371 0872	4.3839 0602	5 5849 2686	7.1066 8335	9.0326 3627
51	2.7454 1979	3.5230 3644	4.5154 2320	5.7803 9930	7.3909 5068	9.4391 0490
52	2.8003 2819	3.6111 1235	4 6508 8590	5.9827 1327	7.6865 8871	9.8638 6463
53	2.8563 3475	3.7013 9016	4.7904 1247	6.1921 0824	7.9940 5226	10.3077 3853
54	2.9134 6144	3.7939 8491	4.9341 2485	6.4088 3202	8.3138 1435	10.7715 8077
55	2.9717 3067	3.8887 7303	5.0821 4859	6.6331 4114	8.6463 6692	11.2563 0817

TABLE.

Amount of $1 at Compound Interest in any number of years.

Yr.	5 per cent.	6 per cent.	7 per cent.	8 per cent.	9 per cent.	10 per cent.
1	1.0500 000	1.0600 000	1.0700 000	1.0800 000	1.0900 000	1.1000 000
2	1.1025 000	1.1236 000	1.1449 000	1.1664 000	1.1881 000	1.2100 000
3	1.1576 250	1.1910 160	1.2250 430	1.2597 120	1.2950 290	1.3310 000
4	1.2155 063	1.2624 770	1.3107 960	1.3604 890	1.4115 816	1.4641 000
5	1.2762 816	1.3382 256	1.4025 517	1.4693 281	1.5386 240	1.6105 100
6	1.3400 956	1.4185 191	1.5007 304	1.5868 743	1.6771 001	1.7715 610
7	1.4071 004	1.5036 303	1.6057 815	1.7138 243	1.8280 391	1.9487 171
8	1.4774 554	1.5938 481	1.7181 862	1.8509 302	1.9925 626	2.1435 888
9	1.5513 282	1.6894 790	1.8384 592	1.9990 046	2.1718 933	2.3579 477
10	1.6288 946	1.7908 477	1.9671 514	2.1589 250	2.3673 637	2.5937 425
11	1.7103 394	1.8982 986	2.1048 520	2.3316 390	2.5804 264	2.8531 167
12	1.7958 563	2.0121 965	2.2521 916	2.5181 701	2.8126 648	3.1384 284
13	1.8856 491	2.1329 283	2.4098 450	2.7196 237	3.0658 046	3.4522 712
14	1.9799 316	2.2609 040	2.5785 342	2.9371 936	3.3417 270	3.7974 983
15	2.0789 282	2.3965 582	2.7590 315	3.1721 691	3.6424 825	4.1772 482
16	2.1828 746	2.5403 517	2.9521 638	3.4259 426	3.9703 059	4.5949 730
17	2.2920 183	2.6927 728	3.1588 152	3.7000 181	4.3276 334	5.0544 703
18	2.4066 192	2.8543 892	3.3799 323	3.9960 195	4.7171 204	5.5599 173
19	2.5269 502	3.0255 995	3.6165 275	4.3157 011	5.1416 613	6.1159 390
20	2.6532 977	3.2071 355	3.8696 845	4.6609 571	5.6044 108	6.7275 000
21	2.7859 626	3.3995 636	4.1405 624	5.0338 337	6.1088 077	7.4002 499
22	2.9252 607	3.6035 374	4.4304 017	5.4365 404	6.6586 004	8.1402 749
23	3.0715 238	3.8197 497	4.7405 299	5.8714 637	7.2578 745	8.9543 024
24	3.2250 999	4.0489 346	5.0723 670	6.3411 807	7.9110 832	9.8497 327
25	3.3863 549	4.2918 707	5.4274 326	6.8484 752	8.6230 807	10.8347 059
26	3.5556 727	4.5493 830	5.8073 529	7.3963 532	9.3991 579	11.9181 765
27	3.7334 563	4.8223 459	6.2138 676	7.9880 615	10.2450 821	13.1099 942
28	3.9201 291	5.1116 867	6.6488 384	8.6271 064	11.1671 395	14.4209 936
29	4.1161 356	5.4183 879	7.1142 571	9.3172 749	12.1721 821	15.8630 939
30	4.3219 424	5.7434 912	7.6122 550	10.0626 569	13.2676 785	17.4494 023
31	4.5380 395	6.0881 006	8.1451 129	10.8676 694	14.4617 695	19.1943 425
32	4.7649 415	6.4533 867	8.7152 708	11.7370 830	15.7633 288	21.1137 768
33	5.0031 885	6.8405 899	9.3253 398	12.6760 496	17.1820 284	23.2251 544
34	5.2533 480	7.2510 253	9.9781 135	13.6901 336	18.7284 109	25.5476 699
35	5.5160 154	7.6860 868	10.6765 815	14.7853 443	20.4139 679	28.1024 369
36	5.7918 161	8.1472 520	11.4239 422	15.9681 718	22.2512 250	30.9126 805
37	6.0814 069	8.6360 871	12.2236 181	17.2456 256	24.2538 353	34.0039 486
38	6.3854 773	9.1542 524	13.0792 714	18.6252 756	26.4366 805	37.4043 434
39	6.7047 512	9.7035 075	13.9948 204	20.1152 977	28.8159 817	41.1447 778
40	7.0399 887	10.2857 179	14.9744 578	21.7245 215	31.4094 200	45.2592 556
41	7.3919 882	10.9028 610	16.0226 699	23.4624 832	34.2362 679	49.7851 811
42	7.7615 876	11.5570 327	17.1442 568	25.3394 819	37.3175 320	54.7636 992
43	8.1496 669	12.2504 546	18.3443 548	27.3666 404	40.6761 098	60.2400 692
44	8.5571 503	12.9854 819	19.6284 596	29.5559 717	44.3369 597	66.2640 761
45	8.9850 078	13.7646 108	21.0024 518	31.9204 494	48.3272 861	72.8904 837
46	9.4342 582	14.5904 875	22.4726 234	34.4740 853	52.6767 419	80.1795 321
47	9.9059 711	15.4659 167	24.0457 070	37.2320 122	57.4176 486	88.1974 853
48	10.4012 697	16.3938 717	25.7289 065	40.2105 731	62.5852 370	97.0172 338
49	10.9213 331	17.3775 040	27.5299 300	43.4274 190	68.2179 083	106.7189 572
50	11.4673 998	18.4201 543	29.4570 251	46.9016 125	74.3575 201	117.3908 529
51	12.0407 698	19.5253 635	31.5190 168	50.6537 415	81.0496 969	129.1299 382
52	12.6428 083	20.6968 853	33.7253 480	54.7060 408	88.3441 696	142.0429 320
53	13.2749 487	21.9386 985	36.0861 224	59.0825 241	96.2951 449	156.2472 252
54	13.9386 961	23.2550 204	38.6121 509	63.8091 260	104.9617 079	171.8719 477
55	14.6356 309	24.6503 216	41.3150 015	68.9138 561	114.4082 616	189.0591 425

TABLE

Showing the present value of $1 per annum from 1 yr. to 55.

Yr.	4 per cent.	5 per cent.	6 per cent.	7 per cent.	8 per cent.	10 per cent.
1	.961538	.952381	.943396	.934579	.925926	.909091
2	1.886095	1.859410	1.833393	1.808018	1.783265	1.735537
3	2.775091	2.723248	2.673012	2.624316	2.577097	2.486852
4	3.629895	3.545951	3.465106	3.387211	3.312127	3.169865
5	4.451822	4.329477	4.212364	4.100197	3.992710	3.790787
6	5.242137	5.075692	4.917324	4.766540	4.622880	4.355261
7	6.002055	5.786373	5.582381	5.389289	5.206370	4.868419
8	6.732745	6.463213	6.209794	5.971299	5.746639	5.334926
9	7.435332	7.107822	6.801692	6.515232	6.246888	5.759024
10	8.110896	7.721735	7.360087	7.023582	6.710081	6.144567
11	8.760477	8.306414	7.886875	7.498674	7.138964	6.495061
12	9.385074	8.863252	8.383844	7.942686	7.536078	6.813692
13	9.985648	9.393573	8.852683	8.357651	7.903776	7.103356
14	10.563123	9.898641	9.294984	8.745468	8.244237	7.366687
15	11.118387	10.379658	9.712249	9.107914	8.559479	7.606080
16	11.652296	10.837770	10.105895	9.446649	8.851369	7.823709
17	12.165669	11.274066	10.477260	9.763223	9.121638	8.021553
18	12.659297	11.689587	10.827603	10.059087	9.371887	8.201412
19	13.133939	12.085321	11.158116	10.335595	9.603599	8.364920
20	13.590326	12.462210	11.469921	10.594014	9.818147	8.513564
21	14.029160	12.821153	11.764077	10.835527	10.016803	8.648694
22	14.451115	13.163003	12.041582	11.061241	10.200744	8.771540
23	14.856842	13.488574	12.303379	11.272187	10.371059	8.883218
24	15.246963	13.798642	12.550358	11.469334	10.528758	8.984744
25	15.622080	14.093945	12.783356	11.653583	10.674776	9.077040
26	15.982769	14.375185	13.003166	11.825779	10.809978	9.160945
27	16.329586	14.643034	13.210534	11.986709	10.935165	9.237223
28	16.663063	14.898127	13.406164	12.137111	11.051078	9.306567
29	16.983715	15.141074	13.590721	12.277674	11.158406	9.369606
30	17.292033	15.372451	13.764831	12.409041	11.257783	9.426914
31	17.588494	15.592811	13.929086	12.531814	11.349799	9.479013
32	17.873552	15.802677	14.084043	12.646555	11.434999	9.526376
33	18.147646	16.002549	14.230230	12.753790	11.513888	9.569432
34	18.411198	16.192904	14.368141	12.854009	11.586934	9.608575
35	18.664613	16.374194	14.498246	12.947672	11.654568	9.644159
36	18.908282	16.546352	14.620987	13.035208	11.717193	9.676508
37	19.142579	16.711287	14.736780	13.117017	11.775179	9.705917
38	19.367864	16.867893	14.846019	13.193473	11.828869	9.732651
39	19.584485	17.017041	14.949075	13.264928	11.878582	9.756956
40	19.792774	17.159086	15.046297	13.331709	11.924613	9.779051
41	19.993052	17.294368	15.138016	13.394120	11.967235	9.799137
42	20.185627	17.423208	15.224543	13.452449	12.006699	9.817397
43	20.370795	17.545912	15.306173	13.506962	12.043240	9.833998
44	20.548841	17.662773	15.383182	13.557908	12.077074	9.849089
45	20.720040	17.774070	15.455832	13.605522	12.108402	9.862808
46	20.884654	17.880067	15.524370	13.650020	12.137409	9.875280
47	21.042936	17.981016	15.589028	13.691608	12.164267	9.886618
48	21.195131	18.077158	15.650027	13.730474	12.189136	9.896926
49	21.341472	18.168722	15.707572	13.766799	12.212163	9.906296
50	21.482185	18.255925	15.761861	13.800746	12.233485	9.914814
51	21.617485	18.338977	15.813076	13.832473	12.253227	9.922559
52	21.747582	18.418073	15.861393	13.862124	12.271506	9.929599
53	21.872675	18.493403	15.906974	13.889836	12.288432	9.935999
54	21.992957	18.565146	15.949976	13.915735	12.304103	9.941817
55	22.108612	18.633472	15.990543	13.939939	12.318614	9.947107

TABLE

Showing the values of Annuities on Single Lives, according to the Carlisle Table of Mortality.

Age.	4 per ct.	5 per ct.	6 per ct.	7 per ct.	Age.	4 per ct.	5 per ct.	6 per ct.	7 per ct.
0	14.28164	12.083	10.439	9.177	52	12.25793	11.154	10.208	9.392
1	16.55455	13.995	12.078	10.605	53	11.94503	10.892	9.988	9.205
2	17.72616	14.983	12.925	11.342	54	11.62673	10.624	9.761	9.011
3	18.71508	15.824	13.652	11.978	55	11.29961	10.347	9.524	8.807
4	19.23133	16.271	14.042	12.322	56	10.96607	10.063	9.280	8.595
5	19.59203	16.590	14.325	12.574	57	10.62559	9.771	9.027	8.375
6	19.74502	16.735	14.460	12.698	58	10.28647	9.478	8.772	8.153
7	19.79019	16.790	14.518	12.756	59	9.96331	9.199	8.529	7.940
8	19.76443	16.786	14.526	12.770	60	9.66333	8.940	8.304	7.743
9	19.69114	16.742	14.500	12.754	61	9.39809	8.712	8.108	7.572
10	19.58339	16.669	14.448	12.717	62	9.13676	8.487	7.913	7.403
11	19.45857	16.581	14.384	12.669	63	8.87150	8.258	7.714	7.229
12	19.33493	16.494	14.321	12.621	64	8.59330	8.016	7.502	7.042
13	19.20937	16.406	14.257	12.572	65	8.30719	7.765	7.281	6.847
14	19.08182	16.316	14.191	12.522	66	8.00966	7.503	7.049	6.641
15	18.95534	16.227	14.126	12.473	67	7.69980	7.227	6.803	6.421
16	18.83636	16.144	14.067	12.429	68	7.37976	6.941	6.546	6.189
17	18.72111	16.066	14.012	12.389	69	7.04881	6.643	6.277	5.945
18	18.60656	15.987	13.956	12.348	70	6.70936	6.336	5.998	5.690
19	18.48649	15.904	13.897	12.305	71	6.35773	6.015	5.704	5.420
20	18.36170	15.817	13.835	12.259	72	6.02548	5.711	5.424	5.162
21	18.23196	15.726	13.769	12.210	73	5.72465	5.435	5.170	4.927
22	18.09386	15.628	13.697	12.156	74	5.45812	5.190	4.944	4.719
23	17.95016	15.525	13.621	12.098	75	5.23901	4.989	4.760	4.549
24	17.80058	15.417	13.541	12.037	76	5.02399	4.792	4.579	4.382
25	17.64486	15.303	13.456	11.972	77	4.82473	4.609	4.410	4.227
26	17.48586	15.187	13.368	11.904	78	4.62166	4.422	4.238	4.067
27	17.32023	15.065	13.275	11.832	79	4.39345	4.210	4.040	3.883
28	17.15412	14.942	13.182	11.759	80	4.18289	4.015	3.858	3.713
29	16.99683	14.827	13.096	11.693	81	3.95309	3.799	3.656	3.523
30	16.85215	14.723	13.020	11.636	82	3.74634	3.606	3.474	3.352
31	16.70511	14.617	12.942	11.578	83	3.53409	3.406	3.286	3.174
32	16.55246	14.506	12.860	11.516	84	3.32856	3.211	3.102	2.999
33	16.39072	14.387	12.771	11.448	85	3.11515	3.009	2.909	2.815
34	16.21943	14.260	12.675	11.374	86	2.92831	2.830	2.739	2.652
35	16.04123	14.127	12.573	11.295	87	2.77593	2.685	2.599	2.519
36	15.85577	13.987	12.465	11.211	88	2.68337	2.597	2.515	2.439
37	15.66586	13.843	12.354	11.124	89	2.57704	2.495	2.417	2.344
38	15.47129	13.695	12.239	11.033	90	2.41621	2.339	2.266	2.198
39	15.27184	13.542	12.120	10.939	91	2.39835	2.321	2.248	2.180
40	15.07363	13.390	12.002	10.845	92	2.49199	2.412	2.337	2.266
41	14.88314	13.245	11.890	10.757	93	2.59955	2.518	2.440	2.367
42	14.69466	13.101	11.779	10.671	94	2.64976	2.569	2.492	2.419
43	14.50529	12.957	11.668	10.585	95	2.67433	2.596	2.522	2.451
44	14.30874	12.806	11.551	10.494	96	2.62779	2.555	2.486	2.420
45	14.10460	12.648	11.428	10.397	97	2.49204	2.428	2.368	2.309
46	13.88928	12.480	11.296	10.292	98	2.33222	2.278	2.227	2.177
47	13.66208	12.301	11.154	10.178	99	2.08700	2.045	2.004	1.964
48	13.41914	12.107	10.998	10.052	100	1.65282	1.624	1.596	1.569
49	13.15312	11.892	10.823	9.908	101	1.21005	1.192	1.175	1.159
50	12.86902	11.660	10.631	9.749	102	0.76183	0.753	0.744	0.735
51	12.56581	11.410	10.422	9.573	103	0.32051	0.317	0.314	0.312

.NSURANCE TABLE FOR COMPUTING SHORT RATES.

Insurances for periods less than one year will be at the following rates. Risks upon Grain, Pork, Wool, and other produce, are sometimes wanted for very short terms.

ANNUAL PREMIUM.	Cts. 30	Cts. 35	Cts. 40	Cts. 45	Cts. 50	Cts. 55	Cts. 60	Cts. 65	Cts. 70	Cts. 75	Cts. 80	Cts. 85	Cts. 90	Cts. 1.00	Cts. 1.10	Cts. 1.25	Cts. 1.50	Cts. 1.75	Cts. 2.00	Cts. 2.25	Cts. 2.50	Cts. 2.75	Cts. 3.00	Cts. 3.25	Cts. 3.50	Cts. 3.75	Cts. 4.00	Cts. 5.00
2 Days or less,	1	1	1	1	2	2	2	2	2	3	3	3	3	3	3	4	5	6	7	8	8	9	10	11	12	13	14	17
5 Days or less,	2	2	3	3	3	4	4	4	5	5	5	6	6	7	7	9	10	12	14	15	17	18	20	22	24	25	27	34
10 Days or less,	3	3	4	4	5	6	6	7	7	8	8	9	9	10	11	12	15	18	20	22	25	27	30	33	35	38	40	50
15 Days or less,	4	4	5	6	6	8	8	8	10	10	10	12	12	14	14	17	20	25	27	30	33	36	40	44	47	50	53	67
20 Days or less,	5	5	6	7	8	9	9	10	11	11	12	13	14	15	17	20	25	30	33	40	42	45	50	55	58	63	66	80
1 Month or less,	6	7	8	9	10	11	12	13	14	15	16	17	18	20	22	25	30	35	40	45	50	55	60	65	70	75	80	1.00
45 Days or less,	8	9	11	12	13	14	16	18	18	19	21	22	24	25	27	31	38	43	50	56	63	68	75	82	88	93	1.00	1.25
2 Months or less,	9	10	12	14	15	16	18	20	21	22	24	25	27	30	33	38	45	52	60	67	75	82	90	98	1.05	1.13	1.20	1.50
75 Days or less,	11	12	14	16	18	20	22	24	25	26	28	30	32	35	38	44	50	60	70	78	88	96	1.05	1.13	1.23	1.31	1.40	1.75
3 Months or less,	12	14	16	18	20	22	24	26	28	30	32	34	36	40	44	50	60	70	80	90	1.00	1.10	1.20	1.30	1.40	1.50	1.60	2.00
4 Months or less,	15	17	20	22	25	27	30	32	35	37	39	42	45	50	55	62	75	88	1.00	1.12	1.25	1.37	1.50	1.62	1.75	1.87	2.00	2.50
5 Months or less,	18	21	24	27	30	33	36	39	42	45	48	51	54	60	66	75	90	1.05	1.20	1.35	1.50	1.65	1.80	1.95	2.10	2.25	2.40	3.00
6 Months or less,	21	24	28	32	35	38	42	46	49	52	55	59	63	70	77	88	1.05	1.22	1.40	1.57	1.75	1.92	2.10	2.27	2.45	2.62	2.80	3.50
7 Months or less,	22	26	30	34	37	41	45	49	52	56	60	63	67	75	82	93	1.12	1.30	1.50	1.68	1.87	2.06	2.25	2.43	2.62	2.81	3.00	3.75
8 Months or less,	24	28	32	36	40	44	48	52	56	60	64	68	72	80	88	1.00	1.20	1.40	1.60	1.80	2.00	2.20	2.40	2.60	2.80	3.00	3.20	4.00
9 Months or less,	26	31	35	40	43	46	51	55	59	64	68	72	76	85	93	1.06	1.27	1.48	1.70	1.91	2.12	2.34	2.55	2.76	2.97	3.17	3.40	4.25
10 Months or less,	27	32	37	42	45	49	54	58	63	67	72	76	81	90	1.00	1.12	1.35	1.57	1.80	2.02	2.25	2.47	2.70	2.92	3.15	3.37	3.60	4.50
11 Months or less,	29	33	39	43	47	52	57	61	66	71	76	80	85	95	1.04	1.18	1.42	1.66	1.90	2.13	2.37	2.61	2.85	3.08	3.32	3.56	3.90	4.75

NOTE.—The upper row of figures shows the rates for the year, from 30 cents up to 500 cents, [illegible] 5 per cent., and the rows of figures below them show the price under each, from 2 days to 11 months. Terms of any intermediate number of days or months are taken at the next higher rates; thus, a [illegible] day policy would command a [illegible] day rate as shown.

TABLE.

Annual Premium Rates for an Insurance of $1000.

	LIFE POLICIES. Payable at death only.					ENDOWMENT POLICIES. Payable as indicated, or at death, if prior		
Age.	Annual Payments. For Life	20 years	10 years	Single Payment	Age.	In 10 years.	In 15 years.	In 20 years.
20 to 25	$19 89	$27 39	$42 56	$326 58	20 to 25	$103 91	$66 02	$47 68
26	20 40	27 93	43 37	332 58	26	104 03	66 15	47 82
27	20 93	28 50	44 22	338 83	27	104 16	66 29	47 98
28	21 48	29 09	45 10	345 31	28	104 29	66 44	48 15
29	22 07	29 71	46 02	352 05	29	104 43	66 60	48 33
30	22 70	30 36	46 97	359 05	30	104 58	66 77	48 53
31	23 35	31 03	47 98	366 33	31	104 75	66 96	48 74
32	24 05	31 74	49 02	373 89	32	104 91	67 16	48 97
33	24 78	32 48	50 10	381 73	33	105 11	67 36	49 [illegible]
34	25 56	33 26	51 22	389 88	34	105 31	67 60	49 49
35	26 38	34 08	52 40	398 34	35	105 53	67 85	49 79
36	27 25	34 93	53 63	407 11	36	105 75	68 12	50 11
37	28 17	35 83	54 91	416 21	37	106 00	68 41	50 47
38	29 15	36 78	56 24	425 64	38	106 28	68 73	50 86
39	30 19	37 78	57 63	435 42	39	106 58	69 09	51 30
40	31 30	38 83	59 09	445 55	40	106 90	69 49	51 78
41	32 47	39 93	60 60	456 04	41	107 26	69 92	52 31
42	33 72	41 10	62 19	466 89	42	107 65	70 40	52 89
43	35 05	42 34	63 84	478 11	43	108 08	70 92	53 54
44	36 46	43 64	65 57	489 71	44	108 55	71 50	54 25
45	37 97	45 03	67 37	501 69	45	109 07	72 14	55 04
46	39 58	46 50	69 26	514 04	46	109 65	72 86	55 91
47	41 30	48 07	71 25	526 78	47	110 30	73 66	56 89
48	43 13	49 73	73 32	539 88	48	111 01	74 54	57 96
49	45 09	51 50	75 49	553 33	49	111 81	75 51	59 15
50	47 18	53 38	77 77	567 13	50	112 68	76 59	60 45
51	49 40	55 38	80 14	581 24	51	113 64	77 77	61 90
52	51 78	57 51	82 63	595 66	52	114 70	79 07	63 48
53	54 31	59 79	85 22	610 36	53	115 86	80 51	63 22
54	57 02	62 22	87 94	625 33	54	117 14	82 09	67 14
55	59 91	64 82	90 79	640 54	55	118 54	83 82	69 24
56	63 00	67 60	93 78	655 99	56	120 09	85 73	
57	66 29	70 59	96 91	671 64	57	121 78	87 84	
58	69 82	73 78	100 21	687 48	58	123 64	90 15	
59	73 60	77 22	103 68	703 49	59	125 70	92 70	
60	77 63	80 91	107 35	719 65	60	127 96	95 50	
61	81 96	84 88	111 23	735 92	61	130 45		
62	86 58	89 16	115 32	752 26	62	133 19		
63	91 54	93 76	119 66	768 67	63	136 20		
64	96 86	98 73	124 28	785 10	64	139 52		
65	102 35	104 10	129 18	801 52	65	143 16		

APPENDIX.

LOGARITHMS.

NOTE.—The following brief treatise on Logarithms is added for the use of teachers and pupils who desire it in connection with this work. The matter is the same as in the author's work on Trigonometry. It may be used here or omitted, at the discretion of the teacher or as the course of study requires.

Logarithms are said to have been invented by Lord Napier of Scotland as early as 1594, and gave the author so high a reputation that Kepler dedicated a work to him in 1617; since which, high compliments have been paid him by succeeding mathematicians. The invention is of inestimable value in abbreviating lengthy calculations. It was improved by Henry Briggs, a contemporary, who, assuming 10 to be a basis, constructed a system much more convenient for ordinary purposes, corresponding to our system of numeration. The two systems are called the Napierian or Hyperbolic, and the Briggean or Common, Logarithms.

1. LOGARITHMS are a species of numbers used to abbreviate Multiplication, Division, Involution, and Evolution.

2. The *logarithm of a number* is the exponent denoting the power to which a fixed number must be raised in order to produce the first number.

3. This *fixed number* is called the *base* of the system. The base of the common system is 10.

4. Raising 10 to different powers, we have,

$10^0 = 1$; hence, 0 is the log of 1;
$10^1 = 10$ " 1 " 10;
$10^2 = 100$ " 2 " 100;
$10^3 = 1000$ " 3 " 1000;
etc.

5. From this we have the following principles:

PRIN. 1. *The logarithm of a number between* 1 *and* 10 *is between* 0 *and* 1, *and is, therefore, a decimal.*

PRIN. 2. *The logarithm of a number between* 10 *and* 100 *is between* 1 *and* 2, *and is, therefore,* 1 *and a decimal.* Thus, it has been found that the log. of 76 is 1.880814.

PRIN. 3. *The logarithm of a number between* 100 *and* 1000 *is between* 2 *and* 3, *and is, therefore,* 2 *and a decimal.* Thus, the log. of 458 is 2.660865.

6. When the logarithm consists of an integer and a decimal,

the integer is called the *characteristic*, and the decimal part the *mantissa*. Thus, in 2.660865 the 2 is the characteristic, and .660865 is the mantissa.

PROPERTIES OF LOGARITHMS.

PRIN. 1.—*The characteristic is always one less than the number of integral places in the number.*

For, from Art. 4, we see that the log. of 100 is 2, the log. of 1000 is 3, and of any number between 100 and 1000 it is 2 and a decimal; hence, the characteristic is one less than the number of integral places.

PRIN. 2.—*The logarithm of the base is* 1, *and the logarithm of* 1 *is zero.*

For, since $10^1 = 10$, the log. of 10 is 1; and since $10^0 = 1$, the logarithm of 1 is 0.

PRIN. 3.—*The characteristic of the logarithm of a decimal is negative, and is numerically one greater than the number of ciphers between the decimal point and the first significant figure.*

For, if we raise the base, 10, to powers which give decimals, we will have,

$$\begin{array}{llll} 10^0 = 1 & ; \text{hence,} & \log 1 & = 0; \\ 10^{-1} = .1 & \text{“} & \log .1 & = -1; \\ 10^{-2} = .01 & \text{“} & \log .01 & = -2; \\ 10^{-3} = .001 & \text{“} & \log .001 & = -3; \\ \quad \text{etc.} & & & \text{etc.} \end{array}$$

which proves the principle. Thus, the log. of .458 is $\bar{1}.660865$.

PRIN. 4.—*The logarithm of the product of two numbers is equal to the sum of the logarithms of those numbers.*

For, let M and N be any two numbers, and m and n their logarithms; then we shall have, according to the definition,

$$10^m = M, \qquad 10^n = N.$$

Multiplying these equations, member by member, we have,

$$10^{m+n} = M \times N.$$

Hence, $\quad \log (M \times N) = m + n;$ or, $= \log M + \log N.$

PRIN. 5.—*The logarithm of the quotient of two numbers equals the difference of the logarithms of those numbers.*

For, from the definition, we have,

$$10^m = M, \qquad 10^n = N.$$

Dividing the first by the second, we have,

$$10^{m-n} = \frac{M}{N}.$$

Hence, $\log\left(\frac{M}{N}\right) = m - n$, or, $= \log M - \log N$.

PRIN. 6.—*The logarithm of any power of a number is equal to the logarithm of the number multiplied by the exponent of the power.*

For, since

$$10^m = M,$$

if we raise both members to the nth power, we have,

$$10^{mn} = M^n.$$

Hence, $\log M^n = mn$, or, $= \log M \times n$.

PRIN. 7.—*The logarithm of the root of any number is equal to the logarithm of the number divided by the index of the root.*

For, since

$$10^m = M,$$

if we take the nth root of both members, we have,

$$10^{\frac{m}{n}} = \sqrt[n]{M}.$$

Hence, $\log \sqrt[n]{M} = \frac{m}{n}$, or, $\log M \div n$.

PRIN. 8.—*The logarithm of the product of any number multiplied by* 10 *is equal to the logarithm of the number increased by* 1.

Suppose $\log M = m$; then, by Prin. 4,

$\log (M \times 10) = \log M + \log 10$. But $\log 10 = 1$;

Hence, $\log (M \times 10) = m + 1$.

Thus, $\log (76 \times 10) = 1.880814 + 1$; or, $\log 760 = 2.880814$.

PRIN. 9.—*The logarithm of the quotient of any number divided by* 10 *is equal to the logarithm of the number diminished by* 1.

Suppose $\log M = m$; then, by Prin. 5,

$$\log (M \div 10) = \log M - \log 10; \text{ from which}$$
$$\log (M \div 10) = m - 1.$$

Thus, $\log (458 \div 10) = 2.660865 - 1$;

or, $\log 45.8 = 1.660865$.

7. The following examples will illustrate Principles 1, 3, 8, and 9.

log 234	is	2.369216,
log 23.4	"	1.369216,
log 2.34	"	0.369216,
log .234	"	$\bar{1}$.369216,
log .0234	"	$\bar{2}$.369216.

From this, we see that when we change the place of the decimal point we change the characteristic, but do not change the decimal part of the logarithm.

The minus sign is written over the characteristic, showing that it only is negative.

TABLE OF LOGARITHMS.

8. A Table of Logarithms is a table by means of which we can find the logarithms of numbers, or the numbers corresponding to given logarithms.

9. In the annexed table the entire logarithms of the numbers up to 100 are given. For numbers greater than 100 the mantissa alone is given; the characteristic being found by Prin. 1.

10. The numbers are placed in the column on the left, headed N; their logarithms are opposite, on the same line. The first two figures of the mantissa are found in the first column of logarithms.

11. The column headed D shows the average differences of the ten logarithms in the same horizontal line. This difference is found by subtracting the logarithm in column 4 from that in column 5, and is very nearly the mean or average difference.

TO FIND THE LOGARITHM OF ANY NUMBER.

12. *To find the logarithm of a number of* ONE *or* TWO *figures.*

Look on the first page of the table, in the column headed N, and opposite the given number will be found its logarithm. Thus,

the logarithm of 25 is 1.397940,
" " 87 is 1.939519.

13. *To find the logarithm of a number of* THREE *figures.*

Look in the table for the given number; opposite this, in column headed 0, will be found the decimal part of the logarithm, to which we prefix the characteristic 2, Prin. 1. Thus,

the logarithm of 325 is 2.511883,
" " 876 is 2.942504.

14. *To find the logarithm of a number of* FOUR *figures.*

Find the three left-hand figures in the column headed N, and opposite to these, in the column headed by the fourth figure, will be found four figures of the logarithm, to which two figures from the column headed 0 are to be prefixed. The characteristic is 3, Prin. 1. Thus,

the logarithm of 3456 is 3.538574,
" " 7438 is 3.871456.

15. In some of the columns, *small dots* are found in the place of figures: these dots mean zeros, and should be written zeros. If the four figures of the logarithm fall where zeros occur, or if, in passing back from the four figures found to the zero column, any of these *dots are passed over*, the two figures to be prefixed must be taken from the line just below. Thus,

the logarithm of 1738 is 3.240050,
" " 2638 is 3.421275.

16. *To find the logarithm of a number of* MORE THAN FOUR *figures.*

Place a decimal point after the fourth figure from the left hand, thus changing the number into an integer and a decimal. Find the mantissa of the entire part by the method just given. Then

from the column headed D take the corresponding *tabular difference*, multiply it by the decimal part, and add the product to the mantissa already found; the result will be the mantissa of the given number. The characteristic is determined by Prin. 1.

If the decimal part of the product exceeds .5, we add 1 to the entire part; if less than .5, it is omitted.

EXAMPLES.

1. Find the logarithm of 234567.

SOLUTION.—The characteristic is 5, Prin. 1. Placing a decimal point after the fourth figure from the left, we have 2345.67. The decimal part of the logarithm of 2345 is .370143; the number in column D is 185; and $185 \times .67 = 123.95$, and since .95 exceeds .5, we have 124, which, added to .370143, gives .370267; hence, log $234567 = 5.370267$.

2. Find the logarithm of 4567. *Ans.* 3.659631.
3. Find the logarithm of 3586. *Ans.* 3.554610.
4. Find the logarithm of 11806. *Ans.* 4.072102.
5. Find the logarithm of .4729. *Ans.* $\bar{1}.674769$.
6. Find the logarithm of 29.337. *Ans.* 1.467416.

17. *To find the number corresponding to a given logarithm.*

1. Find the *two left-hand* figures of the *mantissa* in the column headed 0, and the other four, if possible, in the same or some other column, on the same line; then, in column N, opposite to these latter figures, will be found the *three left-hand* figures, and at the top of the page the other figure of the required number.

2. *When the exact mantissa is not given in the table,* take out the four figures corresponding to the *next less mantissa* in the table; subtract this mantissa from the given one; divide the remainder, with ciphers annexed, by the number in column D, and annex the quotient to the four figures already found.

3. Make the number thus obtained correspond with the characteristic of the given logarithm, by pointing off decimals or annexing ciphers.

EXAMPLES.

1. Find the number whose logarithm is 5.370267.

SOLUTION.—The mantissa of the given logarithm is . . .370267
The mantissa of the next less logarithm of the table is . .370143
and its corresponding number is 2345.
Their difference is 124
The tabular difference is 185
The quotient is . . . 185)124.00(.67
Hence, the required number is . 234567.

NOTE.—If the characteristic had been 2, the number would have been 234.567; if it had been 7, the number would have been 23456700; if it had been $\bar{2}$, the number would have been .0234567, etc.

2. Find the number whose logarithm is 3.659631.

Ans. 4567.

3. Find the number whose logarithm is 2.554610.

Ans. 358.6.

4. Find the number whose logarithm is 1.072102.

Ans. 11.806.

5. Find the number whose logarithm is $\bar{2}$.674769.

Ans. .04729.

6. Find the number whose logarithm is $\bar{3}$.065463.

Ans. .0011627.

MULTIPLICATION BY LOGARITHMS.

18. From Prin. 4, for the multiplication of numbers by means of logarithms, we have the following

RULE.—*Find the logarithms of the factors, take their sum, and find the number corresponding to the result; this number will be the required product.*

NOTE.—The term *sum* is used in its algebraic sense. Hence, when any of the characteristics are negative,—the mantissa is always positive,—we take the difference between the sums of the positive and negative characteristics, and prefix to it the sign of the greater. If any thing is to be carried from the addition of the mantissas, it must be added to a positive characteristic, or subtracted from a negative one.

EXAMPLES.

1. Multiply 35.16 by 8.15.

SOLUTION.

$$\begin{array}{lr} \log 35.16 = & 1.546049 \\ \log 8.15 = & 0.911158 \\ \hline & 2.457207 \\ & 457125 \\ \hline & 152)82.00(.54 \end{array}$$

Product, 286.554

2. Find the product of .7856, 31.42. *Ans.* 24.6835.

3. Find the product of 31.42, 56.13, and 516.78. *Ans.* 911393.7.

4. Find the product of 31.462, .05673, and .006785. *Ans.* .01211168.

5. Product of .06517, 2.16725, .000317, and 42.1234. *Ans.* .001886.

6. Product of 2.3456, .00314, 123.789, .00078, and 67.105. *Ans.* .04772076.

DIVISION BY LOGARITHMS.

19. From Prin. 5, to divide by means of logarithms, we have the following

RULE.—*Find the logarithms of the dividend and divisor, subtract the latter from the former, and find the number corresponding to the result: this number will be the required quotient.*

NOTE.—The term *subtract* is here used in its algebraic sense; hence, we must subtract according to the principles of algebra.

EXAMPLES.

1. Divide 783.5 by 6.25.

SOLUTION.

$$\begin{array}{lr} \log 783.5 = & 2.894039 \\ \log 6.25 = & 0.795880 \\ \hline & 2.098159 \\ & .097951 \\ \hline & 346)208(6 \end{array}$$

Quotient, 125.36

2. Divide 272.636 by 6.37. *Ans.* 42.8.

3. Divide 50.38218 by 67.8. *Ans.* .7431.

4. Divide 155 by .0625. *Ans.* 2480.

ARITHMETICAL COMPLEMENT.

20. The operation of division when combined with multiplication is somewhat simplified by using the principle of the *arithmetical complement.*

21. The ARITHMETICAL COMPLEMENT of a logarithm is the result arising from subtracting the logarithm from 10. Thus, the arithmetical complement of the logarithm 5.623427 is $10 - 5.623427$, or 4.376573.

22. The arithmetical complement may be written directly from the table, *by subtracting each figure of the logarithm from* 9, *except the right-hand figure, which must be taken from* 10. This is the same as subtracting the logarithm from 10.

23. We will now prove that *the difference between two logarithms is equal to the first logarithm, plus the arithmetical complement of the second, minus* 10.

Let $a =$ the first logarithm,

$b =$ the second logarithm,

and $c = 10 - b =$ arith. comp. of b.

The difference is $a - b$.

But, $-b = c - 10$.

Hence, $a - b = a + c - 10$,

which proves the principle.

24. Hence, to divide by means of the arithmetical complement, we have the following

RULE.—*Add the arithmetical complement of the logarithm of the divisor to the logarithm of the dividend, subtract* 10, *and find the number corresponding to the difference, this number will be the required quotient.*

EXAMPLES.

1. Divide 856.3 by 45.32.

SOLUTION.	log 856.3 . .	2.932626
	(a. c.) log 45.32 . .	8.343710
Quotient,	18.8945	1.276336

2. Divide 0.3156 by 78.35.

	log 0.3156 . . .	$\bar{1}.499137$
	(a. c.) log 78.35 . . .	8.105961
Quotient,	.004028	$\bar{3}.605098$

3. Divide 3.7521 by 18.346. *Ans.* .204519.

4. Divide 483.72 by .30751. *Ans.* 1573.02.

5. Multiply 32.16 by 7.856, and divide the product by 45.327. *Ans.* 5.574.

6. Divide the product of 31.57 and 123.4 by the product of 316.2 and .0316. *Ans.* 389.8884.

7. Find by logarithms the first term of the proportion, *x* . 73.15 : : 48.16 : 3167. *Ans.* 1.11237.

INVOLUTION BY LOGARITHMS.

25. From Prin. 6, to raise a number to any power, we have the following

RULE.—*Find the logarithm of the number, multiply it by the exponent of the power, and find the number corresponding to the result.*

EXAMPLES.

1. Find the 4th power of 45.

SOLUTION.

$$\log 45 = 1.653213$$
$$\times 4$$

Power,	4100625	6.612852

2. Find the cube of 0.65. *Ans.* 0.2746

3. Find the 6th power of 1.037. *Ans.* 1.243.

4. Find the 7th power of .4797. *Ans.* 0.005846.

EVOLUTION BY LOGARITHMS.

26. From Prin. 7, to extract any root of a number, we have the following

RULE.—I. *Find the logarithm of the number, divide it by the index of the root, and find the number corresponding to the result.*

II. *If the characteristic is negative and not divisible by the index of the root, add to it the smallest negative number that will make it divisible, prefixing the same number with a plus sign to the mantissa.*

EXAMPLES.

1. Find the square root of 576.

SOLUTION. $\log 576 = 2.760422$

$$2.760422 \div 2 = 1.380211$$

Hence, the root is 24.

2. Find the fourth root of .325.

SOLUTION. $\log .325 = \bar{1}.511883 = \bar{4} + 3.511883.$

Then $(\bar{4} + 3.511883) \div 4 = \bar{1}.877971$

Hence, the quotient is, .75504.

3. Find the fifth root of .0625. *Ans.* .574348.

4. Find the cube root of 7. *Ans.* 1.9129.

5. Find the fifth root of 5. *Ans.* 1.3797.

6. Find the tenth root of 8764.5. *Ans.* 2.479.

CALCULATION OF LOGARITHMS.

The pupil will by this time naturally inquire how these logarithms are calculated. This we have not room to explain here; in fact, an explanation of the modern methods would be almost too difficult for the majority of pupils who study this book. Only a general idea can here be given.

In computing logarithms, it is only necessary to calculate the logarithms of prime numbers, since the logarithms of composite numbers may be obtained by adding the logarithms of their prime factors.

The logarithms of the prime numbers were first computed by com-

paring the geometrical and arithmetical series, 1, 10, 100, etc., and 0, 1, 2, etc., and finding geometrical and arithmetical means; the arithmetical mean being the logarithm of the corresponding geometrical mean. This method was exceedingly laborious, involving so many multiplications and extractions of roots.

The method now generally used is that of series, by which the computations are much more easily made. The following formula is derived by algebraic reasoning.

$$\log(1+x) = A\left(\frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \text{etc.}\right)$$

In this the quantity A is called the *modulus*, which in the Napierian system is *unity*. The series, when A is *one*, put in a more convenient form, becomes,

$$\log.(z+1) - \log.z = 2\left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \text{etc}\right)$$

From which, knowing the logarithm of any number, we readily find the logarithm of the next larger number. The pupil will be interested in finding logarithms by this formula. Begin with 2, in which $z = 1$.

The logarithm found will be the Napierian logarithm, and this multiplied by 0.434294 will give the common logarithm.

A TABLE

OF

LOGARITHMS OF NUMBERS

From 1 to 10,000.

N.	Log.	N.	Log.	N.	Log.	N.	Log.
1	0·000000	26	1·414973	51	1·707570	76	1·880814
2	0·301030	27	1·431364	52	1·716003	77	1·886491
3	0·477121	28	1·447158	53	1·724276	78	1·892095
4	0·602060	29	1·462398	54	1·732394	79	1·897627
5	0·698970	30	1·477121	55	1·740363	80	1·903090
6	0·778151	31	1·491362	56	1·748188	81	1·908485
7	0·845098	32	1·505150	57	1·755875	82	1·913814
8	0·903090	33	1·518514	58	1·763428	83	1·919078
9	0·954243	34	1·531479	59	1·770852	84	1·924279
10	1·000000	35	1·544068	60	1·778151	85	1·929419
11	1·041393	36	1·556303	61	1·785330	86	1·934498
12	1·079181	37	1·568202	62	1·792392	87	1·939519
13	1·113943	38	1·579784	63	1·799341	88	1·944483
14	1·146128	39	1·591065	64	1·806181	89	1·949390
15	1·176091	40	1·602060	65	1·812913	90	1·954243
16	1·204120	41	1·612784	66	1·819544	91	1·959041
17	1·230449	42	1·623249	67	1·826075	92	1·963788
18	1·255273	43	1·633468	68	1·832509	93	1·968483
19	1·278754	44	1·643453	69	1·838849	94	1·973128
20	1·301030	45	1·653213	70	1·845098	95	1·977724
21	1·322219	46	1·662758	71	1·851258	96	1·982271
22	1·342423	47	1·672098	72	1·857333	97	1·986772
23	1·361728	48	1·681241	73	1·863323	98	1·991226
24	1·380211	49	1·690196	74	1·869232	99	1·995635
25	1·397940	50	1·698970	75	1·875061	100	2·000000

Remark.—In the following table, in the nine right-hand columns of each page, where the first or leading figures change from 9's to 0's, points or dots are introduced instead of the 0's, to catch the eye, and to indicate that from thence the two figures of the Logarithm to be taken from the second column, stand in the next line below.

N.	0	1	2	3	4	5	6	7	8	9	D.
100	000000	0434	0868	1301	1734	2166	2598	3029	3461	3891	432
101	4321	4751	5181	5609	6038	6466	6894	7321	7748	8174	428
102	8600	9026	9451	9876	•300	•724	1147	1570	1993	2415	424
103	012837	3259	3680	4100	4521	4940	5360	5779	6197	6616	419
104	7033	7451	7868	8284	8700	9116	9532	9947	•361	•775	416
105	021189	1603	2016	2428	2841	3252	3664	4075	4486	4896	412
106	5306	5715	6125	6533	6042	7350	7757	8164	8571	8978	408
107	9384	9789	•195	•600	1004	1408	1812	2216	2619	3021	404
108	033424	3826	4227	4628	5029	5430	5830	6230	6629	7028	400
109	7426	7825	8223	8620	9017	9414	9811	•207	•602	•998	396
110	041393	1787	2182	2576	2969	3362	3755	4148	4540	4932	393
111	5323	5714	6105	6495	6885	7275	7664	8053	8442	8830	389
112	9218	9606	9993	•380	•766	1153	1538	1924	2309	2694	386
113	053078	3463	3846	4230	4613	4996	5378	5760	6142	6524	382
114	6905	7286	7666	8046	8426	8805	9185	9563	9942	•320	379
115	060698	1075	1452	1829	2206	2582	2958	3333	3709	4083	376
116	4458	4832	5206	5580	5953	6326	6699	7071	7443	7815	372
117	8186	8557	8928	9298	9668	••38	•407	•776	1145	1514	369
118	071882	2250	2617	2985	3352	3718	4085	4451	4816	5182	366
119	5547	5912	6276	6640	7004	7368	7731	8094	8457	8819	363
120	079181	9543	9904	•266	•626	•987	1347	1707	2067	2426	360
121	082785	3144	3503	3861	4219	4576	4934	5291	5647	6004	357
122	6360	6716	7071	7426	7781	8136	8490	8845	9198	9552	355
123	9905	•258	•611	•963	1315	1667	2018	2370	2721	3071	351
124	093422	3772	4122	4471	4820	5169	5518	5866	6215	6562	349
125	6910	7257	7604	7951	8298	8644	8990	9335	9681	••26	346
126	100371	0715	1059	1403	1747	2091	2434	2777	3119	3462	343
127	3804	4146	4487	4828	5169	5510	5851	6191	6531	6871	340
128	7210	7549	7888	8227	8565	8903	9241	9579	9916	•253	338
129	110590	0926	1263	1599	1934	2270	2605	2940	3275	3609	335
130	113943	4277	4611	4944	5278	5611	5943	6276	6608	6940	333
131	7271	7603	7934	8265	8595	8926	9256	9586	9915	•245	330
132	120574	0903	1231	1560	1888	2216	2544	2871	3198	3525	328
133	3852	4178	4504	4830	5156	5481	5806	6131	6456	6781	325
134	7105	7429	7753	8076	8399	8722	9045	9368	9690	••12	323
135	130334	0655	0977	1298	1619	1939	2260	2580	2900	3219	321
136	3539	3858	4177	4496	4814	5133	5451	5769	6086	6403	318
137	6721	7037	7354	7671	7987	8303	8618	8934	9249	9564	315
138	9879	•194	•508	•822	1136	1450	1763	2076	2389	2702	314
139	143015	3327	3639	3951	4263	4574	4885	5196	5507	5818	311
140	146128	6438	6748	7058	7367	7676	7985	8294	8603	8911	309
141	9219	9527	9835	•142	•449	•756	1063	1370	1676	1982	307
142	152288	2594	2900	3205	3510	3815	4120	4424	4728	5032	305
143	5336	5640	5943	6246	6549	6852	7154	7457	7759	8061	303
144	8362	8664	8965	9266	9567	9868	•168	•469	•769	1068	301
145	161368	1667	1967	2266	2564	2863	3161	3460	3758	4055	299
146	4353	4650	4947	5244	5541	5838	6134	6430	6726	7022	297
147	7317	7613	7908	8203	8497	8792	9086	9380	9674	9968	295
148	170262	0555	0848	1141	1434	1726	2019	2311	2603	2895	293
149	3186	3478	3769	4060	4351	4641	4932	5222	5512	5802	291
150	176091	6381	6670	6959	7248	7536	7825	8113	8401	8689	289
151	8977	9264	9552	9839	•126	•413	•699	•985	1272	1558	287
152	181844	2129	2415	2700	2985	3270	3555	3839	4123	4407	285
153	4691	4975	5259	5542	5825	6108	6391	6674	6956	7239	283
154	7521	7803	8084	8366	8647	8928	9209	9490	9771	••51	281
155	190332	0612	0892	1171	1451	1730	2010	2289	2567	2846	279
156	3125	3403	3681	3959	4237	4514	4792	5069	5346	5623	278
157	5899	6176	6453	6729	7005	7281	7556	7832	8107	8382	276
158	8657	8932	9206	9481	9755	••29	•303	•577	•850	1124	274
159	201397	1670	1943	2216	2488	2761	3033	3305	3577	3848	272
N.	0	1	2	3	4	5	6	7	8	9	D.

N.	0	1	2	3	4	5	6	7	8	9	D.
160	204120	4391	4663	4934	5204	5475	5746	6016	6286	6556	271
161	6826	7096	7365	7634	7904	8173	8441	8710	8979	9247	269
162	9515	9783	••51	•319	•586	•853	1121	1388	1654	1921	267
163	212188	2454	2720	2986	3252	3518	3783	4049	4314	4579	266
164	4844	5109	5373	5638	5902	6166	6430	6694	6957	7221	264
165	7484	7747	8010	8273	8536	8798	9060	9323	9585	9846	262
166	220108	0370	0631	0892	1153	1414	1675	1936	2196	2456	261
167	2716	2976	3236	3496	3755	4015	4274	4533	4792	5051	259
168	5309	5568	5826	6084	6342	6600	6858	7115	7372	7630	258
169	7887	8144	8400	8657	8913	9170	9426	9682	9938	•193	256
170	230449	0704	0960	1215	1470	1724	1979	2234	2488	2742	254
171	2996	3250	3504	3757	4011	4264	4517	4770	5023	5276	253
172	5528	5781	6033	6285	6537	6789	7041	7292	7544	7795	252
173	8046	8297	8548	8799	9049	9299	9550	9800	••50	•300	250
174	240549	0799	1048	1297	1546	1795	2044	2293	2541	2790	249
175	3038	3286	3534	3782	4030	4277	4525	4772	5019	5266	248
176	5513	5759	6006	6252	6499	6745	6991	7237	7482	7728	246
177	7973	8219	8464	8709	8954	9198	9443	9687	9932	•176	245
178	250420	0664	0908	1151	1395	1638	1881	2125	2368	2610	243
179	2853	3096	3338	3580	3822	4064	4306	4548	4790	5031	242
180	255273	5514	5755	5996	6237	6477	6718	6958	7198	7439	241
181	7679	7918	8158	8398	8637	8877	9116	9355	9594	9833	239
182	260071	0310	0548	0787	1025	1263	1501	1739	1976	2214	238
183	2451	2688	2925	3162	3399	3636	3873	4109	4346	4582	237
184	4818	5054	5290	5525	5761	5996	6232	6467	6702	6937	235
185	7172	7406	7641	7875	8110	8344	8578	8812	9046	9279	234
186	9513	9746	9980	•213	•446	•679	•912	1144	1377	1609	233
187	271842	2074	2306	2538	2770	3001	3233	3464	3696	3927	232
188	4158	4389	4620	4850	5081	5311	5542	5772	6002	6232	230
189	6462	6692	6921	7151	7380	7609	7838	8067	8296	8525	229
190	278754	8982	9211	9439	9667	9895	•123	•351	•578	•806	228
191	281033	1261	1488	1715	1942	2169	2396	2622	2849	3075	227
192	3301	3527	3753	3979	4205	4431	4656	4882	5107	5332	226
193	5557	5782	6007	6232	6456	6681	6905	7130	7354	7578	225
194	7802	8026	8249	8473	8696	8920	9143	9366	9589	9812	223
195	290035	0257	0480	0702	0925	1147	1369	1591	1813	2034	222
196	2256	2478	2699	2920	3141	3363	3584	3804	4025	4246	221
197	4466	4687	4907	5127	5347	5567	5787	6007	6226	6446	220
198	6665	6884	7104	7323	7542	7761	7979	8198	8416	8635	219
199	8853	9071	9289	9507	9725	9943	•161	•378	•595	•813	218
200	301030	1247	1464	1681	1898	2114	2331	2547	2764	2980	217
201	3196	3412	3628	3844	4059	4275	4491	4706	4921	5136	216
202	5351	5566	5781	5996	6211	6425	6639	6854	7068	7282	215
203	7496	7710	7924	8137	8351	8564	8778	8991	9204	9417	213
204	9630	9843	••56	•268	•481	•693	•906	1118	1330	1542	212
205	311754	1966	2177	2389	2600	2812	3023	3234	3445	3656	211
206	3867	4078	4289	4499	4710	4920	5130	5340	5551	5760	210
207	5970	6180	6390	6599	6809	7018	7227	7436	7646	7854	209
208	8063	8272	8481	8689	8898	9106	9314	9522	9730	9938	208
209	320146	0354	0562	0769	0977	1184	1391	1598	1805	2012	207
210	322219	2426	2633	2839	3046	3252	3458	3665	3871	4077	206
211	4282	4488	4694	4899	5105	5310	5516	5721	5926	6131	205
212	6336	6541	6745	6950	7155	7359	7563	7767	7972	8176	204
213	8380	8583	8787	8991	9194	9398	9601	9805	•••8	•211	203
214	330414	0617	0819	1022	1225	1427	1630	1832	2034	2236	202
215	2438	2640	2842	3044	3246	3447	3649	3850	4051	4253	202
216	4454	4655	4856	5057	5257	5458	5658	5859	6059	6260	201
217	6460	6660	6860	7060	7260	7459	7659	7858	8058	8257	200
218	8456	8656	8855	9054	9253	9451	9650	9849	••47	•246	199
219	340444	0642	0841	1039	1237	1435	1632	1830	2028	2225	198
N.	0	1	2	3	4	5	6	7	8	9	D.

N	0	1	2	3	4	5	6	7	8	9	D.
220	342423	2620	2817	3014	3212	3409	3606	3802	3999	4196	197
221	4392	4589	4785	4981	5178	5374	5570	5766	5962	6157	196
222	6353	6549	6744	6939	7135	7330	7525	7720	7915	8110	195
223	8305	8500	8694	8889	9083	9278	9472	9666	9860	*0054	194
224	350248	0442	0636	0829	1023	1216	1410	1603	1796	1989	193
225	2183	2375	2568	2761	2954	3147	3339	3532	3724	3916	193
226	4108	4301	4493	4685	4876	5068	5260	5452	5643	5834	192
227	6026	6217	6408	6599	6790	6981	7172	7363	7554	7744	191
228	7935	8125	8316	8506	8696	8886	9076	9266	9456	9646	190
229	9835	*0025	*0215	*0404	*0593	*0783	*0972	1161	1350	1539	189
230	361728	1917	2105	2294	2482	2671	2859	3048	3236	3424	188
231	3612	3800	3988	4176	4363	4551	4739	4926	5113	5301	188
232	5488	5675	5862	6049	6236	6423	6610	6796	6983	7169	187
233	7356	7542	7729	7915	8101	8287	8473	8659	8845	9030	186
234	9216	9401	9587	9772	9958	*0143	*0328	*0513	*0698	*0883	185
235	371068	1253	1437	1622	1806	1991	2175	2360	2544	2728	184
236	2912	3096	3280	3464	3647	3831	4015	4198	4382	4565	184
237	4748	4932	5115	5298	5481	5664	5846	6029	6212	6394	183
238	6577	6759	6942	7124	7306	7488	7670	7852	8034	8216	182
239	8398	8580	8761	8943	9124	9306	9487	9668	9849	*0030	181
240	380211	0392	0573	0754	0934	1115	1296	1476	1656	1837	181
241	2017	2197	2377	2557	2737	2917	3097	3277	3456	3636	180
242	3815	3995	4174	4353	4533	4712	4891	5070	5249	5428	179
243	5606	5785	5964	6142	6321	6499	6677	6856	7034	7212	178
244	7390	7568	7746	7923	8101	8279	8456	8634	8811	8989	178
245	9166	9343	9520	9698	9875	*0051	*0228	*0405	*0582	*0759	177
246	390935	1112	1288	1464	1641	1817	1993	2169	2345	2521	176
247	2697	2873	3048	3224	3400	3575	3751	3926	4101	4277	176
248	4452	4627	4802	4977	5152	5326	5501	5676	5850	6025	175
249	6199	6374	6548	6722	6896	7071	7245	7419	7592	7766	174
250	397940	8114	8287	8461	8634	8808	8981	9154	9328	9501	173
251	9674	9847	*0020	*0192	*0365	*0538	*0711	*0883	1056	1228	173
252	401401	1573	1745	1917	2089	2261	2433	2605	2777	2949	172
253	3121	3292	3464	3635	3807	3978	4149	4320	4492	4663	171
254	4834	5005	5176	5346	5517	5688	5858	6029	6199	6370	171
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261	6641	6807	6973	7139	7306	7472	7638	7804	7970	8135	166
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798	2003	2057	2112	2166	2221	2275	2329	2384	2438	2492	54
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894	1338	1386	1435	1483	1532	1580	1629	1677	1726	1775	49
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898	3276	3325	3373	3421	3470	3518	3566	3615	3663	3711	48
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915	1421	1469	1516	1563	1611	1658	1706	1753	1801	1848	47
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921	4260	4307	4354	4401	4448	4495	4542	4590	4637	4684	47
922	4731	4778	4825	4872	4919	4966	5013	5061	5108	5155	47
923	5202	5249	5296	5343	5390	5437	5484	5531	5578	5625	47
924	5672	5719	5766	5813	5860	5907	5954	6001	6048	6095	47
925	6142	6189	6236	6283	6329	6376	6423	6470	6517	6564	47
926	6611	6658	6705	6752	6799	6845	6892	6939	6986	7033	47
927	7080	7127	7173	7220	7267	7314	7361	7408	7454	7501	47
928	7548	7595	7642	7688	7735	7782	7829	7875	7922	7969	47
929	8016	8062	8109	8156	8203	8249	8296	8343	8390	8436	47
930	968483	8530	8576	8623	8670	8716	8763	8810	8856	8903	47
931	8950	8996	9043	9090	9136	9183	9229	9276	9323	9369	47
932	9416	9463	9509	9556	9602	9649	9695	9742	9789	9835	47
933	9882	9928	9975	••21	••68	•114	•161	•207	•254	•300	47
934	970347	0393	0440	0486	0533	0579	0626	0672	0719	0765	46
935	0812	0858	0904	0951	0997	1044	1090	1137	1183	1229	46
936	1276	1322	1369	1415	1461	1508	1554	1601	1647	1693	46
937	1740	1786	1832	1879	1925	1971	2018	2064	2110	2157	46
938	2203	2249	2295	2342	2388	2434	2481	2527	2573	2619	46
939	2666	2712	2758	2804	2851	2897	2943	2989	3035	3082	46
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940	973128	3174	3220	3266	3313	3359	3405	3451	3497	3543	46
941	3590	3636	3682	3728	3774	3820	3866	3913	3959	4005	46
942	4051	4097	4143	4189	4235	4281	4327	4374	4420	4466	46
943	4512	4558	4604	4650	4696	4742	4788	4834	4880	4926	46
944	4972	5018	5064	5110	5156	5202	5248	5294	5340	5386	46
945	5432	5478	5524	5570	5616	5662	5707	5753	5799	5845	46
946	5891	5937	5983	6029	6075	6121	6167	6212	6258	6304	46
947	6350	6396	6442	6488	6533	6579	6625	6671	6717	6763	46
948	6808	6854	6900	6946	6992	7037	7083	7129	7175	7220	46
949	7266	7312	7358	7403	7449	7495	7541	7586	7632	7678	46
950	977724	7769	7815	7861	7906	7952	7998	8043	8089	8135	46
951	8181	8226	8272	8317	8363	8409	8454	8500	8546	8591	46
952	8637	8683	8728	8774	8819	8865	8911	8956	9002	9047	46
953	9093	9138	9184	9230	9275	9321	9366	9412	9457	9503	46
954	9548	9594	9639	9685	9730	9776	9821	9867	9912	9958	46
955	980003	0049	0094	0140	0185	0231	0276	0322	0367	0412	45
956	0458	0503	0549	0594	0640	0685	0730	0776	0821	0867	45
957	0912	0957	1003	1048	1093	1139	1184	1229	1275	1320	45
958	1366	1411	1456	1501	1547	1592	1637	1683	1728	1773	45
959	1819	1864	1909	1954	2000	2045	2090	2135	2181	2226	45
960	982271	2316	2362	2407	2452	2497	2543	2588	2633	2678	45
961	2723	2769	2814	2859	2904	2949	2994	3040	3085	3130	45
962	3175	3220	3265	3310	3356	3401	3446	3491	3536	3581	45
963	3626	3671	3716	3762	3807	3852	3897	3942	3987	4032	45
964	4077	4122	4167	4212	4257	4302	4347	4392	4437	4482	45
965	4527	4572	4617	4662	4707	4752	4797	4842	4887	4932	45
966	4977	5022	5067	5112	5157	5202	5247	5292	5337	5382	45
967	5426	5471	5516	5561	5606	5651	5696	5741	5786	5830	45
968	5875	5920	5965	6010	6055	6100	6144	6189	6234	6279	45
969	6324	6369	6413	6458	6503	6548	6593	6637	6682	6727	45
970	986772	6817	6861	6906	6951	6996	7040	7085	7130	7175	45
971	7219	7264	7309	7353	7398	7443	7488	7532	7577	7622	45
972	7666	7711	7756	7800	7845	7890	7934	7979	8024	8068	45
973	8113	8157	8202	8247	8291	8336	8381	8425	8470	8514	45
974	8559	8604	8648	8693	8737	8782	8826	8871	8916	8960	45
975	9005	9049	9094	9138	9183	9227	9272	9316	9361	9405	45
976	9450	9494	9539	9583	9628	9672	9717	9761	9806	9850	44
977	9895	9939	9983	••28	••72	•117	•161	•206	•250	•294	44
978	990339	0383	0428	0472	0516	0561	0605	0650	0694	0738	44
979	0783	0827	0871	0916	0960	1004	1049	1093	1137	1182	44
980	991226	1270	1315	1359	1403	1448	1492	1536	1580	1625	44
981	1669	1713	1758	1802	1846	1890	1935	1979	2023	2067	44
982	2111	2156	2200	2244	2288	2333	2377	2421	2465	2509	44
983	2554	2598	2642	2686	2730	2774	2819	2863	2907	2951	44
984	2995	3039	3083	3127	3172	3216	3260	3304	3348	3392	44
985	3436	3480	3524	3568	3613	3657	3701	3745	3789	3833	44
986	3877	3921	3965	4009	4053	4097	4141	4185	4229	4273	44
987	4317	4361	4405	4449	4493	4537	4581	4625	4669	4713	44
988	4757	4801	4845	4889	4933	4977	5021	5065	5108	5152	44
989	5196	5240	5284	5328	5372	5416	5460	5504	5547	5591	44
990	995635	5679	5723	5767	5811	5854	5898	5942	5986	6030	44
991	6074	6117	6161	6205	6249	6293	6337	6380	6424	6468	44
992	6512	6555	6599	6643	6687	6731	6774	6818	6862	6906	44
993	6949	6993	7037	7080	7124	7168	7212	7255	7299	7343	44
994	7386	7430	7474	7517	7561	7605	7648	7692	7736	7779	44
995	7823	7867	7910	7954	7998	8041	8085	8129	8172	8216	44
996	8259	8303	8347	8390	8434	8477	8521	8564	8608	8652	44
997	8695	8739	8782	8826	8869	8913	8956	9000	9043	9087	44
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